Linear Stability of Convection in a Viscoelastic Nanofluid Layer

Long Jye Sheu

Abstract—This paper presents a linear stability analysis of natural convection in a horizontal layer of a viscoelastic nanofluid. The Oldroyd B model was utilized to describe the rheological behavior of a viscoelastic nanofluid. The model used for the nanofluid incorporated the effects of Brownian motion and thermophoresis. The onset criterion for stationary and oscillatory convection was derived analytically. The effects of the Deborah number, retardation parameters, concentration Rayleigh number, Prandtl number, and Lewis number on the stability of the system were investigated. Results indicated that there was competition among the processes of thermophoresis, Brownian diffusion, and viscoelasticity which caused oscillatory rather than stationary convection to occur. Oscillatory instability is possible with both bottom- and top-heavy nanoparticle distributions. Regimes of stationary and oscillatory convection for various parameters were derived and are discussed in detail.

Keywords—instability, viscoelastic, nanofluids, oscillatory, Brownian, thermophoresis

I. INTRODUCTION

THE term "nanofluid" was coined by Choi [1] to refer to a fluid containing a dispersion of nanoparticles. Characteristic features of nanofluids are the formation of very stable colloidal systems with very little settling and anomalous enhancement of the thermal conductivity compared to the base fluid [2, 3]. Buongiorno [4] focused on heat transfer enhancement of nanofluids in convective situations. He concluded that in the absence of turbulent effects, only Brownian diffusion and thermophoresis are important slip mechanisms in nanofluids. Based on this finding, Buongiorno [4] wrote down conservation equations of a non-homogeneous equilibrium model of nanofluids for mass, momentum, and heat transport. The onset of convection of the Benard problem of pure nanofluids and nanofluid-saturated layers based on Buongiorno's model has attracted much interest in the past 3 years. The onset of a nanofluid layer was studied by Tzou [5,6] and Nield and Kuznetsov [7]. Convection of non-Newtonian fluids in a porous medium is of considerable importance in several applied fields such as oil recovery, food processing, and the spread of contaminants in the environment, and in various processes in the chemical and materials industry. The onset of thermal convection in a viscoelastic fluid was studied by many authors [8-14]. Since elastic behavior is inherent in non-Newtonian fluids, oscillatory instability can set in before a stationary mode is achieved. It is commonly believed that oscillatory convection is not possible in viscoelastic fluids under realistic experimental conditions [13]. However, experiments with a DNA suspension showed that convection patterns take the form of spatially localized standing and traveling waves which exhibit small amplitudes and extremely long oscillation periods [15]. Those experiments triggered new interest in convection by applying binary aspects to viscoelastic fluids. Rayleigh-Benard convection in binary viscoelastic fluids was studied by some researchers [16-21]. Results show that there is competition among the processes of thermal diffusion, solute diffusion, and viscoelasticity that causes convection to set in through an oscillatory rather than a stationary mode. Results of convection instability in nanofluids indicate that both Brownian diffusion and thermophoresis give rise to cross-diffusion terms that are in some ways analogous to the familiar Soret and Dufour cross-diffusion terms that arise in binary fluids [7,8]. To the author's knowledge, there is only one study on convection instability of non-Newtonian nanofluids. Nield [22] briefly discussed convection instability in a porous medium saturated by a non-Newtonian nanofluid of the power law type. We are unaware of any publication discussing the effect of fluid viscoelasticity on the oscillatory instability of nanofluids. In this present work, the oscillatory instability of a viscoelastic nanofluid layer was studied. Our objective in the present work was to study how the onset criterion for oscillatory convection is affected by interactions among Brownian diffusion, thermophoretic diffusion, and viscoelasticity, and how is it related to the oscillatory instability of a binary viscoelastic base fluid. The Oldroyd-B fluid model was employed to describe the rheological behavior of a viscoelastic nanofluid. In order to assess the effects of viscoelastic parameters through analytical expressions, free-free boundary conditions were used in the first instance.

