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Fourier spectral method for analytic continuation

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Abstract—The numerical analytic continuation of a function $f(z) = f(x + \mathrm{i}y)$ on a strip is discussed in this paper. The data are only given approximately on the real axis. The periodicity of given data is assumed. A truncated Fourier spectral method has been introduced to deal with the ill-posedness of the problem. The theoretic results show that the discrepancy principle can work well for this problem. Some numerical results are also given to show the efficiency of the method.

Keywords—analytic continuation,ill-posed problem, regularization method Fourier spectral method, the discrepancy principle

I. Introduction

The problems of analytic continuation are frequently encountered in many practical applications [10], [11], [13], [14]. In general, this problem is ill-posed and several techniques have been developed for it. In this paper we consider the problem of analytic continuation of periodic analytic function $f(y) = f(x + \mathrm{i}y)$ on a strip domain in the complex plane

$$\Omega = \{z = x + \mathrm{i}y \in \mathbb{C} \,|\, x \in \mathbb{R}, |y| \le y_0, y_0 \text{ is a positive constant}\}, \tag{1}$$

where i is the imaginary unit. The data are only given on the real axis, i.e., $f(z)\,|_{y=0} = f(x) =: f_0$ is known approximately and we would extend f analytically from this data to the whole domain Ω. This problem has been considered by a mollification regularization method in [9]. In [7], [8], Fourier method and Tikhonov regularization method has been developed for solving this problem.

In this paper, the Fourier spectral method will be used to deal with the problem in the case of f_0 is periodic on the real axis. The idea of this paper is analogy to the one in [7]. But in [7], the periodicity of the functions did not be utilized. So only a prior parameter can be used for it and the numerical results in [7] show that the method is sensitive for the choice of the parameter. In this paper, We will point out that the discrepancy principle can be used as the stop rule benefit from the accuracy of Fourier spectral method to periodic functions.

This paper is organized as follows. Some preliminary materials which will be introduced in section 2. In section 3, the developed method and corresponding convergence results will be established. Some numerical results are given in section 4 to show the efficiency of the new method.

II. Preliminaries

In this section, we present some preliminary materials which will be used throughout the paper. Let $\Lambda = (0, 2\pi)$ and

$$L^p(\Lambda) = \{\, v \mid v \text{ is measurable and } \|v\|_{L^p} < \infty \,\},$$

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where

$$\|v\|_{L^p} = \left(\int_\Lambda |v(x)|^p \, dx\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

In particular, $L^2(\Lambda)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{L^2(\Lambda)} = \int_\Lambda u(x)v(x)\, dx.$$

For simplicity, denote the norm$\|v\|_{L^2}$ by $\|v\|$.

The set of functions $e^{\mathrm{i}lx}, l = 0, \pm 1, \ldots$, is an orthogonal system in $L^2(\Lambda)$. The Fourier transformation of a function $v \in L^2(\Lambda)$ is

$$v = \sum_{l=-\infty}^{\infty} \hat{v}_l e^{\mathrm{i}lx} \tag{2}$$

where $\hat{v}$ is the Fourier coefficient,

$$\hat{v}_l = \frac{1}{2\pi}\int_\Lambda v(x)e^{-\mathrm{i}lx}dx, \quad l = 0, \pm 1, \ldots \tag{3}$$

The Parseval equality holds, namely

$$\|v\|^2 = 2\pi \sum_{-\infty}^{\infty} |\hat{v}_l|^2. \tag{4}$$

Now let N be any positive integer and V_N be the set of all trigonometric polynomials of degree at most N, i.e.,

$$V_N = \text{span}\left\{e^{\mathrm{i}lx} \,|\, |l| \le N\right\}. \tag{5}$$

The L^2- orthogonal projection $P_N : L^2(\Omega) \to V_N$ is such a mapping that for any $v \in L^2(\Omega)$,

$$\langle v - P_N v, \phi\rangle = 0, \quad \forall \phi \in V_N. \tag{6}$$

Indeed,

$$P_N v = \sum_{|l|\le N} \hat{v}_l e^{\mathrm{i}lx}. \tag{7}$$

We assume that

$$f(\cdot + \mathrm{i}y) \in L^2(\Lambda) \quad \text{for } |y| \le y_0. \tag{8}$$

