

Fuzzy Bi-ideals in Ternary Semirings

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Abstract—The purpose of the present paper is to study the concept of fuzzy bi-ideals in ternary semirings. We give some characterizations of fuzzy bi-ideals. Characterizations of regular ternary semirings are provided.

Keywords—Fuzzy ternary subsemiring, fuzzy quasi-ideal, fuzzy bi-ideal, regular ternary semiring

I. INTRODUCTION

TERNARY semirings are one of the generalized structures of semirings. The notion of ternary algebraic system was introduced by Lehmer [8]. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. Dutta and Kar [1] introduced the notion of ternary semiring which is a generalization of the ternary ring introduced by Lister [9]. Good and Hughes [3] introduced the notion of bi-ideal and Steinfeld [11], [12] introduced the notion of quasi-ideal. In 2005, Kar [5] studied quasi-ideals and bi-ideals of ternary semirings.

Ternary semiring arises naturally, for instance, the ring of integers \mathbf{Z} is a ternary semiring. The subset \mathbf{Z}^+ of all positive integers of \mathbf{Z} forms an additive semigroup and which is closed under the ring product. Now, if we consider the subset \mathbf{Z}^- of all negative integers of \mathbf{Z} , then we see that \mathbf{Z}^- is closed under the binary ring product; however, \mathbf{Z}^- is not closed under the binary ring product, i.e., \mathbf{Z}^- forms a ternary semiring. Thus, we see that in the ring of integers \mathbf{Z} , \mathbf{Z}^+ forms a semiring whereas \mathbf{Z}^- forms a ternary semiring. More generally; in an ordered ring, we can see that its positive cone forms a semiring whereas its negative cone forms a ternary semiring. Thus a ternary semiring may be considered as a counterpart of semiring in an ordered ring.

The theory of fuzzy sets was first inspired by Zadeh [14]. Fuzzy set theory has been developed in many directions by many scholars and has evoked great interest among mathematicians working in different fields of mathematics. Rosenfeld [13] introduced fuzzy sets in the realm of group theory. Fuzzy ideals in rings were introduced by Liu [10] and it has been studied by several authors. Jun [4] and Kim and Park [7] have also studied fuzzy ideals in semirings. In 2007, [6] we have introduced the notions of fuzzy ideals and fuzzy quasi-ideals in ternary semirings.

Our main purpose in this paper is to introduce the notions of fuzzy bi-ideal in ternary semirings and study regular ternary semiring in terms of these two subsystems of fuzzy subsemirings. We give some characterizations of fuzzy bi-ideals.

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II. PRELIMINARIES

In this section, we review some definitions and some results which will be used in later sections.

Definition 2.1. A set R together with associative binary operations called addition and multiplication (denoted by $+$ and \cdot respectively) will be called a semiring provided:

- (i) Addition is a commutative operation.
- (ii) there exists $0 \in R$ such that $a + 0 = a$ and $a0 = 0a = 0$ for each $a \in R$,
- (iii) multiplication distributes over addition both from the left and the right. i.e., $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$

Definition 2.2. A nonempty set S together with a binary operation, called addition and a ternary multiplication, denoted by juxtaposition, is said to be a ternary semiring if $(S, +)$ is an additive commutative semigroup satisfying the following conditions:

- (i) $(abc)de = a(bcd)e = ab(cde)$
- (ii) $(a + b)cd = acd + bcd$
- (iii) $a(b + c)d = abd + acd$
- (iv) $ab(c + d) = abc + abd$, for all $a, b, c, d, e \in S$.

Definition 2.3. (i) Let S be a ternary semiring. An additive subsemigroup T of S is called a ternary subsemiring of S if $t_1 t_2 t_3 \in T$, for all $t_1, t_2, t_3 \in T$.

