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On the Sphere Method of Linear Programming using Multiple Interior Points Approach

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Abstract**—**The Sphere Method is a flexible interior point algorithm for linear programming problems. This was developed mainly by Professor Katta G. Murty. It consists of two steps, the *centering step* and the *descent step*. The centering step is the most expensive part of the algorithm. In this centering step we proposed some improvements such as introducing two or more initial feasible solutions as we solve for the more favorable new solution by objective value while working with the rigorous updates of the feasible region along with some ideas integrated in the descent step. An illustration is given confirming the advantage of using the proposed procedure.

Keywords**—**Interior point, linear programming, sphere method, initial feasible solution, feasible region, centering and descent steps, optimal solution.

I. Introduction

THE problem of maximizing or minimizing a linear function subject to linear constraints is called linear programming problem . The constraints may be equalities or inequalities. Given a polytope , a linear programming method will find a point in the polytope for the largest or smallest value, respectively, for this function. If the point exists, it can be found through the polytope vertices.

Linear programs are problems that can be expressed in the form:

$$\text{Maximize} \quad c^T x$$
$$\text{Subject to} \quad Ax \le b$$

where x represents the vector of variables which are to be solved, while $\mathbf{c}$ and b are vectors of numerical coefficients and A is a matrix of coefficients. The expressions $Ax \le b$ are the constraints describing a convex polyhedron. The objective $c^T x$ is to be maximized or minimized from these expressions.

The problem of solving a system of linear inequalities dates back at least as far as Fourier, after whom the method of Fourier - Motzkin is named. Linear programming arose as a mathematical model developed during the Second World War to plan expenditures and returns in order to reduce costs to the army and increase losses to the enemy. It was kept secret until 1947. Postwar, many industries found its use in their daily planning.

The founders of the subject are Leonid Kantorovich, a Russian mathematician who developed linear programming problems in 1939, George B. Dantzig, who published the simplex method in 1947, John Von Neumann, who developed the theory of the duality in the same year. The linear programming problem was first shown to be solvable in polynomial time by Leonid Khachiyan in 1979, but a larger theoretical and practical breakthrough in the field came in 1984 when Narenda Kamarkar introduced a new interior point method for solving linear programming problems.

The Simplex Algorithm became the main algorithm used to solve linear programs. As an LP problem solver, this algorithm might be one of the most famous and widely used mathematical tools in the world. Its philosophy is to move on the underlying polyhedron, from a vertex to adjacent vertex, along edges until an optimal vertex is reached. However, the interior point algorithm will take the route that will pass not in the edges or extreme points of the set of feasible solutions. This algorithm will take the interior path towards the optimal extreme point. Thus, finding the optimal extreme point using interior point algorithm will save more time and effort, especially when the data is large. With the advent of fast computers and effective interior point algorithm software, the calculations become more reliable [1].

Among the many interior point methods, the sphere methods were introduced and made popular by Katta Murty. These methods claim the ideas of reaching the optimum values at a reasonable pace.

Hence, this study will try to refine the work of Murty,then use the refined work for a process that uses multiple interior points in reaching the optimal value.

II. Objectives of the Study

The main goal of this study is to be able to address the concern for the refinement of the Interior Point Algorithm of Professor Katta G. Murty on the search for the optimal value of the linear programming problem by the use of *centering* and *descent steps* in the algorithm. Specifically this study will attempt to:

2.1 Determine some innovations in both *centering* and *descent steps* that will further improve the sphere method.

2.2 Establish an improved approach concerning interior point algorithm using the sphere method in solving linear programming problems.

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2.3 Provide results that would lead to less iteration and at minimum computing time.

Theory and Concept

The Sphere Method:

The sphere method is an interior point algorithm. It needs an initial interior feasible solution. Every iteration of the method begins with the best interior feasible solution obtained at the end of the previous iteration. It consists of the *centering* and *descent steps.*

The centering step will find a ball center, which is an interior feasible solution and is the center of a largest ball inside the feasible region of the original LP subject to the constraint on its center. After the ball center is found, the descent step carries out different several descent steps from this ball center, and the iteration stops when the best point is obtained with these descent steps.

