

On the Reduction of Side Effects in Tomography

V. Masilamani, C. Vanniarajan and Kamala Krithivasan

Abstract—As the Computed Tomography(CT) requires normally hundreds of projections to reconstruct the image, patients are exposed to more X-ray energy, which may cause side effects such as cancer. Even when the variability of the particles in the object is very less, Computed Tomography requires many projections for good quality reconstruction. In this paper, less variability of the particles in an object has been exploited to obtain good quality reconstruction. Though the reconstructed image and the original image have same projections, in general, they need not be the same. In addition to projections, if a priori information about the image is known, it is possible to obtain good quality reconstructed image. In this paper, it has been shown by experimental results why conventional algorithms fail to reconstruct from a few projections, and an efficient polynomial time algorithm has been given to reconstruct a bi-level image from its projections along row and column, and a known sub image of unknown image with smoothness constraints by reducing the reconstruction problem to integral max flow problem. This paper also discusses the necessary and sufficient conditions for uniqueness and extension of 2D-bi-level image reconstruction to 3D-bi-level image reconstruction.

Keywords—Discrete Tomography, Image Reconstruction, Projection, Computed Tomography, Integral Max Flow Problem, Smooth Binary Image.

I. INTRODUCTION

OVER the last three decades there has been a revolution in diagnostic radiology as a result of the emergence of computerized tomography(CT), which is the process of obtaining the density distribution within the human body from multiple X-ray projection. More formally, CT attempts to reconstruct a density function $f(x)$ for x in R^2 or R^3 from the knowledge of line integral $X_f(L) = \int_L f(x)dx$ for lines L through the space. This line integral is the X-ray of $f(x)$ along L . The mapping $f \rightarrow X_f$ is known as the radon transform. Since an enormous variety of possible density values may occur in the body, a large number of projections are necessary to ensure the accurate reconstruction of their distribution. There are other situations in which we desire to reconstruct an object from its projections, but in which we know that the object to be reconstructed has only a small number of possible values. For example, a large fraction of objects scanned in industrial CT(for the purpose of non destructive testing or reverse engineering) are made of single material and so that the ideal reconstruction should contain only two values: zero for air and the value associated with material composing the object. similar assumptions may even be made for some specific medical application; for example, in angiography of the heart chambers the value is either zero indicating absence of dye) or the value associated with the dye in the

chamber. Another example arises in the electron microscopy of biological macromolecules, where we may assume that the object to be reconstructed is composed of ice, protein, and RNA. One can also apply electron microscopy to determine the presence or absence of atoms in crystalline structures, which is again a two valued situation. In many of these applications there are strong technical reasons why only a few projections of the objects can be physically collected. This brings us to the following central theme : How to make use of the knowledge that the reconstruction should contain only a few values to make up for the lack of availability of the number of projections typically required in CT Tomography has application in fields such as: image processing [8], statistical data security [9], biplane angiography [1], graph theory, crystallography, medical imaging [2] and Neutron Imaging [3] etc. [4] gives the fundamentals related to this topic.

Here this paper considers the problem of reconstructing smooth bi-level image from its projections along row and column, and a priori information namely a sub image of the unknown image.

An important area where binary image reconstruction obtained is medical imaging, in particular, Digital subtraction angiography [2]. In Digital subtraction angiography, the reconstructed image is the difference between images acquired before and after intra-arterial injection of radio-opaque contrast medium and hence if the difference of a few projections of those two images are given, binary image can be reconstructed.