(ii) Let S be a ternary semiring. If there exists an element $0 \in S$ such that $0 + a = a$ and $0ab = a0b = ab0 = 0$ for all $a, b \in S$, then "0" is called the zero element or simply the zero of the ternary semiring S . In this case we say that S is a ternary semiring with zero.

(iii) Let A, B, C be three subsets of ternary semiring S . Then by ABC , we mean the set of all finite sums of the form $\sum a_i b_j c_k$ with $a_i \in A, b_j \in B, c_k \in C$.

(iv) An additive subsemigroup I of S is called a left (resp., right, and lateral) ideal of S if $s_1 s_2 i$ (resp. $i s_1 s_2, s_1 i s_2$) $\in I$, for all $s_1, s_2 \in S$ and $i \in I$. If I is both left and right ideal of S , then I is called a two-sided ideal of S . If I is a left, a right and a lateral ideal of S , then I is called an ideal of S . An ideal I of S is called a proper ideal if $I \neq S$.

Definition 2.4. (i) An additive subsemigroup $(Q, +)$ of a ternary semiring S is called a quasi-ideal of S if $QSS \cap (SQS + SSQSS) \cap SSQ \subseteq Q$.

(ii) An additive subsemigroup $(Q, +)$ of a ternary semiring S is called a bi-ideal of S if $QSQSQ \subseteq Q$.

Now, we review the concept of fuzzy sets [10], [13], [14]). Let X be a non-empty set. A map $\mu : X \rightarrow [0, 1]$ is called a fuzzy set in X , and the complement of a fuzzy set μ in X ,

denoted by $\bar{\mu}$, is the fuzzy set in X given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.

Let X and Y be two non-empty sets and $f : X \rightarrow Y$ a function, and let μ and ν be any fuzzy sets in X and Y respectively. The image of μ under f , denoted by $f(\mu)$, is a fuzzy set in Y defined by

$$f(\mu)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for each $y \in Y$. The preimage of ν under f , denoted by $f^{-1}(\nu)$, is a fuzzy set in X defined by $(f^{-1}(\nu))(x) = \nu(f(x))$ for each $x \in X$.

Definition 2.5. A fuzzy ideal of a semiring R is a function $A : R \rightarrow [0, 1]$ satisfying the following conditions:

- (i) A is a fuzzy subsemigroup of $(R, +)$; i.e., $A(x - y) \geq \min\{A(x), A(y)\}$,
- (ii) $A(xy) \geq \max\{A(x), A(y)\}$, for all $x, y \in R$

Definition 2.6. Let A and B be any two subsets of S . Then $A \cap B$, $A \cup B$, $A+B$ and $A \circ B$ are fuzzy subsets of S defined by

$$(A \cap B) = \min\{A(x), B(x)\}$$

$$(A \cup B) = \max\{A(x), B(x)\}$$

$$(A + B)(x) = \begin{cases} \sup\{\min\{A(y), A(z)\}, & \text{if } x = y + z, \\ 0 & \text{otherwise} \end{cases}$$

$$(A \circ B)(x) = \begin{cases} \sup\{\min\{A(y), A(z)\}, & \text{if } x = yz, \\ 0 & \text{otherwise} \end{cases}$$

For any $x \in S$ and $t \in (0, 1]$, define a fuzzy point x_t as

$$x_t(y) = \begin{cases} t, & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$$

If x_t is a fuzzy point and A is any fuzzy subset of S and $x_t \leq A$, then we write $x_t \in A$. Note that $x_t \in A$ if and only if $x \in A_t$ where A_t is a level subset of A . If x_r and y_s are fuzzy points, then $x_r y_s = (xy)_{\min\{r,s\}}$.

Definition 2.7. [6]. A fuzzy subset A of a fuzzy subsemigroup of S is called a fuzzy ternary subsemiring of S if:

- (i) $A(x - y) \geq \min\{A(x), A(y)\}$, for all $x, y \in S$
- (ii) $A(-x) = A(x)$
- (iii) $A(xyz) \geq \min\{A(x), A(y), A(z)\}$, for all $x, y, z \in S$.