The sphere method considers Linear Programs in the form:

$$\begin{aligned} &Min && z = cx \\ &subject\ to: \\ & && Ax \geq b \end{aligned} \qquad (1)$$

where A is an m × n data matrix; with a known interior feasible solution x^0 (i.e., satisfying $Ax > b$). Let K denote its set of feasible solutions, and K^0 its interior. We assume that c, and each row vector of A is normalized so that ||c|| = $||A_{i.}||$ = 1 for all i = 1 to m. Here $A_{i.}$ denotes the i-th row vector of A.

The following concepts will be used:

Largest inscribed ball B(x, δ(x)) inside K with x as center, for $x \in K^0$: It is the largest ball with x as center that can be inscribed in K, and δ(x) = min{$A_{i.}x - b_i$: i = 1 to m} is its radius. So, B(x, δ(x)) = {y : ||y − x|| ≤ δ(x)}.

A ball center of K: It is a point $x \in K^0$ such that B(x, δ(x)) is a largest ball that can be inscribed in K, i.e., x maximizes δ(y) over $y \in K^0$.

A ball center of K on the objective plane H = {x : cx = t}: It is a point $x \in H \bigcap K$ that maximizes δ(y) over y ∈ H ∩ K.

The index set of touching constraints in (1), T(x): Defined for $x \in K^0$, is the set of all indices I satisfying: $A_{i.}x - b_i = Min\{A_{p.}x - b_p: p = 1 \text{ to } m\} = \delta(x)$. The facetal hyperplane {x : $A_{i.}x = b_i$} is a tangent plane to B(x, δ(x)) for each $i \in T(x)$.

$$K^{r+1} = \{x : Ax \geq b, A_{m+1.}x \geq b_{m+1}\},$$

where

$$A_{m+1.} = -c\ ,\ b_{m+1} = -A_{m+1.}x_1^r - \in\ \}$$

∈ is a small positive tolerance. K^{r+1}, is the set of feasible solutions of (1) updated corresponding to the current point obtained x_1^r after rth iteration. This updated set of feasible solutions keeps getting smaller during the algorithm [2].

In [2] techniques for computing a ball center of K are discussed, or a ball center of K on a given objective plane H, approximately, using a series of line search steps. In each of these steps, at the current point $\bar{x}$, the algorithm selects a direction y which is a profitable direction to move at $\bar{x}$, i.e., $\delta(\bar{x} + \alpha y)$ strictly increases as α increases from 0; and determines the optimum step length to maximize $\delta(\bar{x} + \alpha y)$ over $\boldsymbol{\alpha \geq 0}$. This optimum step length is obtained by solving a 2-variable LP:

$$\begin{aligned} &Max\ \delta \\ &subject\ to \\ &\delta - \alpha A_{i.}y \leq A_{i.}\bar{x} - b_i\ ,\ i = 1,\ldots,m \\ &\delta \geq 0,\ \alpha\ is\ unrestricted \end{aligned} \qquad (2)$$

and $\bar{\delta}$ is the optimum objective value $\delta(\bar{x} + \bar{\alpha} y)$.

A direction **y** has been shown in [2] to be a profitable direction at $\bar{x} \in K^0$ iff $A_{i.}y > 0$ for all $i \in T(\bar{x})$.

In the descent step, the following concept is used:

Let c^i denote the orthogonal projection of c^T on $\{x : A_{i.}x = 0\}$, i.e, $c^i = (I - (A_{i.})^T A_{i.})c^T$, *for* $i = 1$ *to* m. The directions $-c^i$ for $i \varepsilon T(\bar{x})$ are called GPTC (gradient projection on touching constraint) directions, where $\bar{x}$ is the current ball center [3].