Another area where binary image reconstruction obtained is crystallography. In [10], Peter Schwander and Larry Shepp proposed a model that identifies each possible atom location with a cell of integer lattice Z^3 and the electron beams with lines parallel to a given direction. The value 1 in a cell of Z^3 denotes the presence of atom in the corresponding location of crystal and the value 0 in a cell of Z^3 denotes the absence of atom in the corresponding location of the crystal. The number of atoms that are present in a line passing through the crystal defines the projection of the structure along the line. The set of all projections of the structure along each line parallel to a given direction denotes one projection of the object. The number of atoms present in a line(straight) can be computed by making quantitative analysis of two-dimensional images taken by the transmission electron microscope. The transmission electron microscope uses high energy rays which penetrates the crystal. Hence to get more projections, large amount of energy is to be transmitted through the crystal, which can damage the crystal itself(the atomic configuration may be changed). The conventional *Computed Tomography* needs more projections(usually hundreds of projections) for effective reconstruction of the objects. Hence the *Computed Tomography*(Radon transform based algorithms) is not the desirable reconstruction technique

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to reconstruct crystalline structures. *Discrete Tomography* considers the case where the objects need to be reconstructed with a few projections(usually two to four).

As a crystal is represented by a binary matrix, reconstructing a crystal is the same as reconstructing a 3D-binary matrix. A 3D-binary matrix can be reconstructed by slice-by-slice reconstruction. Hence the problem of reconstructing a 3D-binary matrix is reduced to reconstructing 2D-binary matrices. Reconstructing 2D-binary matrix was studied much before the emergence of its practical application. In 1957 Ryser [11] and Gale [12] gave a necessary and sufficient condition for a pair of vectors being the projections of binary matrices along horizontal and vertical directions. The projections in horizontal and vertical directions are equal to row and column sums of the matrix. They have also given necessary and sufficient conditions for the existence of unique 2D-binary matrix which has a given pair of row sum and column sum. In general, the class of binary matrices having same row and column sums is very large. Though the reconstructed matrix and the original matrix have same projections, they may be very different. One of the main issues in *Discrete Tomography* is to reconstruct the object which is more close to the original object with a few projections only. One approach to reduce the class of possible solutions is to use some a priori information about the objects. For instance, convex binary matrices have been reconstructed uniquely from projections taken in some prescribed set of four directions in [13]. Another approach is given in [14], where the class of binary matrices having same projections is assumed to have some Gibbs distribution. By using this information, object which is close to the original unknown object is reconstructed.

This paper considers the first approach, known sub image and smoothness constraints in particular, to limit the possible solutions of 2D-binary images having given projections. For instance, non invasive imaging techniques such as MRI can be used to get sub image. The more clear portion of MRI image can be considered as sub image of original image to be reconstructed and unclear portion of MRI image may be obtained by X-ray tomography from less number of projections. Algorithm given in [6] considered sub image constraint, but this paper considers smoothness constraint in addition to sub image constraint.

In the next section, a need of unconventional reconstruction algorithms is discussed. In section 3, notations and definitions are given. In section 4, uniqueness problem is discussed. Section 5 contains the reconstruction algorithm and illustrate with an example. In section 6, some results obtained by simulation studies are shown. In section 7, the correctness and complexity of the proposed algorithm are discussed. This paper concludes with a brief remark in section 9.

II. NEED OF UNCONVENTIONAL RECONSTRUCTION

Reconstruction algorithms such as filtered back projection algorithm, convolution back projection algorithm and algorithm based on Fourier slice theorem have been used to reconstruct objects from their projection images. As filtered back projection algorithm and convolution back projection algorithm are different implementation of inverse Radon transform, correctness of these algorithms depends on the existence

of inverse of Radon transform. Existence of inverse Radon transform needs projection images for all angles from 0 to 180 degrees. The reconstruction algorithm based on Fourier slice theorem also needs projection images for all angles(0 to 180 degrees). Hence, in order to find the projections for missing angles, normally various interpolation techniques are used. In order to obtain good interpolation, we need to know a good number of projections. The currently available CT scanners use 700 to 900 projection images. Hence with a few projections(two or three), it is impossible to get reasonable interpolation. Hence all the conventional reconstruction algorithms(Radon transform and Fourier slice theorem based algorithms) fail to reconstruct images of interior of three dimensional objects from their projection images. A conventional algorithm based on Fourier slice theorem for the phantom images(2D-slice images) given in Figure 5 and Figure 7, gives the reconstructed images that are shown in Figure 11 and Figure 12. Hence both theoretically and practically, it is evident that it is impossible to reconstruct objects from very few projection images using conventional reconstruction algorithms, but it is possible to reconstruct some images(images with less variability) with combinatorial image reconstruction approach(proposed approach) from very few projections. The filtered back projection algorithm for three synthesized images give the results that are shown in Fig.A, Fig.B and Fig.C.

