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Two New Collineations of some Moufang-Klingenberg Planes

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***Abstract*—In this paper we are interested in Moufang-Klingenberg planes $\mathbf{M}(\mathcal{A})$ defined over a local alternative ring $\mathcal{A}$ of dual numbers. We introduce two new collineations of $\mathbf{M}(\mathcal{A})$.

***Keywords*—Moufang-Klingenberg planes, local alternative ring, projective collineation.

I. INTRODUCTION

The number of collineations of any projective plane is huge. For example; the Fano plane has 168 collineations, the non-Desarguesian projective Veblen-Wedderburn plane of order 9 (which is denoted by $\pi_N(9)$) has 311,040 collineations [8, p. 366]. It is easy to see that the composite of any two collineations is a collineation, as the invers of any collineation. Function composition is always associative; thus the collineations of any projective or affine plane form a group. For more detailed information about these groups, the reader is referred to the books of [5], [8].

In this paper we deal with the class (which we will denote by $\mathbf{M}(\mathcal{A})$) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ (an alternative field $\mathbf{A}$, $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [3]. We will introduce two collineations of $\mathbf{M}(\mathcal{A})$, different from the collineations given in [4].

The paper is organized as follows. Section 2 includes some basic definitions and results from the literature. In Section 3 we will give two transformations of $\mathbf{M}(\mathcal{A})$ and show that the transformations are collineations $\mathbf{M}(\mathcal{A})$.

II. PRELIMINARIES

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, incidence) and an equivalence relation '$\sim$' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$, respectively. Then $\mathbf{M}$ is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If P, Q are non-neighbour points, then there is a unique line PQ through P and Q.

(PK2) If g, h are non-neighbour lines, then there is a unique point $g \cap h$ on both g and h.

(PK3) There is a projective plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and an incidence structure epimorphism $\Psi : \mathbf{M} \to \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q, \ \Psi(g) = \Psi(h) \Leftrightarrow g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

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A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line $h \sim g$ such that $P \in h$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of $\mathbf{M}$.

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane $\mathbf{M}$ that generalizes a Moufang plane, and for which $\mathbf{M}^*$ is a Moufang plane (for the exact definition see [1]).

An *alternative ring (field)* $\mathbf{R}$ is a not necessarily associative ring (field) that satisfies the alternative laws

$$a(ab) = a^2 b, \quad (ba)a = ba^2, \quad \forall a, b \in \mathbf{R}.$$

An alternative ring $\mathbf{R}$ with identity element 1 is called *local* if the set $\mathbf{I}$ of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

Lemma 2.1: The subring generated by any two elements of an alternative ring is associative (cf. [7, Theorem 3.1]).

Lemma 2.2: The identities

$$x(y(xz)) = (xyx)z$$
$$((yx)z)x = y(xzx)$$
$$(xy)(zx) = x(yz)x$$

which are known as Moufang identities are satisfied in every alternative ring (cf. [6, p. 160]).

We summarize some basic concepts about the coordinatization of MK-planes from [1].

Let $\mathbf{R}$ be a local alternative ring. Then $\mathbf{M}(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$\begin{aligned}
\mathbf{P} &= \{(x,y,1) : \ x, y \in \mathbf{R}\} \\
&\quad \cup \{(1,y,z) : \ y \in \mathbf{R}, \ z \in \mathbf{I}\} \\
&\quad \cup \{(w,1,z) : w, z \in \mathbf{I}\}, \\
\mathbf{L} &= \{[m,1,p] : \ m, p \in \mathbf{R}\} \\
&\quad \cup \{[1,n,p] : \ p \in \mathbf{R}, \ n \in \mathbf{I}\} \\
&\quad \cup \{[q,n,1] : q, n \in \mathbf{I}\}, \\
[m,1,p] &= \{(x, xm+p, 1) : x \in \mathbf{R}\} \\
&\quad \cup \{(1, zp+m, z) : z \in \mathbf{I}\}, \\
[1,n,p] &= \{(yn+p, y, 1) : y \in \mathbf{R}\} \\
&\quad \cup \{(zp+n, 1, z) : z \in \mathbf{I}\}, \\
[q,n,1] &= \{(1, y, yn+q) : y \in \mathbf{R}\} \\
&\quad \cup \{(w, 1, wq+n) : w \in \mathbf{I}\}
\end{aligned}$$

