

Solitary Wave Solutions for Burgers-Fisher type Equations with Variable Coefficients

Amit Goyal, Alka, Rama Gupta and C. Nagaraja Kumar

Abstract—We have solved the Burgers-Fisher (BF) type equations, with time-dependent coefficients of convection and reaction terms, by using the auxiliary equation method. A class of solitary wave solutions are obtained, and some of which are derived for the first time. We have studied the effect of variable coefficients on physical parameters (amplitude and velocity) of solitary wave solutions. In some cases, the BF equations could be solved for arbitrary time-dependent coefficient of convection term.

Keywords—Solitary wave solution, Variable coefficient Burgers-Fisher equation, Auxiliary equation method.

I. INTRODUCTION

THE investigation of solitary wave solutions to nonlinear evolution equations plays an important role in the qualitative description of many nonlinear physical phenomena. Nonlinear wave phenomena of diffusion, dispersion, dissipation, reaction and convection are very important in nonlinear wave equations. Burgers-Fisher (BF) type equations describe the interaction between reaction mechanism, convection effect and diffusion process. Due to this, these equations have a wide range of applications in plasma physics, fluid physics, capillary-gravity waves, nonlinear optics, chemical physics and population dynamics [1]–[3].

Recently, the study of variable coefficient nonlinear equations has grown steadily because most of physical and biological systems are inhomogeneous due to fluctuations in environmental conditions and non-uniform media. Hence, most of real nonlinear physical equations possess spatially and/or temporally varying coefficients [4]–[6]. BF equations with constant coefficients has been well studied by many authors [7], but the effect of inhomogeneities has not been much explored. In this paper, we have studied temporally inhomogeneous BF equations. In these equations, the relevant parameters are time dependent [8], [9] because external factors make the density and/or temperature change in time. The general form of variable coefficient Burgers-Fisher equation is

$$u_t + v(t) u^m u_x = Du_{xx} + \alpha(t) u(1 - u^n), \quad (1)$$

where $v(t)$ & $\alpha(t)$ are convection and reaction terms coefficients, D is diffusion constant and m, n are positive integers. If $v(t) = 0$ & $\alpha(t)$ is constant, Eq. (1) reduces to well known Fisher equation [10] for $n = 1$, and when $\alpha(t) = 0$ & $v(t)$ is constant, then it reduces to Burgers equation [11] for $m = 1$. The Fisher equation describes the dynamics of a

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field subject to diffusive transport and logistic growth, and the Burgers equation is a simple nonlinear model for turbulence.

There are many numerical and analytical techniques to find solutions of variable coefficient nonlinear equations, such as the Painlevé analysis [12], Hirota bilinear method [13], Jacobi elliptic function method [14], homogeneous balance method [15] and exp-function method [16]. We have used auxiliary equation method [17] to solve Eq. (1) and successfully obtained solitary wave solutions.

This paper is organized as follows. In Section 2, we have solved the Eq. (1) for $m = 0$ and $n = 1$, which has applications in the field of plasma dynamics [1] and population biology [18]. In Section 3, we have found the solution of Eq. (1) for $m = 1$ and $n = 1$, which describe the various biological and physical phenomena [see Ref. [2], and references therein]. In Section 4, we have solved the Eq. (1) for quadratic nonlinearity in convective term i.e. $m = 2$ and $n = 3$. Finally, in Section 5, we close this article with concluding remarks.

II. SOLUTIONS OF EQ. (1) FOR $m = 0$ AND $n = 1$

A. Only v is time dependent

For this case, Eq. (1) reads

$$u_t + v(t) u_x = Du_{xx} + \alpha u(1 - u). \quad (2)$$

To begin with, we assume the solution of Eq. (2) in extended Galilean frame of reference [19], $\xi = kx + \eta(t)$, k is constant, as

$$u(\xi) = a(t) + b(t) \phi(\xi) + c(t) \phi^2(\xi), \quad (3)$$

where $\phi(\xi)$ should satisfy an auxiliary ordinary differential equation, viz.