The Centering Step in Iteration r+1 in Sphere method:

Let x^r be the initial interior feasible solution for this iteration. This step consists of a series of line searches in profitable directions with the aim of finding an x that maximizes δ(x) subject to the constraint cx ≤ cx^r. In each of these line searches, given the search direction, the optimum step length to take in that direction is determined by solving a 2-variable LP of the form (2) as described above.

III. Methodology

In this paper LSFN (Line Search in Facetal Normal Directions) is carried out in selecting profitable search directions from the set of facetal normal directions. The following concepts are found in [2]:

Beginning with the initial point x^r, this generates a sequence of points $x^{r,k}$, k = 1, 2, along which the radius of the ball δ is strictly increasing. At the current point $x^{r,k}$, it selects a profitable directions from the set $\Gamma_1 = \{\pm P_{.1}, \ldots, \pm P_{.m}\}$, where $P_{.i} = \left(I - c^T c\right)A_{i.}^T$,the orthogonal projection of $A_{i.}^T$ (the direction normal to the facet of K defined by the i-th constraint in (1)) on the hyperplane {x : cx = 0}, for i = 1 to m. So any step length from a point in the current objective plane, in a direction from Γ_1, will keep the point on the current objective plane. The

procedure continues as long as profitable directions for line search are found in Γ_1, and this sequence terminates with the final point which we denote by $x^{\sim r}$.

Once a profitable direction y at the current point $\bar{x}$ has been determined, the optimum step length α in this direction that maximizes $\delta(\bar{x}+\alpha y)$ over $\alpha \geq 0$ is $\bar{\alpha}$, where $(\delta, \bar{\alpha})$ is the optimum solution of the 2-variable LP (2).

So, the line search for the maximum value of δ in the direction y involves solving this 2-variable LP, which can be carried out efficiently by the simplex algorithm. The following theorems and discussions were shown by Prof. Murty.

Theorem 1: Consider the system of linear inequalities Ax ≥ b where A = (a_{ij}) is an m×n matrix and b = $(b_i) \in \Re_m$.So, the constraints in the system are $A_{i.}x \geq b_i$, i∈{1, . . . ,m}. If this system has a feasible solution, then there exists a subset P = { p_i , . . . , p_s } ⊂ {1, . . . ,m}such that every solution of the system of equations $A_{i.}x = b_i$,i∈P is also a feasible solution of the system of linear inequalities Ax ≥ b.

The radius of the largest ball inside K with the current point x^r as center is $\delta_r = \min\{A_{t.}x^r - b_t : t = 1,...,m\}$, the minimum right hand side (RHS) constant in (2). Let $T = \{t_1,...,t_s\}$be the set of all t that tie for the minimum in the definition of δ_r. Then the minimum RHS constant in (2) is unique only if $|T| = s = 1$, where it is attained at $t = t_1$ only.

Theorem 2: $P_{.i}$ is unprofitable direction to move at the current point x^r *iff* : $s = |T|$ defined above is greater than 1, and the coefficients of α in (2) in rows $t \in T = \{t_1, . . .t_s\}$ have both positive and negative values among them.

A. Descent Steps in Iteration r+1 in Sphere Method

The following discussions are stated in [2].

Let $\bar{x}^r$ denote the approximate ball center obtained in the centering step of this iteration.Each descent step carried out in this iteration requires one minimum ratio computation.For example, consider a descent step from the current center $\bar{x}^r$ in the descent direction y (i.e.,satisfying cy < 0). If the step length is λ, the move leads to the point $\bar{x}^r + \lambda y$. Select a small positive number ∈as the tolerance for minimum $\{A_{i.}x - b_i : i = 1 \text{ to } m\}$ for the point **x** to be in the interior of K. Then we will take the step length from $\bar{x}^r$ in the direction y to be: $-\varepsilon_1$+ (the maximum step length possible while remaining inside K), which is

$$\lambda = \min\left\{\frac{-A_{i.}x + b_i + \varepsilon_1}{A_{i.}y} : \text{such that } A_{i.}y < 0\right\}$$

and then the point obtained at the end of this descent step will be $\bar{x}^r + \lambda y$ if λ is finite. If λ = ∞**,** the objective function **z(x)** is unbounded below in (1). Terminate the method if this occurs. We now list the various descent steps carried out in this iteration. After each descent step, include the point obtained at the end of it, along with its objective value, in a list. These descent steps can be found in [2].