As the conventional smoothing(low pass filtering) alter the projections, a combinatorial approach may be followed for reconstructing smooth binary image from projections.

III. NOTATIONS AND DEFINITIONS

Let $\tau_{m \times n}$ be a family of bi-level images of size $m \times n$, $A \in \tau_{m \times n}$, $R = (r_1, r_2, \dots, r_m)$ and $C = (c_1, c_2, \dots, c_n)$. For each $1 \leq i \leq m$, r_i is the number of 1's in row i . For each $1 \leq j \leq n$, c_j is the number of 1's in column j . vector R and C are said to be projections of the image A along row and column respectively. An image $S = (s_{i,j})$ is said to be a sub image of $A = (a_{i,j})$ if $s_{i,j} = a_{i,j}$ or $s_{i,j} = *$. A **binary matrix with holes** $X = (x_{i,j})$ is defined as $x_{i,j} = 0$ or 1 or h . X is said to have hole at (i, j) if $x_{i,j} = h$. The set $H = \{(i, j) | x_{i,j} = h\}$ is called as hole-set of X . $R = (r_1, r_2, \dots, r_m)$ and $C = (c_1, c_2, \dots, c_n)$ are said to be consistent if $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$.

Smooth-number of a binary matrix is defined as the total number of flips, where flip is two adjacent pixels in row or column directions with different pixel values(i.e if $x_{i,j} = 0$ then $x_{i,j+1} = 1$ or $x_{i,j-1} = 1$ or $x_{i+1,j} = 1$ or $x_{i-1,j} = 1$ or if $x_{i,j} = 1$ then $x_{i,j+1} = 0$ or $x_{i,j-1} = 0$ or $x_{i+1,j} = 0$ or $x_{i-1,j} = 0$)

Example 1:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

For both matrices A and B , $R = (2, 2, 2, 2)$ and $C = (2, 2, 2, 2)$ are the projections along row and column

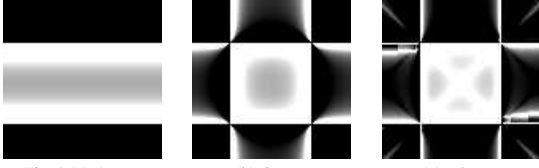
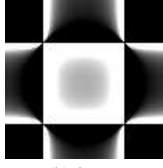
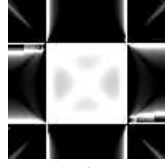


Fig.A(a):1 pro.



(b):2 pro.



(c): 5 pro.

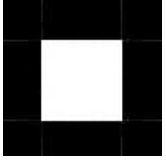
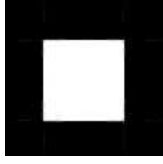
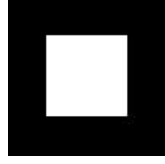


Fig.A (d): 60 pro



(e) 90: pro



(f): 180 pro

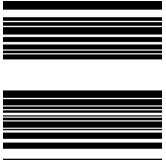
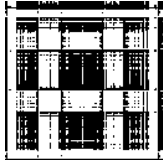
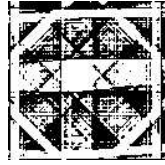


Fig.B(a):1 pro.



(b):2 pro.



(c): 5 pro.