and

$$\begin{aligned} &P=(x_1,x_2,x_3)\sim(y_1,y_2,y_3)=Q\Leftrightarrow\\ &x_i-y_i\in\mathbf{I}\ \ (i=1,2,3)\,,\forall P,Q\in\mathbf{P};\\ &g=[x_1,x_2,x_3]\sim[y_1,y_2,y_3]=h\Leftrightarrow\\ &x_i-y_i\in\mathbf{I}\ \ (i=1,2,3)\,,\forall g,h\in\mathbf{L}. \end{aligned}$$

Now it is time to give the following theorem from [1].

Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Let $\mathbf{A}$ be an alternative field and $\varepsilon\notin\mathbf{A}$. Consider $\mathcal{A}:=\mathbf{A}(\varepsilon)=\mathbf{A}+\mathbf{A}\varepsilon$ with componentwise addition and multiplication as follows:

$$(a_1+a_2\varepsilon)\,(b_1+b_2\varepsilon)=a_1b_1+(a_1b_2+a_2b_1)\,\varepsilon,$$

where $a_i,b_i\in\mathbf{A},\ i=1,2$. Then $\mathcal{A}$ is a local alternative ring with ideal $\mathbf{I}=\mathbf{A}\varepsilon$ of non-units. The set of formal inverses of the non-units of $\mathcal{A}$ is denoted as $\mathbf{I}^{-1}$. Calculations with the elements of $\mathbf{I}^{-1}$ are defined as follows [2]:

$$\begin{aligned} (a\varepsilon)^{-1}+t\quad&:\quad=(a\varepsilon)^{-1}:=t+(a\varepsilon)^{-1}\,,\\ q\,(a\varepsilon)^{-1}\quad&:\quad=\left(aq^{-1}\varepsilon\right)^{-1},\\ (a\varepsilon)^{-1}q\quad&:\quad=\left(q^{-1}a\varepsilon\right)^{-1},\\ \left((a\varepsilon)^{-1}\right)^{-1}\quad&:\quad=a\varepsilon, \end{aligned}$$

where $(a\varepsilon)^{-1}\in\mathbf{I}^{-1},\ t\in\mathcal{A},\ q\in\mathcal{A}\setminus\mathbf{I}$. (Other terms are not defined.). For more information about $\mathcal{A}$ and its relation to MK-planes, the reader is referred to the papers of Blunck [2], [3]. In [3], the centre $\mathbf{Z}\,(\mathcal{A})$ is defined to be the (commutative, associative) subring of $\mathcal{A}$ which is commuting and associating with all elements of $\mathcal{A}$. It is $\mathbf{Z}\,(\mathcal{A}):=\mathbf{Z}\,(\varepsilon)=\mathbf{Z}+\mathbf{Z}\varepsilon$ where $\mathbf{Z}=\{z\in\mathbf{A}:za=az,\ \forall a\in\mathbf{A}\}$ is the centre of $\mathbf{A}$. If $\mathbf{A}$ is not associative, then $\mathbf{A}$ is a Cayley division algebra over its centre $\mathbf{Z}$. Throughout this paper we assume $char\mathbf{A}\neq\mathbf{2}$ and we restrict ourselves to the MK-planes $\mathbf{M}(\mathcal{A})$. In the next section, we will introduce two collineations of $\mathbf{M}(\mathcal{A})$.

III. Two Collineations of $\mathbf{M}(\mathcal{A})$

In this section we will give two transformations. We will show that these are collineations of $\mathbf{M}(\mathcal{A})$.