Definition 2.8 [6]. A fuzzy subsemigroup A of a ternary semiring S called a fuzzy ideal of S if $A : S \rightarrow [0, 1]$ satisfying the following conditions:

- (i) $A(x - y) \geq \min\{A(x), A(y)\}$, for all $x, y \in S$
- (ii) $A(xyz) \geq A(z)$
- (iii) $A(xyz) \geq A(x)$ and
- (iv) $A(xyz) \geq A(y)$, for all $x, y, z \in S$

A fuzzy subset A with conditions (i) and (ii) is called a fuzzy left ideal of S . If A satisfies (i) and (iii), then it is called a fuzzy right ideal of S . Also if A satisfies (i) and (iv), then it

is called a fuzzy lateral ideal of S . A fuzzy ideal is a ternary semiring of S , if A is a fuzzy left, a fuzzy right and a fuzzy lateral ideal of S . It is clear that A is a fuzzy ideal of a ternary semiring S if and only if $A(xyz) \geq \max\{A(x), A(y), A(z)\}$ for all $x, y, z \in S$, and that every fuzzy left (right, lateral) ideal of S is a fuzzy ternary subsemiring of S .

Example 2.9 [6]. Let Z be a ring of integers and $S = \mathbf{Z}^-_0 \subset \mathbf{Z}$ be the set of all negative integers with zero. Then with the binary addition and ternary multiplication, $(\mathbf{Z}^-_0, +, \cdot)$ forms a ternary semiring S with zero. Define a fuzzy subset $A : \mathbf{Z}^- \rightarrow [0, 1]$, we have

$$A(x) = \begin{cases} 1, & \text{if } x \in \mathbf{Z}^-_0 \\ 0, & \text{otherwise} \end{cases}$$

Then A is a fuzzy ternary subsemiring of S .

Example 2.10 [6]. Consider the set integer module 5, non-positive integer $\mathbf{Z}^-_5 = \{0, -1, -2, -3, -4\}$ with the usual addition and ternary multiplication, we have

+	0	-1	-2	-3	-4	·	0	-1	-2	-3	-4
0	0	-1	-2	-3	-4	0	0	0	0	0	0
-1	-1	-2	-3	-4	0	-1	0	1	2	3	4
-2	-2	-3	-4	0	-1	-2	0	2	4	1	3
-3	-3	-4	0	-1	-2	-3	0	3	1	4	2
-4	-4	0	-1	-2	-3	-4	0	4	3	2	1
						·	0	1	2	3	4
						0	0	0	0	0	0
						-1	0	-1	-2	-3	-4
						-2	0	-2	-4	-1	-3
						-3	0	-3	-1	-4	-2
						-4	0	-4	-3	-2	-1

Clearly $(\mathbf{Z}^-_5, +, \cdot)$ is a ternary semiring. Let a fuzzy subset $A : \mathbf{Z}^-_5 \rightarrow [0, 1]$ be defined by $A(0) = t_0$ and $A(-1) = A(-2) = A(-3) = A(-4) = t_1$, where $t_0 \geq t_1$ and $t_0, t_1 \in [0, 1]$. Routine calculations show that A is a fuzzy ideal of \mathbf{Z}^-_5 .

Definition 2.11 [6] Let A be a fuzzy subset of ternary semiring S . We define

$$SAS + SSASS(z) = \begin{cases} \sup\{\min\{A(a), A(b)\}, & \text{if } z = x(a + xby)y, \\ 0, & \text{otherwise} \end{cases}$$

for all $x, y, a, b \in S$

III. FUZZY BI-IDEAL OF TERNARY SEMIRING

Definition 3.1. A fuzzy subsemigroup μ of a ternary semiring S is called a fuzzy quasi-ideal of S [6] if

$$(FQI1)\mu\mathbf{SS} \cap \mathbf{S}\mu\mathbf{S} \cap \mathbf{SS}\mu \leq \mu$$

$$(FQI2)\mu\mathbf{SS} \cap \mathbf{SS}\mu\mathbf{SS} \cap \mathbf{SS}\mu \leq \mu$$

i.e., $\mu(x) \geq \min\{(\mu\mathbf{SS})(x), (\mathbf{S}\mu\mathbf{S} + \mathbf{SS}\mu\mathbf{SS})(x), (\mathbf{SS}\mu)(x)\}$

To strengthen the above definition, we present the following example.