Because the function $f(x)$ is analytic in Ω, the following series converges in Ω:

$$\begin{aligned} f(z) &= f(x + \mathrm{i}y) \\ &= \textstyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!}(\mathrm{i}y)^n \qquad |y| \le y_0. \\ &= \textstyle\sum_{n=0}^{\infty} \frac{(\mathrm{i}y)^n}{n!} D^n f(x) \end{aligned} \tag{9}$$

where $D^n = \frac{\partial^n}{\partial x^n}$. If the data $f(\cdot + \mathrm{i}y) \in L^2(\Lambda)$ for all y, $0 \le y \le y_0$ and we let

$$\hat{f}_l^y = \frac{1}{2\pi}\int_\Lambda f(x + \mathrm{i}y)e^{-\mathrm{i}lx}dx, \quad l = 0, \pm 1, \ldots \tag{10}$$

then we can get

$$\begin{aligned}\hat{f}_l^y &= \frac{1}{2\pi}\int_\Lambda \sum_{n=0}^{\infty}\frac{(\mathrm{i}y)^n}{n!}D^n f(x)e^{-\mathrm{i}lx}dx \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\int_\Lambda \frac{(\mathrm{i}y)^n}{n!}D^n f(x)e^{-\mathrm{i}lx}dx \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\int_\Lambda \frac{(\mathrm{i}y)^n}{n!}(il)^n f(x)e^{-\mathrm{i}lx}dx \\ &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\left(\frac{(-yl)^n}{n!}\int_\Lambda f(x)e^{-\mathrm{i}lx}dx\right) \\ &= e^{-yl}\hat{f}_l^0\end{aligned} \tag{11}$$

that is to say

$$f(z) = f(x+\mathrm{i}y) = \sum_{l=-\infty}^{\infty} e^{-yl}\hat{f}_l^0 e^{\mathrm{i}lx} =: \mathcal{A}_y f_0 \tag{12}$$

Lemma 1: Let $f(\cdot+\mathrm{i}y)\in L^2(\Lambda), \forall |y|\le y_0$, then for any $|y|\le y_0$,

$$\|f(\cdot+\mathrm{i}y)-\mathcal{A}_y(P_N f_0)\| \le e^{N(|y|-y_0)}\left(\|f(\cdot+\mathrm{i}y_0)\|+\|f(\cdot-\mathrm{i}y_0)\|\right). \tag{13}$$

Proof:

$$\begin{aligned}&\|f(\cdot+\mathrm{i}y)-\mathcal{A}_y P_N f_0\|^2 \\ &= 2\pi\left(\textstyle\sum_{l>N}|e^{-yl}\hat{f}_l|^2+\sum_{l<-N}|e^{-yl}\hat{f}_l|^2\right) \\ &\le 2\pi\left(\textstyle\sum_{l>N}|e^{|y|l}\hat{f}_l|^2+\sum_{l<-N}|e^{|y|l}\hat{f}_l|^2\right) \\ &\le 2\pi e^{2N(|y|-y_0)}\left(\textstyle\sum_{l>N}|e^{l(y_0-|y|)}e^{|y|l}\hat{f}_l|^2\right. \\ &\qquad\left.+\textstyle\sum_{l<-N}|e^{l(y_0-|y|)}e^{|y|l}\hat{f}_l|^2\right) \\ &\le 2\pi e^{2N(|y|-y_0)}\left(\textstyle\sum_{l>N}|e^{y_0 l}\hat{f}_l|^2+\sum_{l<-N}|e^{y_0 l}\hat{f}_l|^2\right) \\ &\le e^{2N(|y|-y_0)}\left(\|f(\cdot+\mathrm{i}y_0)\|+\|f(\cdot-\mathrm{i}y_0)\|\right).\end{aligned}$$

■

III. The method and convergence results

We assume the exact data f_0 and the measured data f^δ belong to $L^2(\Lambda)$ and satisfies

$$\|f_0 - f^\delta\| \le \delta, \tag{14}$$

where $\delta>0$ denotes the noisy level.