II. MATHEMATICAL FORMULATION

Conservation equations for a viscoelastic nanofluid layer;

We consider an infinite horizontal layer of a viscoelastic nanofluid subjected to a vertical temperature gradient, confined between the plane z=0 and z=d. According to the work of Buongiorno [4], the momentum equation for a nanofluid is of the same form as that of a pure fluid. The viscoelastic fluid of the Oldroyd type was used to model the momentum equation. We assumed that when the Boussinesq approximation is adopted, the basic governing equations are:

$$\nabla \cdot \mathbf{q}^* = 0 , \qquad (1)$$

$$\left(1+\bar{\lambda}_1\frac{\partial}{\partial t^*}\right)\left(\rho_0\left(\frac{\partial \mathbf{q}^*}{\partial t^*}+\mathbf{q}^*\cdot\nabla\mathbf{q}^*\right)+\nabla p^*-\rho\mathbf{g}\right)=\mu\left(1+\bar{\lambda}_2\frac{\partial}{\partial t^*}\right)\nabla^2\mathbf{q}^* , (2)$$

$$(\rho c)_f\left(\frac{\partial T^*}{\partial t^*}+\mathbf{q}^*\cdot\nabla T^*\right)=k\nabla^2 T^*+(\rho c)_p[D_B\nabla\phi^*\cdot\nabla T^*+(D_T/T_u^*)\nabla T^*\cdot\nabla T^*] (3)$$

$$\frac{\partial\phi^*}{\partial t^*}+\mathbf{q}^*\cdot\nabla\phi^*=D_B\nabla^2\phi^*+\frac{D_T}{T_u^*}\nabla T^* , \qquad (4)$$

$$\rho\mathbf{g}\cong\{\phi\rho_p+(1-\phi)\rho_{f0}[1-\beta(T-T_u)]\}\mathbf{g} ; \qquad (5)$$

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where $\bar{\lambda}_1$ is the stress relaxation time, $\bar{\lambda}_2$ is the strain retardation time, $\mathbf{q}^*$ is the velocity vector, p^* is the hydrostatic pressure, $\mathbf{g}$ is the gravitation acceleration vector, D_B is the Brownian diffusion coefficient, D_T is the thermophoretic diffusion coefficient, and ϕ is the nanoparticle volume fraction in the nanofluid. Notice that $\bar{\lambda}_1 > \bar{\lambda}_2$ and that when $\bar{\lambda}_2 = 0$, we recover the Maxwell viscoelastic model; while for $\bar{\lambda}_1 = \bar{\lambda}_2$, the model reduces to that of a Newtonian nanofluid. It should be noted that in writing down Eqs. (2) and (3), we assumed that the spatial variations in μ and k are negligible.

We assumed that the temperature and volumetric fraction of the nanoparticles are constant at the boundaries. Thus the boundary conditions are:

$$w^* = 0, \frac{\partial^2 w^*}{\partial z^{*2}} = 0, T^* = T_l^*, \phi^* = \phi_0^* \quad at \ z^* = 0 \tag{6}$$

$$w^* = 0, \frac{\partial^2 w^*}{\partial z^{*2}} = 0, T^* = T_u^*, \phi^* = \phi_1^* \quad at \ z^* = d. \tag{7}$$

A derivation of the hydrodynamic boundary conditions can be found in [23], for example. The validity of the boundary conditions of nanoparticle volume fractions (6) and (7) are discussed in [7].

We introduce the dimensionless variables as follows:

$$(x, y, z) = \left(x^*, y^*, z^*\right)/d, t = t^* \alpha_f / d^2,$$
$$\mathbf{q} = \mathbf{q}^* d / \alpha_f, p = p^* d^2 / \mu \alpha_f, \tag{8}$$
$$\phi = (\phi^* - \phi_0^*)/(\phi_1^* - \phi_0^*), T = (T^* - T_u^*)/(T_l^* - T_u^*)$$

Then Eqs. (1)~(4) take the form:

$$\nabla \ \mathbf{q} = 0, \tag{9}$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\left(\frac{1}{\Pr}\left(\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \ \nabla \mathbf{q}\right) + \nabla p + R_M \vec{k} - Ra T \vec{k} + R_N \phi \vec{k}\right)$$
$$= \left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\nabla^2 \mathbf{q} \tag{10}$$

$$\frac{\partial T}{\partial t} + \mathbf{q} \ \nabla T = \nabla^2 T + \frac{N_B}{Le}\nabla \phi \ \nabla T + \frac{N_A N_B}{Le}\nabla T \ \nabla T, \tag{11}$$

$$\frac{\partial \phi}{\partial t} + \mathbf{q} \ \nabla \phi = \frac{1}{Le}\nabla^2 \phi + \frac{N_A}{Le}\nabla^2 T, \tag{12}$$

with the dimensionless boundary conditions of

$$w = 0, \frac{\partial^2 w}{\partial z^2} = 0, T = 1, \phi = 0 \quad at \ z = 0 \tag{13}$$