Example 3.2. Consider the ternary semiring $(\mathbf{Z}_5^-, +, \cdot)$ as defined in Example 2.10 in this paper. Let $A = \{0, -2, -3\}$. Then $SSA = \{-2, -3, -4\}$, $(SAS + SSASS) = \{0, -1, -2, -3\}$ and $ASS = \{-1, -2, -3\}$. Therefore $ASS \cap (SAS + SSASS) \cap SSA = \{-2, -3\} \subseteq A$. Hence A is a quasi-ideal of \mathbf{Z}_5^- . Define a fuzzy subset $A : \mathbf{Z}_5^- \rightarrow [0, 1]$ by $A(0) = A(-2) = A(-3) = 1$ and $A(-1) = A(-4) = 0$. Clearly A is a fuzzy quasi-ideal of \mathbf{Z}_5^- .

Definition 3.3. A fuzzy ternary subsemiring μ of S is called a fuzzy bi-ideal of S if

$$\mu \mathbf{S} \mu \mathbf{S} \mu \leq \mu$$

$$\text{i.e., } \mu(x s_1 y s_2 z) \geq \min\{\mu(x), \mu(y), \mu(z)\} \quad \forall x, y, z, w, v \in S$$

Example 3.4 Let $\mathbf{Z}^- = \mathbf{S}$ be the set of all negative integers. Then \mathbf{Z}^- is a ternary semiring under usual addition and ternary multiplication. Let $B = 2\mathbf{S}$. Thus $B\mathbf{S}B\mathbf{S}B = 2\mathbf{S}2\mathbf{S}2\mathbf{S}2\mathbf{S} = 6(\mathbf{S}\mathbf{S}\mathbf{S})\mathbf{S} = 6(\mathbf{S}\mathbf{S}\mathbf{S}) = 6\mathbf{S} \subseteq 2\mathbf{S} = B$. Hence B is a bi-ideal of \mathbf{Z}^- .

Define $\mu : S \rightarrow [0, 1]$ by

$$\mu(x) = \begin{cases} t, & \text{if } x \in 2\mathbf{S} \\ 0, & \text{otherwise} \end{cases}$$

For any $t \in [0, 1]$, $\mu_t = \{2\mathbf{S}\}$, since $\{2\mathbf{S}\}$ is a bi-ideal in \mathbf{Z}^- , μ_t is the bi-ideal in \mathbf{Z}^- for all t . Hence μ is a fuzzy bi-ideal of \mathbf{Z}^- .

Lemma 3.5. Let μ be a fuzzy subset of S . If μ is a fuzzy left ideal, fuzzy right ideal and lateral ideal of ternary semiring of S , then μ is a fuzzy quasi-ideal of S .

Proof: Let μ be a fuzzy left ideal, fuzzy right ideal and fuzzy lateral ideal of S . Let $x = a s_1 s_2 = s_1 (b_1 + s_1 c s_2) s_2 = s_1 s_2 d$ where $a, b, c, d, s_1, s_2 \in S$.