In addition, note that for any ill posed problem some a priori assumption on the exact solution is needed and necessary, otherwise, the convergence of the regularization approximate solution will not be obtained or the convergence rate can be arbitrary slow[3]. In this paper, we will assume there hold the following a priori bounds

$$\|f(\cdot+\mathrm{i}y_0)\| \le E, \tag{15}$$

and

$$\|f(\cdot-\mathrm{i}y_0)\| \le E, \tag{16}$$

We want to find a function φ^δ such that

$$\lim_{\delta\to 0}\|f(\cdot+\mathrm{i}y)-\mathcal{A}_y\varphi^\delta\| = 0, \quad \forall |y|\le y_0.$$

In the following, we propose a scheme to attain the function φ^δ from the perturbed data f^δ. We can give the approximate function as follows:

$$\varphi_m^\delta(x) = P_m f^\delta = \sum_{|l|\le m}\hat{f}_l^\delta e^{\mathrm{i}lx} \tag{17}$$

where $\hat{f}_l^\delta$ are the Fourier coefficients of f^δ and $m=m(\delta, f_0^\delta)$ is determined by the discrepancy principle

$$\|(I-P_m)f^\delta\| \le \tau\delta < \|(I-P_{m-1})f^\delta\| \tag{18}$$

with $\tau>1$.

In the following, we will prove a convergence estimate.

Theorem 2: Suppose that φ_m^δ is defined by (17) and (18) with $\tau>1$ and the conditions (14) and (15),(16)are hold, then for any $|y|\le y_0$, we have

$$\|f(\cdot+\mathrm{i}y)-\mathcal{A}_y\varphi_m^\delta\| \le E^{\frac{|y|}{y_0}}\left[(\tau+1)^{\frac{y_0-|y|}{y_0}}+2^{\frac{|y|}{y_0}}e^{|y|}(\tau-1)^{\frac{|y|}{y_0}}\right]\delta^{\frac{y_0-|y|}{y_0}}. \tag{19}$$

Proof:

Suppose $0<y\le y_0$, the proof for the case $-y_0\le y<0$ will be analogous. From (18), we can get

$$\begin{aligned}\|(I-P_m)f\| &= \|(I-P_m)f^\delta+(I-P_m)(f-f^\delta)\| \\ &\le \|(I-P_m)f^\delta\|+\|(I-P_m)(f-f^\delta)\| \\ &\le (\tau+1)\delta.\end{aligned}$$

And the following inequality is hold by Hölder inequality

$$\begin{aligned}&\|f(\cdot+\mathrm{i}y)-\mathcal{A}_y P_m f\|^2 = \textstyle\sum_{|l|>m}|e^{-yl}\hat{f}_l^0|^2 \\ &= \textstyle\sum_{|l|>m}e^{-2yl}|\hat{f}_l^0|^{\frac{2y}{y_0}}|\hat{f}_l^0|^{\frac{2(y_0-y)}{y_0}} \\ &\le \left(\textstyle\sum_{|l|>m}e^{-2y_0 l}|\hat{f}_l^0|^2\right)^{\frac{y}{y_0}}\left(\textstyle\sum_{|l|>m}|\hat{f}_l^0|^2\right)^{\frac{y_0-y}{y_0}} \\ &\le E^{\frac{2y}{y_0}}\|(I-P_m)f\|^{\frac{2(y_0-y)}{y_0}}.\end{aligned} \tag{20}$$

Hence

$$\|f(\cdot+\mathrm{i}y)-\mathcal{A}_y P_m f\| \le E^{\frac{y}{y_0}}(\tau+1)^{\frac{(y_0-y)}{y_0}}\delta^{\frac{(y_0-y)}{y_0}}. \tag{21}$$

On the other hand,

$$\begin{aligned}\|P_{m-1}f-f\| &= \|(P_{m-1}f^\delta-f^\delta)-(I-P_{m-1})(f-f^\delta)\| \\ &\ge \|P_{m-1}f^\delta-f^\delta\|-\|(I-P_{m-1})(f-f^\delta)\|.\end{aligned} \tag{22}$$