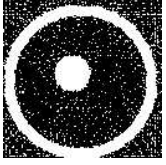
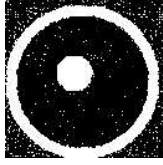


Fig.B(d): 60 pro



(e) 90: pro



(f): 180 pro

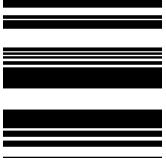
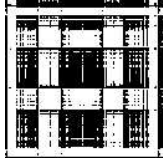
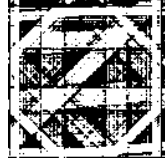


Fig.C(a):1 pro.



(b):2 pro.



(c): 5 pro.

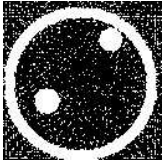
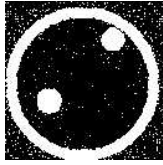
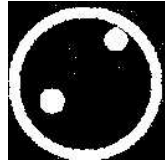


Fig.C (d): 60 pro



(e) 90: pro



(f): 180 pro

respectively.

$$\text{and a sub image : } S = \begin{bmatrix} 1 & * & * & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & 1 \end{bmatrix}$$

, but smooth numbers for A and B are 8 and 16 respectively.

IV. UNIQUENESS PROBLEM

Given row projection $R = (r_i)$, column projection $C = (c_j)$ and a sub image $S = (s_{i,j})$ of an unknown bi-level image $A = (a_{i,j})$, the goal is to check whether or not there is unique smooth bi-level image $B = (b_{i,j})$ such that R and C are the

row and column projections of $B = (b_{i,j})$ respectively, and $S = (s_{i,j})$ is a sub image of B .

In [11], a necessary and sufficient conditions for unique reconstruction of a binary matrix from two orthogonal projections (not with a priori information) is given. The necessary and sufficient condition is the presence of switching components in the binary matrix. The switching components are defined as follows:

$$\begin{bmatrix} a_{i,j} & a_{i,j'} \\ a_{i',j} & a_{i',j'} \end{bmatrix}$$

where $a_{i,j} = a_{i',j'} = 1$ and $a_{i,j'} = a_{i',j} = 0$ or $a_{i,j} = a_{i',j'} = 0$ and $a_{i,j'} = a_{i',j} = 1$

It is obvious that a necessary and sufficient conditions for unique reconstruction of smooth bi-level image from two orthogonal projections and sub image is the absence of switching components that do not increase the smooth number, and $a_{i,j}$, $a_{i,j'}$, $a_{i',j}$ and $a_{i',j'}$ are not in sub image .

By switching 0's to 1's and 1's to 0's in the switching components, we can get another bi-level image with the same projections.

Example 2: Consider the the following images:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection along row $R = (2, 1, 1, 2)$, projection along column: $C = (2, 0, 1, 1, 2)$

$$\text{sub image of A: } S = \begin{bmatrix} * & * & * & * & 1 \\ 0 & * & * & 0 & 0 \\ * & * & * & * & 0 \\ 1 & * & 0 & * & 1 \end{bmatrix}$$

Here even though both the matrices A and B have same orthogonal projections R and C , the sub image S , and same smooth-number(8), they are different. The switching component

$$\begin{bmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{bmatrix}$$

is switched to

$$\begin{bmatrix} b_{2,2} & b_{2,3} \\ b_{3,2} & b_{3,3} \end{bmatrix}$$

and the remaining entries of matrices A and B are the same.

V. RECONSTRUCTION PROBLEM

Given row projection $R = (r_i)$, column projection $C = (c_j)$ and a sub image $S = (s_{i,j})$ of an unknown bi-level image $A = (a_{i,j})$, the goal is to obtain a bi-level image $B = (b_{i,j})$ such that R and C are row and column projections of $B = (b_{i,j})$ respectively, the smooth number for B is minimum, and $S = (s_{i,j})$ is a sub image of B .