Now we start with giving the transformations, where $w,z,q,n\in\mathbf{A}$: For any $s\notin\mathbf{I}$, the map J_s transforms points and lines as follows:

$$\begin{aligned} &(x,y,1)\rightarrow\left(ys^{-1},xs,1\right),\\ &(1,y,z\varepsilon)\rightarrow\left(1,sy^{-1}s,s(y^{-1}z)\right)\ \ if\quad y\notin\mathbf{I},\\ &(1,y,z\varepsilon)\rightarrow\left(s^{-1}ys^{-1},1,s^{-1}z\right)\quad if\ y\in\mathbf{I},\\ &(w\varepsilon,1,z\varepsilon)\rightarrow(1,sws,sz) \end{aligned}$$

and

$$\begin{aligned} &[m,1,k]\rightarrow\left[sm^{-1}s,1,-\left(km^{-1}\right)s\right]\ \ if\ m\notin\mathbf{I},\\ &[m,1,k]\rightarrow\left[1,s^{-1}ms^{-1},ks^{-1}\right]\ \ if\ m\in\mathbf{I},\\ &[1,n\varepsilon,p]\rightarrow[sns,1,ps]\,,\\ &[q\varepsilon,n\varepsilon,1]\rightarrow\left[sn,s^{-1}q,1\right]. \end{aligned}$$

For any $s\notin\mathbf{I}$, the map H_s transforms points and lines as follows:

$$\begin{aligned} &(x,y,1)\rightarrow\left(s\left((y+s)^{-1}x\right),\left(s\,(y+s)^{-1}\right)y,1\right)\\ &\quad if\quad y+s\notin\mathbf{I},\\ &(x,y,1)\rightarrow\left(1,x^{-1}y,\left(x^{-1}\,(y+s)\right)s^{-1}\right)\\ &\quad if\quad y+s\in\mathbf{I}\wedge x\notin\mathbf{I},\\ &(x,y,1)\rightarrow\left(y^{-1}x,1,y^{-1}\left((y+s)\,s^{-1}\right)\right)\\ &\quad if\quad y+s\in\mathbf{I}\wedge x\in\mathbf{I},\\ &(1,y,z\varepsilon)\rightarrow\left(s\,(y+zs)^{-1},\left(s\,(y+zs)^{-1}\right)y,1\right)\\ &\quad if\quad y\notin\mathbf{I},\\ &(1,y,z\varepsilon)\rightarrow\left(1,y,z+ys^{-1}\right)\\ &\quad if\quad y\in\mathbf{I},\\ &(w\varepsilon,1,z\varepsilon)\rightarrow\left(\left(s\,(1+zs)^{-1}\right)w,s\,(1+zs)^{-1},1\right) \end{aligned}$$

and

$$\begin{aligned} &[m,1,k]\rightarrow\left[\begin{array}{c} m-\left(ms^{-1}\right)\left(\left(s\,(s+k)^{-1}\right)k\right),\\ 1,\left(s\,(s+k)^{-1}\right)k \end{array}\right]\\ &if\quad s+k\notin\mathbf{I},\\ &[m,1,k]\rightarrow\left[1,s^{-1}\left((s+k)\,m^{-1}\right),-km^{-1}\right]\\ &if\quad s+k\in\mathbf{I}\wedge m\notin\mathbf{I},\\ &[m,1,k]\rightarrow\left[-mk^{-1},k^{-1}\left((s+k)\,s^{-1}\right),1\right]\\ &if\quad s+k\in\mathbf{I}\wedge m\in\mathbf{I},\\ &[1,n\varepsilon,p]\rightarrow\left[(sn-p)^{-1}s,1,-p\left((sn-p)^{-1}s\right)\right]\\ &if\quad p\notin\mathbf{I},\\ &[1,n\varepsilon,p]\rightarrow\left[1,n-s^{-1}p,p\right]\\ &if\quad p\in\mathbf{I},\\ &[q\varepsilon,n\varepsilon,1]\rightarrow\left[-q\left(s\,(1+ns)^{-1}\right),1,s\,(1+ns)^{-1}\right]. \end{aligned}$$

Now we are ready to give the main result of the paper.

Theorem 3.1: The transformations J_s and H_s, defined above, are collineations of $\mathbf{M}(\mathcal{A})$.

Proof: The proof can be done by direct computation with using Moufang identities and properties of the local alternative rings (cf [1]). We will only show that J_s preserves the incidence relation (i.e. $P\in l\Leftrightarrow\mathrm{J}_s\,(P)\in\mathrm{J}_s\,(l)$) and the neighbour relation (i.e. $P\sim Q\Leftrightarrow\mathrm{J}_s\,(P)\sim\mathrm{J}_s\,(Q)$ and $g\sim h\Leftrightarrow\mathrm{J}_s\,(g)\sim\mathrm{J}_s\,(h)$).