Consider $(\mu \mathbf{S} \mathbf{S} \cap (\mathbf{S} \mu \mathbf{S} + \mathbf{S} \mathbf{S} \mu \mathbf{S} \mathbf{S}) \cap \mathbf{S} \mathbf{S} \mu)(x)$

$$= \min\{(\mu \mathbf{S} \mathbf{S})(x), (\mathbf{S} \mu \mathbf{S} + \mathbf{S} \mathbf{S} \mu \mathbf{S} \mathbf{S})(x), (\mathbf{S} \mathbf{S} \mu)(x)\}$$

$$= \min\left\{ \sup_{x=as_1s_2} \{\mu(a)\}, \sup_{x=s_1(b+s_1cs_2)s_2} \{\mu(b), \mu(c)\}, \right.$$

$$\left. \sup_{x=s_1s_2d} \{\mu(d)\} \right\}$$

$$\leq \min\left\{ 1, \sup_{x=s_1(b+s_1cs_2)s_2} \{\mu(s_1(b+s_1cs_2)s_2)\}, 1 \right\}$$

(as μ is a fuzzy left, fuzzy right and fuzzy lateral ideal,

$$\mu\{s_1(b+s_1cs_2)s_2\} \geq \min\{\mu(b), \mu(c)\}$$

$= \mu(b)$ if $\mu(b) < \mu(c)$, ($= \mu(c)$ if $\mu(b) > \mu(c)$)) we get,

$$(\mu \mathbf{S} \mathbf{S} \cap (\mathbf{S} \mu \mathbf{S} + \mathbf{S} \mathbf{S} \mu \mathbf{S} \mathbf{S}) \cap \mathbf{S} \mathbf{S} \mu)(x) \leq \mu(x)$$

We remark that if x is not expressed as $x = a s_1 s_2 = s_1 (b_1 + s_1 c s_2) s_2 = s_1 s_2 d$, then

$$(\mu \mathbf{S} \mathbf{S} \cap (\mathbf{S} \mu \mathbf{S} + \mathbf{S} \mathbf{S} \mu \mathbf{S} \mathbf{S}) \cap \mathbf{S} \mathbf{S} \mu)(x) = 0 \leq \mu(x).$$

Thus,

$$(\mu \mathbf{S} \mathbf{S} \cap (\mathbf{S} \mu \mathbf{S} + \mathbf{S} \mathbf{S} \mu \mathbf{S} \mathbf{S}) \cap \mathbf{S} \mathbf{S} \mu)(x) \leq \mu(x).$$

Hence μ is a fuzzy quasi-ideal of S . ■

Lemma 3.6. For any non-empty subsets A, B and C of S ,

- (1) $f_A f_B f_C = f_{ABC}$
- (2) $f_A \cap f_B \cap f_C = f_{A \cap B \cap C}$
- (3) $f_A + f_B = f_{A+B}$

Proof: Proof is straight forward. ■

Lemma 3.7. Let Q be an additive subsemigroup of S .

- (1) Q is a quasi-ideal of S if and only if f_Q is a fuzzy quasi-ideal of S .
- (2) Q is a bi-ideal of S if and only if f_Q is a fuzzy bi-ideal of S .

Proof: Proof of (1) can seen in [8].

Proof of (2) Assume that Q is a bi-ideal of S . Then f_Q is a fuzzy ternary subsemiring of S .

$$f_Q f_S f_Q f_S f_Q \leq f_Q s_Q s_Q \leq f_Q$$

This means that f_Q is a fuzzy bi-ideal of S .

Conversely, let us assume that f_Q is a fuzzy bi-ideal of S . Let x be any element of $Q S Q S Q$. Then, we have

$$f_Q(x) \geq (f_Q f_S f_Q f_S f_Q)(x) = f_Q s_Q s_Q(x) = 1$$

Thus $x \in Q$ and $Q S Q S Q \subseteq Q$. Hence Q is a bi-ideal of S . ■

Lemma 3.8. Any fuzzy quasi-ideal of S is a fuzzy bi-ideal of S .