We construct network G for the given hole-set H and the projections $R = (r_1, r_2, \dots, r_m)$ and $C = (c_1, c_2, \dots, c_n)$ as follows:

$G = (V, E, C)$ be a weighted directed graph where

$$\begin{aligned}
V &= U \cup W \cup \{s, t\} \\
U &= \{u_i \mid 1 \leq i \leq m\} \\
W &= \{w_j \mid 1 \leq j \leq n\} \\
E &= \{(u_i, w_j) \mid (i, j) \notin H, 1 \leq i \leq m, 1 \leq j \leq n\} \\
&\quad \cup \{(s, u_i) \mid 1 \leq i \leq m\} \cup \{(w_j, t) \mid 1 \leq j \leq n\}
\end{aligned}$$

For each $1 \leq i \leq m$ and $1 \leq j \leq n$, if $(u_i, w_j) \in E$,

$$C(u_i, w_j) = 1; C(s, u_i) = r_i; C(w_j, t) = c_j$$

where s is the source and t is the sink.

Theorem 1: Let H be a hole-set and $R = (r_1, r_2, \dots, r_m)$ and $C = (c_1, c_2, \dots, c_n)$ be two integral vectors. There is a binary matrix $X = (x_{i,j})$ with hole-set H such that row and column projections of X are R and C respectively iff R and C are consistent and max flow value for the network G corresponds to R , C and H is $|f| = \sum_{i=1}^m r_i$.

Proof: Assume that $X = (x_{i,j})$ is binary matrix with hole-set H and having row and column projections $R = (r_1, r_2, \dots, r_m)$ and $C = (c_1, c_2, \dots, c_n)$ respectively. Then, the network G corresponds to R , C and H has flow $f: V \times V \rightarrow \mathbb{Z}^*$ such that

For each $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$f(s, u_i) = r_i, f(w_j, t) = c_j,$$

$$f(u_i, w_j) = 1 \text{ if } x_{i,j} = 1,$$

$$f(u_i, w_j) = 0 \text{ if } x_{i,j} \neq 1,$$

$$f(s, u_i) = -f(u_i, s), f(w_j, t) = -f(t, w_j),$$

$$f(u_i, w_j) = -f(w_j, u_i)$$

and $f(e) = 0$ for all $e \in V \times V$ such that $f(e)$ is not defined above. The flow f has the following properties:

Capacity constraint: $f(e) \leq C(e)$ for all $e \in V \times V$. From our definition of f , capacity constraint is evident.

Skew symmetry: $f(u, v) = -f(v, u)$ for all $(u, v) \in V \times V$.

Skew symmetry is also evident from our definition of flow f

Flow conservation: For all $u \in V - \{s, t\}$ $\sum_{v \in V} f(u, v) = 0$. Since for each $1 \leq i \leq m$, row i has r_i 1's, the number of outgoing edges with capacity 1 from u_i is r_i . Hence total amount outgoing flow from u_i is r_i . The only incoming flow to vertex u_i is from the source, which is also r_i . Since f is skew symmetry and incoming flow is same as outgoing flow at node u_i , $\sum_{v \in V} f(u_i, v) = 0$ where $1 \leq i \leq m$. Since for each $1 \leq j \leq n$, column j has c_j 1's, the number of incoming edges with capacity 1 to w_j is c_j . Hence total amount outgoing flow from w_j is c_j . The only outgoing flow from vertex w_j is to the sink, which is also c_j . Since f is skew symmetry and incoming flow is same as outgoing flow at node w_j , $\sum_{v \in V} f(w_j, v) = 0$ where $1 \leq j \leq n$. Hence the flow conservation.

The value of flow f is $|f| = \sum_{v \in V} f(s, v) = \sum_{i=1}^m f(s, u_i)$. Since $|f| \leq \sum_{v \in V} C(s, v)$ for any f and for our f , $|f| = \sum_{v \in V} C(s, v)$, $|f| = \sum_{i=1}^m r_i$ is the maximum flow.

Since $|f| = \sum_{i=1}^m f(s, u_i)$ and $|f| = \sum_{i=1}^n f(w_j, t)$, $\sum_{i=1}^m r_i = \sum_{j=1}^n c_j$. Hence R and C are consistent.