From (18), we have

$$\|P_{m-1}f^\delta - f^\delta\| > \tau\delta, \tag{23}$$

and it is obvious that

$$\|(I-P_{m-1})(f-f^\delta)\| \le \delta.$$

So we can obtain

$$\|P_{m-1}f-f\| \ge (\tau-1)\delta. \tag{24}$$

From Lemma 1

$$(\tau-1)\delta \le \|P_{m-1}f-f\| \le 2e^{(1-m)y_0}E. \tag{25}$$

So we can obtain

$$m \le \frac{\ln 2E - \ln(\tau-1)\delta}{y_0}+1.$$

TABLE I
RESULTS OF EXAMPLE 1 WITH $y = 0.5$

δ_1	m	$\frac{\| f(\cdot+iy)-\mathcal{A}_y\varphi_m^\delta \|}{\| f(\cdot+iy)\|}$	$\frac{\| f(\cdot+iy)-\mathcal{A}_y\varphi_m^\delta \|_\infty}{\| f(\cdot+iy)\|_\infty}$
10^{-1}	3	0.0101	0.0102
10^{-2}	4	0.0018	0.0022
10^{-3}	5	0.0003	0.0004
10^{-4}	6	6.12e-5	0.0001

TABLE II
RESULTS OF EXAMPLE 1 WITH $y = 1$

δ_1	m	$\frac{\| f(\cdot+iy)-\mathcal{A}_y\mathbf{f}_\delta \|}{\| f(\cdot+iy)\|}$	$\frac{\| f(\cdot+iy)-\mathcal{A}_y\mathbf{f}_\delta \|_\infty}{\| f(\cdot+iy)\|_\infty}$
10^{-1}	3	0.0573	0.0390
10^{-2}	4	0.0161	0.0135
10^{-3}	5	0.0042	0.0037
10^{-4}	6	0.0008	0.0008

Therefore

$$\begin{aligned}
&\| f(\cdot+iy) - \mathcal{A}_y\varphi_m^\delta \| \\
&= \| f(\cdot+iy) - \mathcal{A}_y P_m f \| + \| \mathcal{A}_y P_m f - \mathcal{A}_y\varphi_m^\delta \| \\
&\le E^{\frac{y}{y_0}}(\tau+1)^{\frac{y_0-y}{y_0}}\delta^{\frac{y_0-y}{y_0}} + \| \mathcal{A}_y P_m(f-f^\delta) \| \\
&\le E^{\frac{y}{y_0}}(\tau+1)^{\frac{y_0-y}{y_0}}\delta^{\frac{y_0-y}{y_0}} + e^{my}\delta \\
&\le E^{\frac{y}{y_0}}\left[(\tau+1)^{\frac{y_0-y}{y_0}} + 2^{\frac{y}{y_0}}e^y(\tau-1)^{\frac{-y}{y_0}}\right]\delta^{\frac{y_0-y}{y_0}}.
\end{aligned}$$

■

IV. NUMERICAL IMPLEMENTATION

In this section, we present numerical results of one example to check the efficiency of the method. . In all the cases, the discretization knots are $t_i = ih$, $i = 1, \cdots, N$, with $N = 256$, $h = 1/N$ and . The perturbed discrete data are given by

$$f^\delta(t_i) = f(t_i) + \epsilon_i,$$

where $\{\epsilon_i\}_{i=0}^N$ are generated by Function $\mathrm{randn}(N+1,1)\times\delta_1$ in Matlab. All examples are computed by using Matlab with parameters $\tau = 1.01$.

Example 1 The function

$$f(z) = \exp(\cos(z)).$$

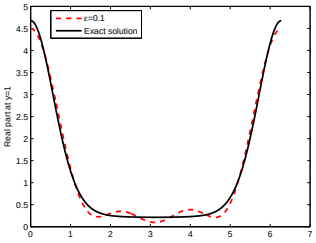
is a periodic analytic function with

$$f(z)|_{y=0} = \exp(\cos(x))$$

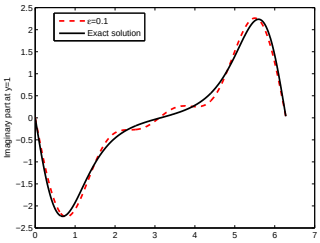
The L^2–norm relative errors and the maximum–norm relative errors are given in the tables 1, 2 to verify the theoretical results. Fig. 1 is also given to compare qualitatively the computed solutions and the exact ones. All of the results show that the new method works well.

V. CONCLUTION

In this paper the truncated Fourier spectral method is used to give a stable analytic continuation of the periodic analytic function. The theoretical and numerical results indicate that the discrepancy principle can work well if we can find a suitable approximation even for severely ill-posed problem.



(a) real part at $y = 1$



(b) imaginary part at $y = 1$

Fig. 1. Results of Example1 with $\delta = 0.1$

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