Case 1. Let $P=(x,y,1)$. Then $\mathrm{J}_s\,(P)=\left(ys^{-1},xs,1\right)$.

1.1. Let $l=[m,1,k]$.

1.1.1. If $m\in\mathbf{I}$, then since $\mathrm{J}_s(P)=(ys^{-1},xs,1)$ and $\mathrm{J}_s(l)=[1,s^{-1}ms^{-1},ks^{-1}]$, we have $\mathrm{J}_s\,(P)\in\mathrm{J}_s\,(l)\Leftrightarrow ys^{-1}=(xs)\left(s^{-1}ms^{-1}\right)+ks^{-s}$. By Lemma 2.1 and 2.2, we get $\mathrm{J}_s\,(P)\in\mathrm{J}_s\,(l)\Leftrightarrow ys^{-1}=\left(\left(xss^{-1}\right)m\right)s^{-1}+ks^{-1}=(xm)s^{-1}+ks^{-1}$ and multiplying by s on the right, we find $y=xm+k\Leftrightarrow P\in l$.

1.1.2. If $m\notin\mathbf{I}$, then since $\mathrm{J}_s(P)=(ys^{-1},xs,1)$ and $\mathrm{J}_s(l)=[sm^{-1}s,1,-(km^{-1})s]$, we have $\mathrm{J}_s\,(P)\in\mathrm{J}_s\,(l)\Leftrightarrow xs=\left(ys^{-1}\right)\left(sm^{-1}s\right)+-\left(km^{-1}\right)s$. Again by Lemma 2.1 and 2.2, we get $\mathrm{J}_s\,(P)\in\mathrm{J}_s\,(l)\Leftrightarrow xs=\left(\left(ys^{-1}s\right)m^{-1}\right)s+$

$-\left(km^{-1}\right)s=\left(ym^{-1}\right)s-\left(km^{-1}\right)s$ and multiplying by s^{-1} on the right, we obtain $\mathrm{J}_s(P)\in\mathrm{J}_s(l)\Leftrightarrow x=ym^{-1}-km^{-1}$. Finally, by multiplying both sides on the right by m, $y=xm+k\Leftrightarrow P\in l$.

1.2. Let $l=[1,n\varepsilon,p]$ where $n\in\mathbf{A}$. Then since $\mathrm{J}_s(P)=\left(ys^{-1},xs,1\right)$ and $\mathrm{J}_s(l)=[sns,1,ps]$, we have $\mathrm{J}_s(P)\in\mathrm{J}_s(l)\Leftrightarrow xs=\left(ys^{-1}\right)(sns)+ps\Leftrightarrow xs=\left(\left(ys^{-1}s\right)n\right)s+ps\Leftrightarrow x=yn+p\Leftrightarrow P\in l$.

1.3. Let $l=[q\varepsilon,n\varepsilon,1]$ where $q,n\in\mathbf{A}$. In this case $P\notin l$. Since $\mathrm{J}_s(P)=\left(ys^{-1},xs,1\right)$ and $\mathrm{J}_s(l)=\left[sn,s^{-1}q,1\right]$ then $\mathrm{J}_s(P)\notin\mathrm{J}_s(l)$. So, $P\notin l\Leftrightarrow\mathrm{J}_s(P)\notin\mathrm{J}_s(l)$.

Case 2. Let $P=(1,y,z\varepsilon)$ where $z\in\mathbf{A}$.

2.1. Let $l=[m,1,k]$.

2.1.1. If $m\in\mathbf{I}$ and $y\in\mathbf{I}$ then since $\mathrm{J}_s(P)=\left(s^{-1}ys^{-1},1,s^{-1}z\right)$ and $\mathrm{J}_s(l)=\left[1,s^{-1}ms^{-1},ks^{-1}\right]$. In this case, we have $\mathrm{J}_s(P)\in\mathrm{J}_s(l)\Leftrightarrow s^{-1}ys^{-1}=s^{-1}ms^{-1}+\left(s^{-1}z\right)\left(ks^{-1}\right)$. By Lemma 2.1, we obtain $\mathrm{J}_s(P)\in\mathrm{J}_s(l)\Leftrightarrow s^{-1}ys^{-1}=s^{-1}ms^{-1}+s^{-1}(zk)s^{-1}$. By multiplying both sides on the right and left by s we find $\mathrm{J}_s(P)\in\mathrm{J}_s(l)\Leftrightarrow y=m+zk$ and so $\mathrm{J}_s(P)\in\mathrm{J}_s(l)\Leftrightarrow P\in l$.