Assume that H is a hole-set and $R = (r_1, r_2, \dots, r_m)$ and $C = (c_1, c_2, \dots, c_n)$ are two consistent integral vectors, and the max flow value for the network G corresponds to R , C and H is $|f| = \sum_{i=1}^m r_i$. $X = (x_{i,j})$ is defined as follows.

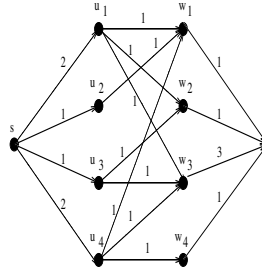


Fig. 1 Flow Network

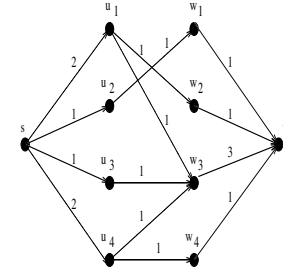


Fig. 2 Max Flow

For each $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$x_{i,j} = 1 \text{ if } f(u_i, w_j) = 1$$

$$x_{i,j} = h \text{ if } (i, j) \in H$$

$$x_{i,j} = 0 \text{ otherwise}$$

From the definition of X , the hole-set of X is H . Since max flow value $|f| = \sum_{i=1}^m r_i$ and $C(s, u_i) = r_i$, the amount of flow goes through u_i from the source s is exactly r_i . Therefore there are exactly r_i outgoing edges from vertex u_i have flow value 1. Hence there will be r_i cells in row i in X filled with 1's.

Since $|f| = \sum_{j=1}^n c_j$ and $C(w_j, t) = c_j$, the amount of flow goes from w_j to the sink t is exactly c_j . Therefore there are exactly c_j incoming edges to the vertex w_j have flow value 1. Hence there will be c_j cells in column j in X filled with 1's. Hence the theorem.

Example 3:

Consider the projections $R = (2, 1, 1, 2)$, $C = (1, 1, 3, 1)$ and the hole-set $H = \{(1,4), (2,2), (2,3), (2,4), (3,1), (3,4), (4,2)\}$. The binary matrix with projections R and C and hole-set H is

$$X = \begin{bmatrix} 0 & 1 & 1 & h \\ 1 & h & h & h \\ h & 0 & 1 & h \\ 0 & h & 1 & 1 \end{bmatrix},$$

Figure 1 is the corresponding flow network and figure 2 is a maximum flow in the flow network.

Algorithm: Constrained binary image reconstruction

Input: A sub image S of an unknown bi-level image A and row and column projections $R = (r_1, r_2, \dots, r_m)$ and $C = (c_1, c_2, \dots, c_n)$ of A

Output: Smooth bi-level image $B = (b_{i,j})$ such that R and C are the row and column projections of $B = (b_{i,j})$ respectively, and $S = (s_{i,j})$ is a sub image of B .

Initialisation: $m :=$ the number of components in R , $n :=$ the number of components in C . For each $1 \leq i \leq m$ and $1 \leq j \leq n$, $b_{i,j} := 0$

Step 1: For each $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$b_{i,j} := s_{i,j} \text{ if } s_{i,j} \neq *$$

Step 2: Compute $R' = (r'_1, \dots, r'_m)$ $C' = (c'_1, \dots, c'_n)$ where $r'_i = \sum_{j=1}^n b_{i,j}$, $c'_j = \sum_{i=1}^m b_{i,j}$ and $1 \leq i \leq m$, $1 \leq j \leq n$

Step 3. Construct integral weighted directed graph as given below.