D1, Descent Step 1: From the ball center $\bar{x}^r$ take a descent step in the direction $d_1 = -c^T$.

D2, Descent Step 2: From the ball center $\bar{x}^r$ take a descent step in the direction $d_2 = x^r - x^{r-1}$, *where* x^{r-1} denotes the ball center computed in the previous iteration **r.** So, this direction is the direction of the path of ball centers generated in the algorithm.

D3, Descent Steps 3: Carry out descent steps from the ball center $\bar{x}^r$ in each of the GPTC directions at $\bar{x}^r$. After these descent steps are carried out, define $d_3 =$ direction among the GPTC directions that gives maximum reduction in objective value when the descent step is taken from the center $\bar{x}^r$.

D4, Descent Step 4: From the ball center $\bar{x}^r$ take a descent step in the direction

$$d_4 = \frac{\sum\left(-c^i : for\ i \in T(\bar{x}^r)\right)}{|T(\bar{x}^r)|},$$

the average direction of all the GPTC(gradient projection on touching constraints) directions at $\bar{x}^r$.

D5.1, Descent Steps 5.1: For $\bar{x}^r$, let x^{ir} denote the orthogonal projection of the center $\bar{x}^r$ on the touching facetal hyperplane $\{x : A_{i.}x = b_i\}$; it is the point where this facetal hyperplane touches the ball $B(\bar{x}^r, \delta(\bar{x}^r))$. The points x^{ir} for $i \in T(\bar{x}^r)$) are called the touching points of the ball $B(\bar{x}^r, \delta(\bar{x}^r))$ with its touching facetal hyperplanes of K.

Let $0 < \varepsilon < 1$ be a small positive tolerance $(\varepsilon = 0.1\ can do)$.Then for $i \in T(\bar{x}^r)$**,** the point on the line segment joining $\bar{x}^r$ and x^{ir} close to the touching point x^{ir}, $\hat{x}^{ir} = \varepsilon\bar{x}^r + (1-\varepsilon)x^{ir}$ is called the near touching point (NTP) corresponding to the tangent plane $\{x : A_{i.}x = b_i\}$ of

the ball $B(\bar{x}^r, \delta(\bar{x}^r))$. The *D5.1* consists of $|T(\bar{x}^r)|$ for each descent steps: $i\varepsilon T(\bar{x}^r)$), it carries out a descent step in the GPTC direction $-c^i$ from the NTP $\overset{=ir}{x}$. The output of *D5.1*, denoted by $\overset{\sim r1}{x}$ is the best point obtained in it.

After all these descent steps are carried out, the best feasible solution in objective value from all these descent steps, is the output of this iteration. With that interior feasible solution we go to the next iteration.

B. Termination Condition

Just as other IPMs, this method also terminates when the change in the final points obtained in successive iterations is smaller than some tolerance (i.e., it terminates at the end of the iteration $r+1$ if $\frac{\| x_{r+1} - x_r \|}{\| x_r \|} < 1$, concluding that x_{r+1} an optimum solution of (1)). This means that after having two successive iterations, if the successive outputs have negligible difference, then the approximate optimal value is reached.

C. The Case when K is Unbounded

Suppose K, the set of feasible solutions of (1) is an unbounded polyhedron. The method works even when K is unbounded. If cx is unbounded below on K, it will terminate in some iteration with the step length in one of the descent steps as ∞, this is the indication that cx is unbounded below on K.