2.1.2. If $m\in\mathbf{I}$ and $y\notin\mathbf{I}$ then $y=m+zk\in\mathbf{I}$, which is a contradiction. That is, $P\notin l$. Also as a direct consequence of the coordinatization of $\mathbf{M}(\mathcal{A})$, $\mathrm{J}_s(P)\notin\mathrm{J}_s(l)$.

2.1.3. If $m\notin\mathbf{I}$ and $y\in\mathbf{I}$ then $y-zk=m\in\mathbf{I}$, which is a contradiction. That is, $P\notin l$. Also as a direct consequence of the coordinatization of $\mathbf{M}(\mathcal{A})$, $\mathrm{J}_s(P)\notin\mathrm{J}_s(l)$.

2.1.4. If $m\notin\mathbf{I}$ and $y\notin\mathbf{I}$ then since $\mathrm{J}_s(P)=\left(1,sy^{-1}s,s(y^{-1}z)\right)$ and $\mathrm{J}_s(l)=\left[sm^{-1}s,1,-\left(km^{-1}\right)s\right]$ .In this case, we have $P\in l\Rightarrow y=m+zk$. where $y^{-1}=m^{-1}-m^{-1}(zk)m^{-1}$. By Lemma 2.2, we get $y^{-1}=m^{-1}-\left(m^{-1}z\right)\left(km^{-1}\right)$. Note that $m^{-1}z=y^{-1}z$ where $z\in\mathbf{I}$. So, $y^{-1}=m^{-1}-\left(y^{-1}z\right)\left(km^{-1}\right)$. By multiplying both sides on the right and left by s, we find $sy^{-1}s=sm^{-1}s-s\left((y^{-1}z)\right)\left(\left(km^{-1}\right)\right)s$. By Lemma 2.2, we obtain $sy^{-1}s=sm^{-1}s-\left(s(y^{-1}z)\right)\left(\left(km^{-1}\right)s\right)$ which means that $\mathrm{J}_s(P)\in\mathrm{J}_s(l)$. Conversely, let $\mathrm{J}_s(P)\in\mathrm{J}_s(l)\Rightarrow sy^{-1}s=sm^{-1}s-\left(s(y^{-1}z)\right)\left(\left(km^{-1}\right)s\right)$. Since $m\notin\mathbf{I}$ and $y\notin\mathbf{I}$, there exists $m_1,\ m_2,\ y_1,\ y_2\in\mathbf{A}$ such that $m_1\neq 0\neq\ y_1$ and $m=m_1+m_2\varepsilon,\ y=y_1+y_2\varepsilon$. Then using the inverses $m^{-1}=m_1^{-1}-m_1^{-1}m_2m_1^{-1}\varepsilon$ and $y^{-1}=y_1^{-1}-y_1^{-1}y_2y_1^{-1}\varepsilon$ of m and y, respectively;

$$\begin{array}{rcl}\left(1,sy^{-1}s,s(y^{-1}z)\right) & \in & \left[sm^{-1}s,1,-\left(km^{-1}\right)s\right]\\ & \Leftrightarrow & \left\{\begin{array}{l}y_1^{-1}=m_1^{-1}\Leftrightarrow y_1=m_1\\ y_1^{-1}y_2y_1^{-1}=m_1^{-1}m_2m_1^{-1}\\ +\left(y_1^{-1}z\right)\left(k_1m_1^{-1}\right)\end{array}\right.\end{array}$$

(in which k has the form $k_1+k_2\varepsilon$ where $k_1,k_2\in\mathbf{A}$) and so the solution of this equation system is

$$y_1^{-1}y_2y_1^{-1}=y_1^{-1}m_2y_1^{-1}+\left(y_1^{-1}z\right)\left(k_1y_1^{-1}\right).$$