$$w = 0, \frac{\partial^2 w}{\partial z^2} = 0, T = 0, \phi = 1 \quad at \ z = 1, \tag{14}$$

where the nondimensional parameters are

Darcy-Prandtl number, $\quad \Pr = \dfrac{\mu}{\rho \alpha_f}$, $\tag{15}$

Lewis number, $\quad Le = \dfrac{\alpha_f}{D_B}$, $\tag{16}$

thermal Darcy-Rayleigh number, $Ra = \dfrac{\rho_{f0} g \beta d^3 (T_l^* - T_u^*)}{\mu \alpha_f}$, $\tag{17}$

basic density Rayleigh number, $R_M = \dfrac{[\rho_p \phi_0 + \rho_{f0}(1 - \phi_0)]g d^3}{\mu \alpha_f}$, $\tag{18}$

concentration Rayleigh number, $R_N = \dfrac{(\rho_p - \rho_{f0})(\phi_1^* - \phi_0^*)g d^3}{\mu \alpha_f}$, $\tag{19}$

Deborah number, $\quad \lambda_1 = \dfrac{\bar{\lambda}_1 \alpha_f}{d^2}$, $\tag{20}$

retardation parameter, $\lambda_2 = \dfrac{\bar{\lambda}_2 \alpha_f}{d^2}$, $\tag{21}$

modified diffusivity ratio, $N_A = \dfrac{D_T(T_l^* - T_u^*)}{D_B T_u(\phi_1^* - \phi_0^*)}$, and $\tag{22}$

modified particle-density increment; $N_B = \dfrac{(\rho c)_p}{(\rho c)_f}(\phi_1^* - \phi_0^*)$. $\tag{23}$

In the spirit of the Oberbeck-Boussinesq approximation, Eq. (11) was linearized by neglecting a term proportional to the product of ϕ and T. This is valid in the case of small temperature gradients in a dilute suspension of nanoparticles.

Basic solutions

The basic state was assumed to be quiescent and is given by

$$u = v = w = 0, T = T_b(z), \phi = \phi_b(z), p = p_b(z). \tag{24}$$

The basic states of the temperature and nanoparticle volume fraction satisfy the equations

$$\frac{d^2 T_b}{dz^2} + \frac{N_B}{Le}\frac{d\phi_b}{dz}\frac{dT_b}{dz} + \frac{N_A N_B}{Le}\left(\frac{dT_b}{dz}\right)^2 = 0 \ \text{ and} \tag{25}$$

$$\frac{d^2 \phi_b}{dz^2} + N_A \frac{d^2 T_b}{dz^2} = 0. \tag{26}$$

Using boundary conditions in Eqs. (13) and (14), Eq. (26) can be integrated to give

$$\phi_b = -N_A T_b + (1 - N_A)z + N_A. \tag{27}$$

Substituting this into Eq. (25) gives

$$\frac{d^2 T_b}{dz^2} + \frac{(1 - N_A)N_B}{Le}\frac{dT_b}{dz} = 0. \tag{28}$$

The solution of Eq. (28) satisfying boundary conditions Eq. (13) and (14) is

$$T_b = \frac{1 - e^{-(1-N_A)N_B(1-z)/Le}}{1 - e^{-(1-N_A)N_B/Le}}. \tag{29}$$

The basic solution of ϕ_b can easily be obtained by substituting this into Eq. (26). According to Buongiorno [4], Nield and Kuznetsov [7,8] discussed the exponents in Eq. (29) and found that they are small. Hence, to a good approximation, one has

$$T_b = 1 - z \tag{30}$$

and so

$$\phi_b = z. \tag{31}$$

Perturbed state

To study the stability of the system, we superimpose infinitesimal perturbations onto the basic state, which are of the forms

$$\mathbf{q} = 0 + \mathbf{q}', T = T_b + T', \phi = \phi_b + \phi', p = p_b + p'. \quad (32)$$

Using Eq. (32) in Eqs. (9)~(12) and the basic state solutions, and neglecting the nonlinear terms, we obtain the linearized equations governing infinitesimal perturbations in the form:

$$\nabla \mathbf{q}' = 0, \quad (33)$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\left(\frac{1}{\Pr}\frac{\partial \mathbf{q}'}{\partial t} + \nabla p' - RaT'\vec{k} + R_N \phi'\vec{k}\right) = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\nabla^2 \mathbf{q}' \quad (34)$$

$$\frac{\partial T'}{\partial t} - w' = \nabla^2 T' + \frac{N_B}{Le}\left(\frac{\partial \phi'}{\partial z} - \frac{\partial T'}{\partial z}\right) - \frac{2N_A N_B}{Le}\frac{\partial T'}{\partial z}, \text{ and} \quad (35)$$

$$\frac{\partial \phi'}{\partial t'} + w' = \frac{1}{Le}\nabla^2 \phi' + \frac{N_A}{Le}\nabla^2 T'. \quad (36)$$