$$(\phi_\xi)^2 = p\phi^2 + q\phi^3 + r\phi^4, \quad (4)$$

where p, q, r are constants. By substituting Eqs. (3) and (4) in Eq. (2), and then setting the coefficients of ϕ^i ($i = 0, 1, 2, 3, 4$), ϕ_ξ and $\phi\phi_\xi$ equal to zero, we get the following relations for a, b, c, k and η :

$$\begin{aligned} \phi^4 & : -6crDk^2 + \alpha c^2 = 0, \\ \phi^3 & : -2brDk^2 - 5cqDk^2 + 2bca = 0, \\ \phi^2 & : 2aca + b^2\alpha + c_t - \frac{3}{2}bqDk^2 - 4cpDk^2 - \alpha c = 0, \\ \phi^1 & : 2ab\alpha + b_t - bpDk^2 - \alpha b = 0, \\ \phi^0 & : a^2\alpha + a_t - \alpha a = 0, \\ \phi\phi_\xi & : 2c\eta_t + 2vck = 0, \\ \phi_\xi & : b\eta_t + vbk = 0. \end{aligned} \quad (5)$$

Solving these equations consistently, we get

$$a = 1, \quad b = \frac{3q}{p}, \quad c = \frac{6r}{p},$$

$$k = \pm \sqrt{\frac{\alpha}{pD}}, \quad \eta(t) = -k \int v(t) dt, \quad (6)$$

along with a constraining condition $q^2 = 4pr$. From Eq. (6) it is clear that a, b, c are constants. As $q^2 = 4pr$, the general solution for Eq. (4) can be obtained by quadrature method in a straightforward way as [20]

$$\phi(\xi) = -\frac{q}{r} \left[\frac{\operatorname{sech}^2\left(\frac{\sqrt{p}}{2}\xi\right)}{4 - \left[1 - \tanh\left(\frac{\sqrt{p}}{2}\xi\right)\right]^2} \right], \quad \text{for } p > 0, \quad (7)$$

and finally, we get the solution for Eq. (2) using Eq. (3) as

$$u(x, t) = 1 - 12 \left[\frac{\operatorname{sech}^2\left(\frac{\sqrt{p}}{2}\xi\right)}{4 - \left[1 - \tanh\left(\frac{\sqrt{p}}{2}\xi\right)\right]^2} \right] + 24 \left[\frac{\operatorname{sech}^2\left(\frac{\sqrt{p}}{2}\xi\right)}{4 - \left[1 - \tanh\left(\frac{\sqrt{p}}{2}\xi\right)\right]^2} \right]^2, \quad (8)$$

where $\xi = k(x - \int v(t) dt)$. It means, non-trivial time dependance of the ξ variable is entirely expressed in terms of function of $v(t)$, ie wave is moving with variable velocity as a function of convection coefficient. Typical profile of Eq. (8) is shown in Figure 1, for $\alpha = 1, D = 1, p = 1$ and $v(t) = \sin(2t)$. It is interesting to note that for small magnitude of x the function $u(x, t)$ has periodic structure in time but as magnitude of x increases, $u(x, t)$ approaches a constant value, ie, amplitude of wave becomes constant.

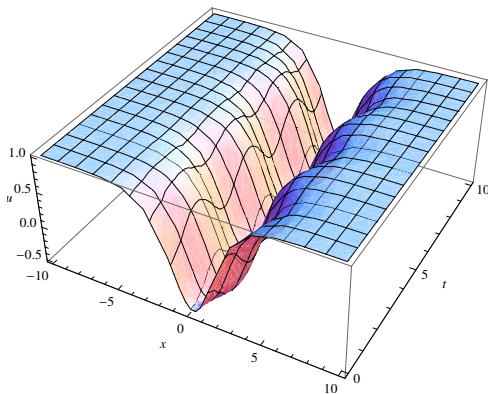


Fig. 1. Typical form of $u(x,t)$, Eq. (8) for values mentioned in the text.

For this case, Eq. (2) with constant convection coefficient has same solution as Eq. (8) where $\xi = k(x - c_0 t)$. It means solitary wave is moving with a constant velocity in this case.