Since all terms of this equation are elements of Cayley division ring $\mathbf{A}$, Moufang identities are valid. Therefore,

$$\begin{array}{rcl}y_1^{-1}y_2y_1^{-1} & = & y_1^{-1}m_2y_1^{-1}+y_1^{-1}(zk_1)y_1^{-1}\\ & = & y_1^{-1}(m_2+zk_1)y_1^{-1}\\ & = & y_1^{-1}\left((m_2+zk_1)y_1^{-1}\right)\end{array}$$

is obtained. Then we have

$$y_1^{-1}y_2y_1^{-1}=y_1^{-1}\left((m_2+zk_1)y_1^{-1}\right)\Leftrightarrow y_2=m_2+zk_1$$

from Lemma 2.1. Finally we have $y_1=m_1$ and $y_2=m_2+zk_1$ which means that $P\in l$.

2.2. Let $l=[1,n\varepsilon,p]$ where $n\in\mathbf{A}$. In this case, $\mathrm{J}_s(l)=[sns,1,ps]$ and $P\notin l$.

2.2.1. If $y\in\mathbf{I}$, then $\mathrm{J}_s(P)=\left(s^{-1}ys^{-1},1,s^{-1}z\right)$. Also as a direct consequence of the coordinatization of $\mathbf{M}(\mathcal{A})$, $\mathrm{J}_s(P)\notin\mathrm{J}_s(l)$.

2.2.2. If $y\notin\mathbf{I}$, then $\mathrm{J}_s(P)=(1,sy^{-1}s,s(y^{-1}z))$. In this case we have $J_s(P)\in J_s(l)\Leftrightarrow sy^{-1}s=sns+\left(s(y^{-1}z)\right)(ps)$. By Lemma 2.2, $J_s(P)\in J_s(l)\Leftrightarrow sy^{-1}s=sns+s\left((y^{-1}z)p\right)s$. By multiplying both sides on the right and left by s^{-1}, we find $J_s(P)\in J_s(l)\Leftrightarrow y^{-1}=n+(y^{-1}z)p\in\mathbf{I}$ which contradicts with our hypothesis $y\notin\mathbf{I}$. That is, $J_s(P)\notin J_s(l)$.

2.3. Let $l=[q\varepsilon,n\varepsilon,1]$ where $q,n\in\mathbf{A}$. In this case, $\mathrm{J}_s(l)=\left[sn,s^{-1}q,1\right]$.

2.3.1. If $y\in\mathbf{I}$, then $\mathrm{J}_s(P)=\left(s^{-1}ys^{-1},1,s^{-1}z\right)$. So we have $J_s(P)\in J_s(l)\Leftrightarrow s^{-1}z=(s^{-1}ys^{-1})(sn)+s^{-1}q$. By Lemma 2.1 and 2.2, we get $J_s(P)\in J_s(l)\Leftrightarrow s^{-1}z=s^{-1}\left(y\left(s^{-1}sn\right)\right)+s^{-1}q$. By multiplying both sides on the left by s, we find $J_s(P)\in J_s(l)\Leftrightarrow z=yn+q$. So, $J_s(P)\in J_s(l)\Leftrightarrow P\in l$.

2.3.2. If $y\notin\mathbf{I}$, then $\mathrm{J}_s(P)=(1,sy^{-1}s,s(y^{-1}z))$. In his case, we have $J_s(P)\in J_s(l)\Leftrightarrow s(y^{-1}z)=sn+\left(sy^{-1}s\right)\left(s^{-1}q\right)$. By Lemma 2.1 and 2.2, we get $J_s(P)\in J_s(l)\Leftrightarrow s(y^{-1}z)=sn+s\left(y^{-1}\left(ss^{-1}q\right)\right)$. By multiplying both sides on the left by s, we find $J_s(P)\in J_s(l)\Leftrightarrow y^{-1}z=n+y^{-1}q$. By multiplying both sides on the left by y, we obtain $J_s(P)\in J_s(l)\Leftrightarrow z=yn+q$. So, we get $J_s(P)\in J_s(l)\Leftrightarrow P\in l$.

Case 3. Let $P=(w\varepsilon,1,z\varepsilon)$ where $w,z\in\mathbf{A}$. Then $\mathrm{J}_s(P)=(1,sws,sz)$.