IV. Summary of Findings

The following improvements on the sphere method provided by Prof. Murty in Section IV are expected to reduce the number of computing time and iterations in reaching the optimal.

A. Updating the Feasible Region

Let us use two points x_1^0 and x_2^0, where $x_1^0 \neq x_2^0$ as initial interior feasible solutions. Then after the first iteration with the sphere method for each of these, we arrived with two new interior feasible solutions x_1^1 and x_2^1. We choose the best solution in objective value to be used for the next iteration, say x_1^1. Then we update the feasible region in accordance to [2]. This means that for the succeeding iterations when $x_1^r, r \geq 1$ is the current initial feasible solution, the updated feasible region is

$$K^{r+1} = \{x : Ax \geq b, A_{m+1.}x \geq b_{m+1},$$

where

$$A_{m+1.} = -c \ , \ b_{m+1} = -A_{m+1.}x_1^r - \in \ \}$$

$\in$ is a small positive tolerance. K^{r+1}, is the set of feasible solutions of (1) updated by the current value is strictly monotonic decreasing in the algorithm, and hence this updated set of feasible solutions keeps getting smaller during the algorithm. Note the decrease of the feasible region compare to that of [2] that uses only a single initial feasible solution. We used two initial interior feasible solutions, and whichever comes out as the best in objective value, that becomes our point for the updating of the feasible region.

In general, the feasible region of (1) consists of infinitely many interior feasible solutions. If only we are lucky to hit that point where upon the use of sphere method only one iteration is required to reach the solution for the *Approximate Optimal Value* (AOV) then after the second iteration, we are through. Our goal is to reach this point. If we use three, four or more initial interior feasible solutions the more chances we have to hit or get closer and closer to that point for the AOV. But since every initial point means an iteration, as we increase the number of initial interior feasible solutions the number of iterations increases (if we count the number of iterations this way). Thus, it is recommended that when the number of constraints in (1) is greater than 100 (the number of variables may not do much problem), try to use three or four points as initial feasible solutions for the sphere method, otherwise use two or three points. Observed that if we consider three initial interior feasible solutions x_1^0, x_2^0 *and* x_3^0. Then after the first iteration with the sphere method for each of these, we arrived with three new interior feasible solutions x_1^1, x_2^1 *and* x_3^1. We choose the best solution in objective value to be used for the next iteration, say x_1^1. Then we update the feasible region in accordance to [2]. This means that for the succeeding iterations when $x_1^r, r \geq 1$ is the current initial feasible solution, the updated feasible region is

$$K^{r+1} = \{x : Ax \geq b, A_{m+1.}x \geq b_{m+1},$$

where

$$A_{m+1.} = -c \ , \ b_{m+1} = -A_{m+1.}x_1^r - \in \ \}$$

$\in$ is a small positive tolerance. Note the further decrease (in general) of the feasible region compare to that of [2] that uses only a single initial interior feasible solution. We used three initial interior feasible solutions, and whichever comes out as the best in objective value, that becomes our point for the updating of the feasible region

Consider n initial interior feasible solutions $x_1^0, x_2^0, \ldots, x_n^0$. Then after the first iteration with the sphere method for each of these, we arrived with new interior feasible solutions $x_1^1, x_2^1, \ldots, x_n^1$. We choose the best solution in objective value to be used for the next iteration, say x_1^1. Then we update the feasible region in accordance to [2]. This means that for the succeeding iterations when $x_1^r, r \geq 1$ the current initial feasible solution is, the updated feasible region is

$$K^{r+1} = \{x : Ax \geq b, A_{m+1.}x \geq b_{m+1},$$

where

$$A_{m+1.} = -c \ , \ b_{m+1} = -A_{m+1.}x_1^r - \in \ \}$$

$\in$ is a small positive tolerance. Note again the much decrease (in general) of the feasible region compare to that of [2] that uses only a single initial feasible solution. We used n initial feasible solutions, and whichever comes out as the best in objective value, that becomes our point for the updating of the feasible region