B. v and α both are time dependent

In order to solve Eq. (2) for the case when v and α both are time dependent, we follow the same procedure as in last section and get the same set of equations as Eq. (5) with $v(t)$ and $\alpha(t)$. Solving these equations consistently, we get the following relations

$$a(t) = b(t) = \frac{q c(t)}{2r}, \quad a(t) = 1 + e^{-pDk^2 t},$$

$$\alpha(t) = \frac{3qDk^2}{a(t)}, \quad \eta(t) = -k \int v(t) dt, \quad (10)$$

such that p, q, r satisfying the conditions, $q^2 = 4pr$ and $p = 3q$. Here a, b and c are turns out to be a function of time. The complete solution for $u(\xi)$, Eq. (1) for $m = 0, n = 1$ reads

$$u(\xi) = (1 + e^{-pDk^2 t}) - 12(1 + e^{-pDk^2 t}) \left(\frac{\operatorname{sech}^2\left(\frac{\sqrt{p}}{2}\xi\right)}{4 - \left[1 - \tanh\left(\frac{\sqrt{p}}{2}\xi\right)\right]^2} \right) \left[1 - \frac{2\operatorname{sech}^2\left(\frac{\sqrt{p}}{2}\xi\right)}{4 - \left[1 - \tanh\left(\frac{\sqrt{p}}{2}\xi\right)\right]^2} \right], \quad (11)$$

where $\xi = k(x - \int v(t) dt)$. Here, amplitude of wave is changing with time. Typical profile of Eq. (11) is shown in Figure 2, for $D = 1, p = 1, k = 1$ and $v(t) = \tanh(t)$. If we compare both the solutions, Eqs. (8) and (11), we can say that convection coefficient effects the velocity of wave whereas reaction coefficient effects the amplitude of wave.

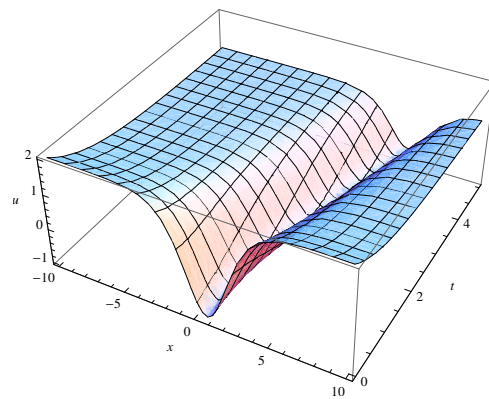


Fig. 2. Typical form of $u(x,t)$, Eq. (11) for values mentioned in the text.

Special case

Let $v(t) = c_0$ (a constant)
Then, from Eq. (6)

$$\eta(t) = -kc_0 t. \quad (9)$$

III. SOLUTIONS OF EQ. (1) FOR $m = 1$ AND $n = 1$

A. Only v is time dependent

For this case, Eq. (1) reads

$$u_t + v(t) uu_x = Du_{xx} + \alpha u(1 - u). \quad (12)$$

Now, we assume the solution of Eq. (12) in extended Galilean frame of reference, $\xi = kx + \eta(t)$, k is constant, as

$$u(\xi) = a(t) + b(t) \phi(\xi), \tag{13}$$

where $\phi(\xi)$ satisfies an auxiliary equation, viz.

$$\phi_\xi = p\phi + \phi^2, \tag{14}$$

where p is constant. Substituting Eqs. (13) and (14) in Eq. (12), and equating the coefficients of ϕ^i 's ($i = 0, 1, 2, 3$) to zero, we get a set of equations given as

$$\begin{aligned} \phi^3 : & b^2kv - 2bDk^2 = 0, \\ \phi^2 : & \alpha b^2 + abkv + pb^2kv - 3pbDk^2 + b\eta_t = 0, \\ \phi^1 : & -\alpha b + 2\alpha ab + pabkv - p^2bDk^2 + b_t + pb\eta_t = 0, \\ \phi^0 : & -\alpha a + \alpha a^2 + a_t = 0. \end{aligned} \tag{15}$$

Solving these equations consistently, we obtain the following relations

$$\begin{aligned} a(t) &= p b(t), \\ b(t) &= \frac{1}{2p} \left[1 - \tanh \left(\frac{-\alpha t}{2} \right) \right] = \frac{1}{p(1 + e^{-\alpha t})}, \\ v(t) &= \frac{2Dk}{b(t)} = 2pDk(1 + e^{-\alpha t}), \\ \eta(t) &= \int -(pDk^2 + \alpha b(t)) dt \\ &= - \left(pDk^2 + \frac{\alpha}{2p} \right) t - \frac{1}{p} \ln \cosh \left(\frac{-\alpha t}{2} \right). \end{aligned} \tag{16}$$