Proof: Let μ be any fuzzy quasi-ideal of S . Then, we have

$$\begin{aligned} \mu \mathbf{S} \mu \mathbf{S} \mu &\subseteq \mu(\mathbf{S}\mathbf{S}\mathbf{S})\mathbf{S} \subseteq \mu \mathbf{S} \mathbf{S}, \\ \mu \mathbf{S} \mu \mathbf{S} \mu &\subseteq \mathbf{S}(\mathbf{S}\mathbf{S}\mathbf{S})\mu \subseteq \mathbf{S} \mathbf{S} \mu, \\ \mu \mathbf{S} \mu \mathbf{S} \mu &\subseteq \mathbf{S} \mathbf{S} \mu \mathbf{S} \mathbf{S} \text{ and taking } \{0\} \subseteq \mathbf{S} \mu \mathbf{S} \end{aligned}$$

$$\text{so, } \mu \mathbf{S} \mu \mathbf{S} \mu \subseteq \mathbf{S} \mu \mathbf{S} + \mathbf{S} \mathbf{S} \mu \mathbf{S} \mathbf{S}$$

$$\text{we have, } \mu \mathbf{S} \mu \mathbf{S} \mu \subseteq \mu \mathbf{S} \mathbf{S} \cap (\mathbf{S} \mu \mathbf{S} + \mathbf{S} \mathbf{S} \mu \mathbf{S} \mathbf{S}) \cap \mathbf{S} \mathbf{S} \mu \subseteq \mu$$

Hence, μ is a fuzzy bi-ideal of S . ■

Remark 3.9. The converse of Lemma 3.8 does not hold, in general, that is, a fuzzy bi-ideal of a ternary semiring S may not be a fuzzy quasi-ideal of S .

Theorem 3.10. Let μ be a fuzzy subset of S . If μ is a fuzzy left, fuzzy right and lateral ideal of ternary semiring of S , then μ is a fuzzy bi-ideal of S .

Proof: As μ is fuzzy left, right, lateral ideal of S and Lemma 3.5, μ is a fuzzy quasi-ideal of S . Hence by Lemma 3.8, μ is a fuzzy bi-ideal of S . ■

Theorem 3.11.[6] Let μ be a fuzzy subset of S . Then μ is a fuzzy quasi-ideal of S , if and only if μ_t is a quasi-ideal of S , for all $t \in \text{Im}(\mu)$.

Theorem 3.12. Let μ be a fuzzy subset of S . Then μ is a fuzzy bi-ideal of S , if and only if μ_t is a bi-ideal of S , for all $t \in \text{Im}(\mu)$.

Proof: Let μ be a fuzzy bi-ideal of S . Let $t \in \text{Im}(\mu)$. Suppose $x, y, z \in S$ such that $x, y, z \in \mu_t$. Then

$$\mu(x) \geq t, \mu(y) \geq t, \mu(z) \geq t$$

, and

$$\min\{\mu(x), \mu(y), \mu(z)\} \geq t.$$

As μ is a fuzzy bi-ideal, $\mu(x-y) \geq t$ and thus $x-y \in \mu_t$. Let $u \in S$. Suppose $u \in \mu_t \mathbf{S} \mu_t \mathbf{S} \mu_t$. Then there exist $x, y, z \in \mu_t$ and $s_1, s_2 \in S$ such that $u = xs_1ys_2z$. Then,

$$\begin{aligned} (\mu \mathbf{S} \mu \mathbf{S} \mu)(u) &= \mu(xs_1ys_2z) \\ &\geq \min\{\mu(x), \mu(y), \mu(z)\} \geq \min\{t, t, t\} = t. \end{aligned}$$

Therefore, $(\mu \mathbf{S} \mu \mathbf{S} \mu)(u) \geq t$. As μ is a bi-ideal of S , $\mu(u) \geq t$ implies $u \in \mu_t$. Hence μ_t is a bi-ideal of S .