3.1. Let $l=[m,1,k]$. Then from the coordinatization of $\mathbf{M}(\mathcal{A})$ we obviously have $P\notin l$.

3.1.1. If $m\in\mathbf{I}$, then $J_s(l)=[1,s^{-1}ms^{-1},ks^{-1}]$. Also as a direct consequence of the coordinatization of $\mathbf{M}(\mathcal{A})$, $\mathrm{J}_s(P)\notin\mathrm{J}_s(l)$.

3.1.2. If $m\notin\mathbf{I}$, then $J_s(l)=[sm^{-1}s,1,-(km^{-1})s]$. In this case, we have $J_s(P)\in J_s(l)\Leftrightarrow sws=sm^{-1}s-(sz)\left(\left(km^{-1}\right)s\right)$. By Lemma 2.2, we get $J_s(P)\in J_s(l)\Leftrightarrow sws=sm^{-1}s-s\left(z\left(km^{-1}\right)\right)s$. By multiplying both sides on the right and left by s^{-1}, we find $J_s(P)\in J_s(l)\Leftrightarrow w=m^{-1}-z\left(km^{-1}\right)$. So, we obtain $J_s(P)\in J_s(l)\Leftrightarrow m^{-1}=w+z\left(km^{-1}\right)\in\mathbf{I}$ which contradicts with our hypothesis $m\notin\mathbf{I}$. That is, $J_s(P)\notin J_s(l)$.

3.2. Let $l=[1,n\varepsilon,p]$ where $n\in\mathbf{A}$. Then $J_s(P)=[sns,1,ps]$. In this case, $J_s(P)\in J_s(l)\Leftrightarrow sws=sns+(sz)(ps)$. By Lemma 2.2, we have $J_s(P)\in J_s(l)\Leftrightarrow sws=sns+s(zp)s$. By multiplying both sides on the right and left by s^{-1}, we obtain $J_s(P)\in J_s(l)\Leftrightarrow w=n+zp$ which means that $P\in l$.

3.3. Let $l = [q\varepsilon, n\varepsilon, 1]$ where $q, n \in \mathbf{A}$. Then $\mathrm{J}_s(l) = \left[sn, s^{-1}q, 1\right]$. In this case we have $J_s(P) \in J_s(l) \Leftrightarrow sz = sn + (sws)\left(s^{-1}q\right) = sn + s(wq)$ by Lemma 2.1 and 2.2. By multiplying both sides on the left by s^{-1}, we find $J_s(P) \in J_s(l) \Leftrightarrow z = n + wq$ which means that $P \in l$.

Now, we will show that J_s preserves the neighbour relation for the point and the lines by using properties of ideals. The case in which the most complicated computations arise is when $m, u \notin \mathbf{I}$ for the lines $[m, 1, k]$ and $[u, 1, v]$. Therefore we give the proof for only this case. Then $\mathrm{J}_s([m, 1, k]) = \left[sm^{-1}s, 1, -\left(km^{-1}\right)s\right]$ and $\mathrm{J}_s([u, 1, v]) = \left[su^{-1}s, 1, -\left(vu^{-1}\right)s\right]$ and also

$$\begin{aligned}
&\left[sm^{-1}s, 1, -\left(km^{-1}\right)s\right] \sim \left[su^{-1}s, 1, -\left(vu^{-1}\right)s\right] \\
&\Leftrightarrow m^{-1} - u^{-1} \in \mathbf{I} \wedge vu^{-1} - km^{-1} \in \mathbf{I} \\
&\Leftrightarrow m_1^{-1} - u_1^{-1} = 0,\ v_1u_1^{-1} - k_1m_1^{-1} = 0 \\
&\Leftrightarrow m_1 = u_1,\ v_1 = k_1 \\
&\Leftrightarrow m_1 - u_1 = 0,\ v_1 - k_1 = 0 \\
&\Leftrightarrow m - u \in \mathbf{I} \wedge v - k \in \mathbf{I}\ \ (or\ k - v \in \mathbf{I}) \\
&\Leftrightarrow [m, 1, k] \sim [u, 1, v].
\end{aligned}$$

■

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