Integrating Eq. (14), we get the solution for $\phi(\xi)$ as

$$\phi(\xi) = -\frac{p}{2} \left[1 + \tanh \left(\frac{p}{2} \xi \right) \right]. \tag{17}$$

Using Eqs. (13), (16) and (17), we get the solution for Eq. (12), as

$$u(\xi) = \frac{1}{4} \left(1 - \tanh \left(\frac{-\alpha t}{2} \right) \right) \left[1 - \tanh \left(\frac{p}{2} \xi \right) \right]. \tag{18}$$

Typical profile of Eq. (18) is shown in Figure 3, for $\alpha = 1$, $D = 1$, $p = 1$ and $k = 1$. It is a kink-type solitary wave solution, whose amplitude and velocity are changing with time due to time-dependent coefficient of convection term.

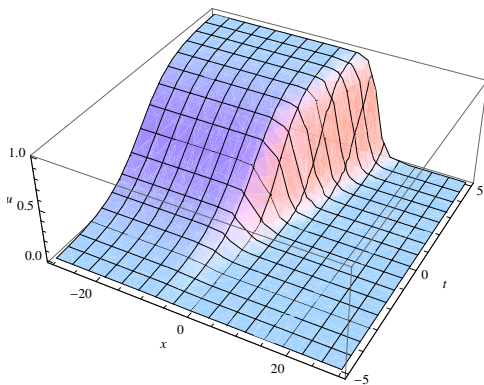


Fig. 3. Typical form of $u(x,t)$, Eq. (18) for values mentioned in the text.

B. v and α both are time dependent

In order to solve Eq. (12) for the case when v and α both are time dependent, we follow the same procedure as in last section and get the same set of equations as Eq. (15) with $v(t)$ and $\alpha(t)$. Solving these equations consistently, we get the following relations

$$\begin{aligned} a(t) &= p b(t), \\ b(t) &= \frac{1}{2p} \left[1 - \tanh \left(\frac{-\int \alpha(t) dt}{2} \right) \right] = \frac{1}{p(1 + e^{-\int \alpha(t) dt})}, \\ v(t) &= \frac{2Dk}{b(t)} = 2pDk(1 + e^{-\int \alpha(t) dt}), \\ \eta(t) &= \int -(pDk^2 + \alpha(t) b(t)) dt \\ &= -pDk^2 t - \frac{1}{2p} \int \alpha(t) dt - \frac{1}{p} \ln \cosh \left(-\frac{1}{2} \int \alpha(t) dt \right). \end{aligned} \tag{19}$$

The complete solution for $u(\xi)$, Eq. (1) for $m = 1$ and $n = 1$, comes out to be

$$u(\xi) = \frac{1}{4} \left(1 - \tanh \left(\frac{-\int \alpha(t) dt}{2} \right) \right) \left[1 - \tanh \left(\frac{p}{2} \xi \right) \right]. \tag{20}$$

Here, $v(t)$ is fixed but $\alpha(t)$ can be chosen arbitrarily which effects the both parameters, amplitude and velocity, of wave. Typical profile of Eq. (20) is shown in Figure 4, for $p = 1$, $D = 1$, $k = 1$, and $\alpha(t) = \cos(t)$.

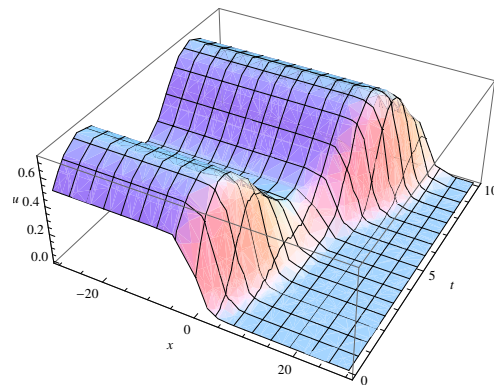


Fig. 4. Typical form of $u(x,t)$, Eq. (20) for values mentioned in the text.