Let $G = (V, E, C)$ be a weighted directed graph where

$$V = U \cup W \cup \{s, t\}$$

$$U = \{u_i \mid 1 \leq i \leq m\}$$

$$W = \{w_i \mid 1 \leq i \leq n\}$$

$$E = \{(u_i, w_j) \mid s_{i,j} = *, 1 \leq i \leq m, 1 \leq j \leq n\} \\ \cup \{(s, u_i) \mid 1 \leq i \leq m\} \cup \{(w_j, t) \mid 1 \leq j \leq n\}$$

$$C(u_i, w_j) = 1, C(s, u_i) = r_i - r'_i \text{ and } C(w_j, t) = c_j - c'_j \\ \text{where } 1 \leq i \leq m \text{ and } 1 \leq j \leq n$$

Step 4. Compute the integer max flow f for the network $G = (V, E, C)$ constructed in step 3 by using Ford-Fulkerson max-flow algorithm, in which the augmenting path is chosen as given below:

Step 4(a). Find shortest path P_i for each pair (u_i, t) where $1 \leq i \leq n$

Step 4(b). For each $1 \leq i \leq n$ Compute augmenting path P'_i which is from s to u_i and then along the path P_i

Step 4(c). Choose the augmenting path from all augmenting paths computed in step 4(b) such that the augmenting path that updates the image being constructed should result in less smooth-number.

Step 5. Construct matrix B as follows.

For each $1 \leq i \leq m$ and $1 \leq j \leq n$, $b_{i,j} := 1$ if $f(i, j) = 1$

Note: It is to be noted that step 4 is iterative, which terminates when there is no augmenting path.

EXAMPLE 4:

The above algorithm is illustrated with an example,

$$\text{Input : } R = (2, 3, 1, 2) \quad C = (1, 3, 3, 1)$$

$$\text{and a sub image } \begin{bmatrix} * & 1 & * & * \\ 0 & * & 1 & * \\ * & 0 & * & * \\ * & * & 1 & 0 \end{bmatrix}$$

Initialisation:

$$m = 4, n = 4 \text{ and } B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 1:

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 2: The row and column projections of the matrix B are $R' = (1, 1, 0, 1)$, $C' = (0, 1, 2, 0)$ respectively

Step 3: The network that corresponds to the given instance of the problem is given in figure 3.

Step 4: The maximum flow of the network is given in figure 4.

The reconstructed matrix is:

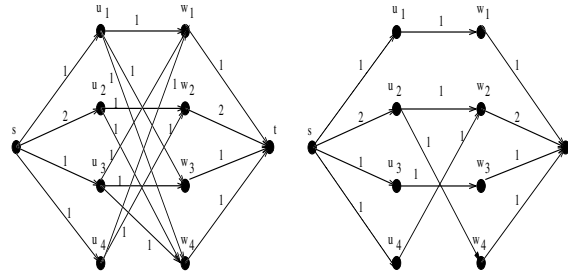


Fig. 3 Flow Network

Fig. 4 Max Flow

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

VI. SIMULATION STUDIES

A phantom image (figure 5) was considered, and row and column projections of the phantom image were computed. The algorithm given in [11] is implemented, and the reconstructed image is given in figure 6. For the implementation of proposed algorithm, the row and column projections of the same phantom image and a priori sub image given in figure 9 are considered as input. The reconstructed image obtained by implementing proposed algorithm is exactly same as the input phantom image (figure 5). But for the phantom image given in figure 7 and for the known portion of the image given in figure 9, the reconstructed image by proposed algorithm (figure 10) is not the same as the original image. The reconstructed image obtained by implementing algorithm given in [11] for the phantom image given in figure 7, is given in figure 8. For phantom image considered in Fig.A(square with black background), the reconstructed image from two projections is same as the unknown image, but the conventional algorithms takes 90 projections to get the perfect reconstruction.

It should be noted that figures 13 and 14 have same projections along the row and column directions, however, figure 14 has lesser smooth number than that of figure 13. (Figure 14 has been obtained using our algorithm).

VII. CORRECTNESS AND COMPLEXITY

By Theorem 1, a bi-level image with holes can be reconstructed from its orthogonal projections if the number of 1 's in the image is the max flow value and no bi-level-image exists if the number of 1 's in the image is not same as the max flow value. The algorithm considers the non $*$ locations of the known sub image as the hole-set. The algorithm copies the sub image and then unknown portion of the unknown image is reconstructed by considering non $*$ locations of the known sub image as the hole-set.