B. Another Approach in the Update of the Feasible Region

Let us use two points x_1^0 and x_2^0, where $x_1^0 \neq x_2^0$ as initial interior feasible solutions. Then after the first iteration with the sphere method for each of these, we arrived with two new interior feasible solutions x_1^1 and x_2^1. We take the average, $\bar{x} = \frac{x_1^1 + x_2^1}{2}$, apply the descent method to $\bar{\mathbf{x}}$ then the resulting new interior feasible solution, say x^2 will be compared to x_1^1 and x_2^1. The best of the three in objective value, say $\mathbf{x^2}$ will be our choice for the next iteration and the updated feasible region is

$$K^{r+1} = \{x : Ax \geq b, A_{m+1.}x \geq b_{m+1},$$

where

$$A_{m+1.} = -c \ , \ b_{m+1} = -A_{m+1.}x^2 - \in \ \},$$

$\in$ is a small positive tolerance. Note the decrease of the feasible region, especially when x_1^1 and x_2^1 are located in the opposite parts of the feasible region. Observed that in this case the $\bar{\mathbf{x}}$ is also on top of the feasible solution for the optimal value. Thus, $\mathbf{x^2}$ is expected to be close to the said feasible solution. Now, we can do similar process to three, four,. . . n initial feasible solutions for the sphere method. Note that if the two or three initial feasible solutions were spread (scattered far from each other) and will have their paths in the different faces of the convex polyhedron (feasible region) during the algorithm, the location of the average point would be much better.

C. Trial and Error Approach

Now, as the strategy in selecting three interior points that are scattered far from each other is still under development, let us *consider trial and error approach* in collecting as many interior points from the feasible region, may be a set of ten interior points that are scattered far from each other, then using *intelligent guesses,* we select two or three from the set. Then proceed with the process. For the meantime, this strategy may be an expensive part in the process.

D. Improvement on the Descent Steps

In order to minimize the number of iterations in reaching the optimum value of the objective function is proposed to implement the following steps.

In the descent step, we include the following:

1. D4.1 From the ball center $\bar{x}^r$ take a descent step in the direction $d_{4.1} = \frac{\sum\left(-c^i : for\ i \in T(\bar{x}^r)\right)}{2}$ the average direction of the two GPTC directions at $\bar{x}^r$ that give the best two objective values.

2. D4.2 From the ball center $\bar{x}^r$ take a descent step in the direction $d_{4.2} = \frac{\sum\left(-c^i : for\ i \in T(\bar{x}^r)\right)}{3}$ the average direction of the three GPTC directions at $\bar{x}^r$ that give the best three objective values.

E. Illustration on the Advantage of the Proposed Procedure

We will now present an example of a simple application of LP from the class of product mix models from [2], [4], A fertilizer company makes two kinds of fertilizers called Hi-phosphate (Hiph) and Lo-phosphate (Lo-ph). The manufacture of these fertilizers requires three raw materials called RM 1, 2, 3. At present their supply of these raw materials comes from the company's own quarry which is only able to supply maximum amounts of 1500, 1200, 500 tons/day respectively of RM 1, RM 2,RM 3. Even though there are other vendors who can supply these raw materials if necessary, at the moment they are not using these outside suppliers. They sell their output of Hi-ph and Lo-ph fertilizers to a wholesaler who is willing to buy any amount that they can produce, so there are no upper bounds on the amounts of Hi-ph and Lo-ph manufactured daily. At the present rates of operation their Cost Accounting Department estimates that it is costing the quarry \$50, 40, 60/ton respectively to produce and deliver RM 1, RM 2, RM 3 at the fertilizer plant. Also, at the present rates of operation, all other production costs (for labor, power, water, maintenance, depreciation of plant and equipment, floor space, insurance, shipping to the wholesaler, etc.) come to \$7/ton to manufacture Hi-ph or Lo-ph and to deliver them to the wholesaler. The sale price of the manufactured fertilizers to the wholesaler fluctuates daily, but their averages over the last one month have been \$222, 107/ton respectively for Hi-Ph, Lo-ph fertilizers. The Hi-ph manufacturing process needs as inputs 2 tons of RM 1, and 1 ton each of RM 2, RM 3 for each ton of Hi-ph manufactured. Similarly the Lo-ph manufacturing process needs as inputs 1 ton of RM 1, and 1 ton of RM 2 for each ton of Lo-ph manufactured. So, the net profit/ton of fertilizer manufactured is \$(222 − 2 × 50 − 1 × 40 − 1 × 60 − 7) = 15, (107 − 1 × 50 − 1 × 40 − 7) = 10 /respectively for Hi-ph, Lo-ph.