Special case

If we assume Eq. (12) for the case when v and α both are constant, then by following the same procedure as in section 3.1, we obtain the various parameters as

$$\begin{aligned} a &= 1, \quad b = \frac{1}{p}, \quad v = 2pDk, \\ \eta(t) &= - \left(pDk^2 + \frac{\alpha}{p} \right) t. \end{aligned} \tag{21}$$

Using Eqs. (13), (17) and (21), the solution for constant coefficient BF equation becomes

$$u(\xi) = \frac{1}{2} \left[1 - \tanh \left(\frac{p}{2} \xi \right) \right], \quad (22)$$

which is of same type as obtained by Chen et al. [7].

IV. SOLUTIONS OF EQ. (1) FOR $m = 2$ AND $n = 3$

This equation can be solved only for one case, when v and α both are time dependent. Here, Eq. (1) reads

$$u_t + v(t) u^2 u_x = D u_{xx} + \alpha(t) u(1 - u^3). \quad (23)$$

Again, substituting Eqs. (13) and (14) in Eq. (23), and equating the coefficients of ϕ^i 's ($i = 0, 1, 2, 3, 4$) to zero, we get the following set of equations

$$\begin{aligned} \phi^4 : \quad & \alpha b^4 + b^3 k v = 0, \\ \phi^3 : \quad & 4\alpha a b^3 - 2b D k^2 + p b^3 k v + 2a b^2 k v = 0, \\ \phi^2 : \quad & 6\alpha a^2 b^2 - 3p b D k^2 + b \eta_t + a^2 b k v + 2p a b^2 k v = 0, \\ \phi^1 : \quad & b_t - \alpha b + 4\alpha a^3 b - p^2 b D k^2 + p b \eta_t + p a^2 b k v = 0, \\ \phi^0 : \quad & -\alpha a + \alpha a^4 + a_t = 0. \end{aligned} \quad (24)$$

Solving these equations consistently, we obtain the following relations :

$$\begin{aligned} a(t) &= p b(t), \quad b^3(t) = \frac{1}{p^3} \left(1 + e^{-6p^2 D k^2 t} \right), \\ \alpha(t) &= \frac{2Dk^2}{p b^3(t)}, \quad v(t) = -\frac{\alpha(t) b(t)}{k} = -\frac{2Dk}{p b^2(t)}, \\ \eta(t) &= \int (-3p D k^2) dt = -3p D k^2 t. \end{aligned} \quad (25)$$

Using Eqs. (13), (17) and (25), we get the solution for Eq. (23), as

$$u(\xi) = \frac{1}{2} \left(1 + e^{-6p^2 D k^2 t} \right)^{\frac{1}{3}} \left[1 - \tanh \left(\frac{p}{2} \xi \right) \right]. \quad (26)$$

It is a kink-type solitary wave solution which moves with a constant velocity. But its amplitude is changing with time due to variable coefficients. Typical profile of Eq. (26) is shown in Figure 5, for $p = 1$, $D = 1$, $k = 1$.

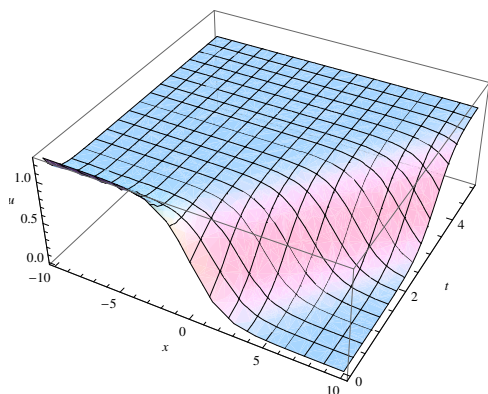


Fig. 5. Typical form of $u(x,t)$, Eq. (26) for values mentioned in the text.

V. CONCLUSION

In this paper, we have studied the BF type equations which are prototype model for the reaction, convection and diffusion processes. These equations can be used to model various physical phenomena, like to study the role of waterways in the spread of the Neolithic [21] and for swarming behavior based on non-local interactions [22]. We have obtained the solitary wave-type solutions for variable coefficient BF equations by using the auxiliary equation method. We have showed the effect of time-dependent coefficients on wave parameters. For $m = 0$, we have found that convection term coefficient can be choosed an arbitrary function of time, which effects the velocity of wave. Whereas for $m = 1$ case, reaction term coefficient can be choosed arbitrarily which effect the both parameters, amplitude and velocity of wave. But for the case $m = 2$, it is interesting to note that velocity of wave does not depend on any variable coefficient. These observations may be useful while working on specific phenomena based on BF type equations.