The time complexity of Step 1, step 2 and step 3 of our algorithm is $O(mn)$ where m is number of rows and n is number of columns. The dominant part of proposed algorithm is computing max flow (step 4). The FORD-FULKERSON algorithm for solving max flow problem can be implemented in $O(V^3)$ where V is the number of vertices in the network.

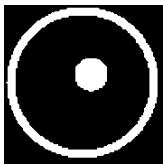


Fig. 5: Unknown(a)

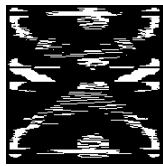


Fig. 6: Ryser(a)



Fig. 7: Unknown(b)

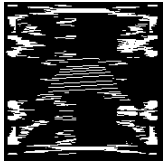


Fig. 8: Ryser(b)



Fig. 9: Sub Image



Fig. 10: prop. Rec

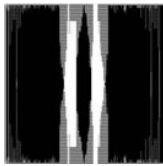


Fig. 11: Fourier(a)

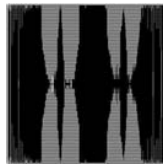


Fig. 12: Fourier(b)

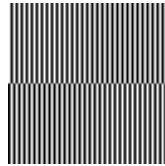


Fig. 13: Smooth



Fig. 14: Non-smooth

The augmenting path that minimizes smooth-number in every iteration is chosen. The augmenting path is computed in $\mathcal{O}(n^3)$ time. As this algorithm computes n augmenting paths when a single path is computed in max-flow algorithm, the time complexity of step 4 of proposed algorithm is $\mathcal{O}(V^3 * n^3)$ where $V = m + n + 2$. Hence the time complexity of our algorithm is $\mathcal{O}(V^3 * n^3)$ where $V = m + n + 2$.

VIII. CONCLUSION

In this paper, reconstruction of a smooth 2D-bi-level image from its two orthogonal projections and a priori sub image is considered. 2D-bi-level image reconstruction from two orthogonal projections has polynomial time algorithm [11], 2D-tri-level image reconstruction from two orthogonal projections is still open, 2D-four-level image reconstruction from two orthogonal projection is NP-hard. The problem that we have solved is more complex than 2D-bi-level image reconstruction from two orthogonal projections and less complex than 2D-tri-level image reconstruction from two orthogonal projections. The proposed algorithm is implemented and the quality of reconstructed image by the proposed algorithm is compared with the quality of reconstructed image by algorithm given in [6], and it is noticed that the reconstructed image by proposed algorithm is more close to original image even outside the known sub image portion of the original image than 2D-bi-level image reconstruction from two orthogonal projections without smoothness constraint (algorithm given in [6]). The reconstruction of 3D-bi-level smooth image from two

orthogonal projections with a priori sub image can be done by slice-by-slice reconstruction using the proposed algorithm. One of the possible areas in which our algorithm can be used is medical imaging. Non-invasive imaging techniques such as MRI can be used to get a sub image. The more clear portion of MRI image can be considered as sub image of original image to be reconstructed and unclear portion of MRI image may be obtained by X-ray tomography from less number of projections. Another area of application is crystallography. The proposed algorithm can be used to reconstruct crystalline structure from two projections without damaging the crystal. Though the proposed algorithm always gives solutions that satisfy projection constraints, and sub image constraints, the smoothness is not optimum, and we conjecture that obtaining optimum smoothness is NP-hard.

Future work will be on exploring possibilities of Simulated Annealing based approach to get better smooth binary images. As a new binary image can be obtained from current binary image by switching the switching component, and all switching components can be computed in polynomial time, we can get a new binary image by switching the switching component that results in less *smooth-no*. As the above greedy method may not lead to binary image with least *smooth-no*, Simulated Annealing which makes use of above method may give binary image with minimum *smooth-no*.

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