We will model the problem with the aim of determining how much of Hi-p and Lo-ph to make daily to maximize the

total daily net profit from these fertilizer operations. There are clearly two decision variables; these are:

$\mathbf{x_1}$ = the tons of Hi-ph made per day

$\mathbf{x_2}$ = the tons of Lo-ph made per day

Since all the data is given on a per ton basis, it provides an indication that the linearity assumptions (proportionality, additivity) are quite reasonable in this problem to express each of the constraint and the objective functions. Also, the amount of each fertilizer manufactured can vary continuously within its present range. So, LP is an appropriate model for this problem. The LP formulation of this fertilizer product mix problem is given below. Each constraint in the model is the material balance inequality of the item shown against it.

$$\begin{aligned} &\textbf{Maximize} \quad z(x) = 15x_1 + 10x_2 && \text{Item} \\ &\text{Subject to } 2x_1 + x_2 \le 1500 && \text{RM 1} \\ &\qquad x_1 + x_2 \le 1200 && \text{RM 2} \qquad (1) \\ &\qquad x_1 \le 500 && \text{RM 3} \\ &\qquad x_1 \ge 0,\ x_2 \ge 0 \end{aligned}$$

In minimization, this is:

$$\begin{aligned} &\textbf{Minimize} \quad z(\mathbf{x}) = -15x_1 - 10x_2 \\ &\text{Subject to } 1500 - 2x_1 - x_2 \ge 0 \qquad (2) \\ &\qquad 1200 - x_1 - x_2 \ge 0 \\ &\qquad 500 - x_1 \ge 0 \\ &\qquad x_1 \ge 0 \\ &\qquad x_2 \ge 0 \end{aligned}$$

Normalizing the coefficient vectors of all the constraints and the objective function to Euclidean norm 1, here it is gain:

$$\begin{aligned} &\textbf{Minimize} \quad z(\mathbf{x}) = -0.832x_1 - 0.555x_2 \\ &\text{subject to } 670.820 - 0.894x_1 - 0.447x_2 \ge 0 \\ &\qquad 848.530 - 0.707x_1 - 0.707x_2 \ge 0 \qquad (3) \\ &\qquad 500 - x_1 \ge 0 \\ &\qquad x_1 \ge 0 \\ &\qquad \mathbf{x_2} \ge 0 \end{aligned}$$

1. The Centering Step

Let K denote the set of feasible solutions, and let $x^0 = (10, 1)^T$ be the initial interior feasible solution. When we plug in x^0 in the constraints in (3), the left hand side expressions have values 661.433, 840.753, 490, 10, 1 respectively. So, the radius of the largest ball inside K with x^0 as center is $\delta^0 = \min\{661.433, 840.753, 490, 10, 1\} = 1$.