Conversely, let us assume that μ_A is a bi-ideal of S , $t \in \text{Im}(\mu)$. Let $p \in S$. Consider

$$(\mu \mathbf{S} \mu \mathbf{S} \mu)(p) = \sup_{p=xs_1ys_2z} \left\{ \min\{\mu(x), \mu(y), \mu(z)\} \right\}$$

Let $\mu(x) = t_1 < \mu(y) = t_2 < \mu(z) = t_3$. Then, $\mu_{t_1} \supseteq \mu_{t_2} \supseteq \mu_{t_3}$. Thus $x, y, z \in \mu_{t_1}$ and $p = xs_1ys_2z \in \mu_{t_1} \mathbf{S} \mu_{t_1} \mathbf{S} \mu_{t_1} \subseteq \mu_{t_1}$. This implies $\mu(p) \geq t_1$ and hence $\mu \mathbf{S} \mu \mathbf{S} \mu \leq \mu$. Therefore, μ is a fuzzy bi-ideal of S . ■

Definition 3.13 Let S and T be two ternary semirings. Let f be a mapping which maps from S into T . Then f is called a homomorphism of S into T if

- (i) $f(a + b) = f(a) + f(b)$ and
- (ii) $f(abc) = f(a)f(b)f(c)$ for all $a, b, c \in S$

Theorem 3.14. If λ is a fuzzy bi-ideal of a ternary semiring S and μ is a fuzzy ternary subsemiring of S , then $(\lambda \cap \mu)$ is a fuzzy bi-ideal of S .

Proof: Let λ be a fuzzy bi-ideal and μ be a fuzzy ternary subsemiring of S . Clearly $(\lambda \cap \mu)$ is a fuzzy ternary subsemiring of S . Next we prove that $(\lambda \cap \mu)$ is a fuzzy bi-ideal of ternary semiring S . Let $t \in S$ and $s_1, s_2, x, y, z \in S$ such that $t = xs_1ys_2z$.

Consider

$$\begin{aligned} &((\lambda \cap \mu) \mathbf{S} (\lambda \cap \mu) \mathbf{S} (\lambda \cap \mu))(t) \\ &= \sup_{t=xs_1ys_2z} \left\{ \min\{(\lambda \cap \mu)(x), \mathbf{S}(s_1), (\lambda \cap \mu)(y), \mathbf{S}(s_2), \right. \\ &\quad \left. (\lambda \cap \mu)(z)\} \right\} \\ &= \sup_{t=xs_1ys_2z} \left\{ \min\{(\lambda \cap \mu)(x), (\lambda \cap \mu)(y), (\lambda \cap \mu)(z)\} \right\} \end{aligned}$$

Let $\min\{(\lambda \cap \mu)(x), (\lambda \cap \mu)(y), (\lambda \cap \mu)(z)\} = t$. This implies that $(\lambda \cap \mu)(x) \geq t$, $(\lambda \cap \mu)(y) \geq t$ and $(\lambda \cap \mu)(z) \geq t$. Then $x, y, z \in (\lambda_t \cap \mu_t)$. As λ is the fuzzy bi-ideal and μ is the fuzzy ternary subsemiring, $(\lambda_t \cap \mu_t)$ is a bi-ideal of S . Hence, $xs_1ys_2z \in (\lambda_t \cap \mu_t)$. This implies

$$\begin{aligned} (\lambda \cap \mu)(xs_1ys_2z) &\geq t \\ &= \min\{(\lambda \cap \mu)(x), (\lambda \cap \mu)(y), (\lambda \cap \mu)(z)\}. \end{aligned}$$

Thus,

$$\begin{aligned} &\min\{(\lambda \cap \mu)(x), (\lambda \cap \mu)(y), (\lambda \cap \mu)(z)\} \\ &\leq (\lambda \cap \mu)(xs_1ys_2z) \end{aligned}$$