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REFERENCES

- [1] H. Wilhelmsson, Simultaneous diffusion and reaction processes in plasma dynamics, *Phys. Rev. A* **38** (1988) 1482-1489.
- [2] R.D. Benguria, M.C. Depassier and V. Mendez, Minimal speed of fronts of reaction-convection-diffusion equations, *Phys. Rev. E* **69** (2004) 031106.
- [3] J.D. Murray, *Mathematical Biology I & II* (Springer-Verlag, New York, 2002).
- [4] C. Sophocleous, Further transformation properties of generalised inhomogeneous nonlinear diffusion equations with variable coefficients, *Physica A* **345** (2005) 457-471.
- [5] L. Shaoyong, L. Xiumei and W. Yonghong, Explicit solutions of two nonlinear dispersive equations with variable coefficients, *Phys. Lett. A* **372** (2008) 7001-7006.
- [6] A.M. Wazwaz and H. Triki, Bright solitons and multiple soliton solutions for coupled modified KdV equations with time-dependent coefficients, *Phys. Scr.* **82** (2010) 015001.
- [7] H. Chen and H. Zhang, New multiple soliton solutions to the general Burgers-Fisher equation and the Kuramoto-Sivashinsky equation, *Chaos Solitons Fractals* **19** (2004) 71-76.
- [8] M.G. Neubert, M. Kot and M.A. Lewis, Invasion speeds in fluctuating environments, *Proc. R. Soc. Lond. B* **267** (2000) 1603-1610.
- [9] V. Méndez, J. Fort and T. Pujol, The speed of reaction-diffusion wavefronts in nonsteady media, *J. Phys. A: Math. Gen.* **36** (2003) 3983-3993.
- [10] R.A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics* **7** (1937) 355-369.
- [11] J.M. Burgers, A mathematical model illustrating the theory of turbulence, *Adv. Appl. Mech.* **1** (1948) 171-199.
- [12] P.S. Sundaram, A. Mahalingam and T. Alagesan, Solitary wave solution for inhomogeneous nonlinear Schrödinger system with loss/gain, *Chaos Solitons Fractals* **36** (2008) 1412-1418.
- [13] W.J. Liu et al., A new approach to the analytic soliton solutions for the variable-coefficient higher-order nonlinear Schrödinger model in inhomogeneous optical fibers, *J. Mod. Optic.* **57** (2010) 309-315.
- [14] X. Zhao, D. Tang and L. Wang, New soliton-like solutions for KdV equation with variable coefficient, *Phys. Lett. A* **346** (2005) 288-291.
- [15] C.L. Zheng and L.Q. Chen, Solitons with fission and fusion behaviors in a variable coefficient Broer-Kaup system, *Chaos Solitons Fractals* **24** (2005) 1347-1351.

- [16] S.A. El-Wakil, M.A. Madkour and M.A. Abdou, Application of Exp-function method for nonlinear evolution equations with variable coefficients, *Phys. Lett. A* **369** (2007) 62-69.
- [17] Sirendaoreji and S. Jiong, Auxiliary equation method for solving nonlinear partial differential equations, *Phys. Lett. A* **309** (2003) 387-396.
- [18] D.R. Nelson and N.M. Shnerb, Non-Hermitian localization and population biology, *Phys. Rev. E* **58** (1998) 1383-1403.
- [19] D.M. Greenberger, Some remarks on the extended Galilean transformation, *Am. J. Phys.* **47** (1979) 35-38.
- [20] E. Yomba, Construction of new soliton-like solutions for the (2 + 1) dimensional KdV equation with variable coefficients *Chaos Solitons Fractals* **21** (2004) 75-79.
- [21] K. Davison et al., The role of waterways in the spread of the Neolithic, *J. Archaeol. Sci.* **33** (2006) 641-652.
- [22] A. Mogilner and L.E. Keshet, A non-local model for a swarm, *J. Math. Biol.* **38** (1999) 534-570.