The objective plane through x^0 is the straight line in $\Re^2$ defined by

$$-0.832x_1 - 0.555x_2 = -8.875.$$

This is the straight line joining $(10.667, 0)^T$ and $(0,15.991)^T$ in the x_1, x_2 - plane. So, the only direction on it is $P_{.1} = (10.667, -15.991)^T$. Moving from x^0 in the direction of $P_{.1}$ a step length α leads to the new point $(10 + 10.667\alpha, 1 - 15.991\alpha)^T$. Finding the optimum step length α leads to the following 2-variable LP in variables θ, α:

$$\begin{aligned} &\textbf{Maximize} \quad \theta \\ &\text{Subject to:} \\ &\qquad \theta + 2.388\alpha \le 661.433 \\ &\qquad \theta - 3.765\alpha \le 840.753 \\ &\qquad \theta + 10.667\alpha \le 490 \qquad (4) \\ &\qquad \theta - 10.667\alpha \le 10 \\ &\qquad \theta + 15.991\alpha \le 1 \\ &\qquad \theta \ge 0,\ \alpha \text{ is unrestricted in sign} \end{aligned}$$

Since the minimum RHS constant in this problem occurs in only one row, the optimum value of α in this problem will be nonzero. Actually the optimum solution of this problem is $(\bar{\theta}, \bar{\alpha})^T = (6.4, -0.338)^T$. The new position for the center is $\overline{x_1} = x^0 - 0.338P_{.1}$

$= (10, 1)^T - 0.338(10.667, -15.991)^T = (6.4, 6.4)^T$, and the maximum radius ball with it as center has radius 6.4. Since $P_{.1}$ is the only direction in $K \cap \{\mathbf{x} : c\mathbf{x} = cx^0\}$ in this case, this ball is the maximum radius ball inside K with center on the objective plane through x^0. If we try to get a larger ball by moving from $\overline{x_1}$ in the direction $P_{.1}$ a step length of α, it can be verified that in the 2-variable LP to find the optimum step length α, the entries in the RHS vector are: 662.238, 839.48, 493.6, 6.4,6.4; and the coefficient vector of α remains the same as in the above table. In this problem the minimum RHS constant occurs in both rows 4 and 5; and the coefficients of α in these two rows have opposite signs, so the optimum value for step length α will be 0. This indicates that $\overline{x_1}$ is the best position for the center of the ball on the objective plane through x^0 in this problem.

2. Descent Move Following Centering

The current center is $\overline{x_1} = (6.4, 6.4)^T$. In this initial iteration, the only descent direction we have available at $\overline{x_1}$ is $-c^T = (0.832, 0.555)^T$. Moving from $\overline{x_1}$ a step length γ in the direction $-c^T$ leads to the point $(6.4 + 0.832\gamma, 6.4 +$

$0.555\gamma)^T$. Taking the tolerance $\in = 1$, we see that the maximum step length is $\gamma = \min\{666.571, 854.72, 592.067\} = 592.067$. Fixing $\gamma = 592.067$, we get the new interior feasible solution $x_1^1 = (499, 335)^T$. The objective value is $c(x_1^1) = -10{,}835$. So in {[2] Murty 2009]}, this is the result up to this point. The next iteration will use x_1^1 as the new interior feasible solution.

Now if consider another interior feasible solution such as $x_2^0 = (100, 150)$. Then using similar process we arrive with $x_2^1 = (450.07, 600.208)$. The objective value is

$$c(x_2^1) = -12{,}753.13$$

The results show that the initial feasible solution x_2^0 is a better choice than x^0. Confirming the advantage of employing the proposed procedure in the updating of the feasible region. From this result, our new interior feasible solution for the next iteration is x_2^1. The optimal value for this minimization problem is c(z) = −13,500. Thus, the optimal value for the original maximation problem (fertilizer problem) is c(z) = 13,500.

V. Conclusion

In the updating of the feasible region, though the two points are using possibly different directions or path, these are approaching the same approximate objective value (if the optimal value exists) and with the same objective function. The best initial feasible solution for the sphere method is just there in the feasible region.

VI. Recommendation

Let the optimal value exists for the given minimization linear programming problem. The reader may find interest in determining the set of points that can trigger for less iteration using the sphere method.

Note that choosing different set of initial interior feasible solutions will result to either improve or no improvement than the current set. Thus, a strategy on how to select favorable initial interior feasible solutions for this process may be considered.

References

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