This shows that

$$\begin{aligned} &\sup_{t=xs_1ys_2z} \min\{(\lambda \cap \mu)(x), (\lambda \cap \mu)(y), (\lambda \cap \mu)(z)\} \\ &\leq (\lambda \cap \mu)(xs_1ys_2z) \end{aligned}$$

Thus, we have

$$((\lambda \cap \mu) \mathbf{S} (\lambda \cap \mu) \mathbf{S} (\lambda \cap \mu))(t) \leq (\lambda \cap \mu)(t)$$

Hence,

$$((\lambda \cap \mu) \mathbf{S} (\lambda \cap \mu) \mathbf{S} (\lambda \cap \mu)) \leq (\lambda \cap \mu)$$

and $(\lambda \cap \mu)$ is a fuzzy ideal of S . ■

IV. REGULAR TERNARY SEMIRING

A ternary semiring S is called regular if for every $a \in S$, there exists an x in S such that $axa = a$. **Lemma 4.1.** A ternary semiring S is regular if and only if

$$\mu * \gamma * \lambda = \mu \cap \gamma \cap \lambda$$

for every fuzzy right ideal μ , fuzzy left ideal λ and fuzzy lateral ideal γ of S .

Proof: Straight forward from Theorem 5.1 in [5] ■

Theorem 4.2. For a ternary semiring S , the following conditions are equivalent:

- (1) S is regular
- (2) $\mu = \mu * S * \mu * S * \mu$, for every fuzzy bi-ideal μ of S .
- (3) $\mu = \mu * S * \mu * S * \mu$, for every fuzzy quasi-ideal μ of S

Proof: (1)⇒(2) First assume that (1) holds. Let μ be any fuzzy bi-ideal of S , and a any element of S . Then since S is regular, there exists an element x in S such that $a = axa (= axaxa)$. Then we have

$$\begin{aligned} &(\mu * S * \mu * S * \mu)(a) \\ &= \sup_{a=\sum_{finite} x_i y_i z_i} \min \{ \mu(x_i), (S * \mu * S)(y_i), (\mu)(z_i) \} \\ &\geq \min\{ \mu(a), (S * \mu * S)(axa), (\mu)(a) \} \\ &= \min \left\{ \mu(a), \sup_{axa=\sum_{finite} p_i q_i r_i} \{ \min\{S(p_i), \mu(q_i), \right. \\ &\quad \left. S(r_i)\}, \mu(a) \} \right\} \\ &\geq \min \left\{ \mu(a), \min\{S(x), \mu(a), S(x)\}, \mu(a) \right\} \\ &= \min \left\{ \mu(a), \min\{1, \mu(a), 1\}, \mu(a) \right\} = \mu(a), \end{aligned}$$

and so $\mu * S * \mu * S * \mu \subseteq \mu$. Since μ is a fuzzy bi-ideal of S , the converse inclusion holds. Thus we have $\mu * S * \mu * S * \mu = \mu$

(2)⇒(3) Since any fuzzy quasi-ideal of S is a fuzzy bi-ideal of S by Lemma 3.8.

(3)⇒(1) Assume (3) holds. Let Q be any quasi-ideal of S , and a any element of Q . Then it follows from Lemma 3.7 (1)

that the characteristic function f_Q is a quasi-ideal of S . Then we have

$$f_{QSQSQ}(a) = (f_Q * f_S * f_Q * f_S * f_Q)(a) = f_Q(a) = 1$$

and so, $a \in QSQSQ$. Thus $Q \subseteq QSQSQ$. On the other hand, Q is a quasi-ideal of S

$$QSQSQ \subseteq (QSS \cap SQS \cap SSQ)$$

$$QSQSQ \subseteq (QSS \cap SSQSS \cap SSQ)$$

then,

$$QSQSQ \subseteq (QSS \cap (SQS + SSQSS) \cap SSQ) \subseteq Q$$

and so we have $QSQSQ = Q$ and hence, by [5, Theorem 3.4], S is a regular ternary semiring. ■

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