Eliminating p' by operating a curl twice on it, one has

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\left[\frac{1}{\Pr}\frac{\partial}{\partial t}(\nabla^2 w') - Ra\nabla_1^2 T' + R_N \nabla_1^2 \phi'\right]$$
$$= \left(1 + \lambda_2 \frac{\partial}{\partial t}\right)\nabla^4 w' \quad ; \quad (37)$$

where $\nabla_1^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplacian operator on the horizontal plane. The boundary conditions for the infinitesimal perturbations are given by

$$w' = 0, \frac{\partial^2 w'}{\partial z^2} = 0, T' = 1, \phi' = 0 \quad at \ z = 0 \text{ and} \quad (38)$$

$$w' = 0, \frac{\partial^2 w'}{\partial z^2} = 0, T' = 0, \phi' = 1 \quad at \ z = 1. \quad (39)$$

III. LINEAR STABILITY ANALYSIS

The differential Eqs. (35)~(37) and boundary conditions (38) and (39) constitute a linear boundary-value problem that can be solved using the method of normal modes in the form

$$\begin{pmatrix} w' \\ T' \\ \phi' \end{pmatrix} = \begin{pmatrix} W(z) \\ \Theta(z) \\ \Phi(z) \end{pmatrix} \exp[i(lx + my) + \omega t]; \quad (40)$$

where l and m are the wavenumber in the x- and y-directions and ω is the growth rate. Substituting Eq. (40) into Eqs. 35)~(37) one has

$$(1 + \lambda_1 \omega)\left[\frac{\omega}{\Pr}(D^2 - a^2)W + Ra a^2 \Theta - R_n a^2 \Phi\right],$$
$$= (1 + \lambda_2 \omega)(D^2 - a^2)^2 W \quad (41)$$

$$W + \left(D^2 - \frac{N_B}{Le}D + \frac{2N_A N_B}{Le}D - a^2 - \omega\right)\Theta + \frac{N_B}{Le}D\Phi = 0, \text{ and} \quad (42)$$

$$W - \frac{N_A}{Le}(D^2 - a^2)\Theta - \left(\frac{1}{Le}(D^2 - a^2) - \omega\right)\Phi = 0, \quad (43)$$

with boundary conditions

$$W = 0, D^2 W = 0, \Theta = 0, \Phi = 0 \quad at \ z = 0 \text{ and} \quad (44)$$

$$W = 0, D^2 W = 0, \Theta = 0, \Phi = 0 \quad at \ z = 1; \quad (45)$$

where $a^2 = l^2 + m^2$ is the horizontal wavenumber and $D = d/dz$.

We assume the solution to W, Θ and Φ is in the form

$$W = W_0 \sin \pi z, \quad \Theta = \Theta_0 \sin \pi z, \quad \Phi = \Phi_0 \sin \pi z \quad (46)$$

which satisfy boundary conditions (44) and (45). Substituting Eq. (46) into Eqs. (41)~(43), multiplying the resulting equations by $\sin \pi z$, and integrating each equation from $z = 0$ to $z = 1$ and performing some integration by parts, one obtains the following matrix equation:

$$\begin{bmatrix} M_{11} & -Ra a^2 & R_N a^2 \\ -1 & \delta^2 + \omega & 0 \\ 1 & N_A Le^{-1}\delta^2 & Le^{-1}\delta^2 + \omega \end{bmatrix}\begin{bmatrix} W_0 \\ \Theta_0 \\ \Phi_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad (47)$$

where $\delta^2 = \pi^2 + a^2$ is the total wavenumber, and

$$M_{11} = \{\omega/\Pr + [(1 + \lambda_2 \omega)/(1 + \lambda_1 \omega)]\delta^2\}\delta^2. \quad (48)$$

The nontrivial solution of the above matrix requires that

$$Ra = (\omega + \delta^2)\frac{\delta^2}{a^2}\left(\frac{\omega}{\Pr} + \frac{(1 + \lambda_2 \omega)}{(1 + \lambda_1 \omega)}\delta^2\right)$$
$$- \frac{\omega + Le^{-1}\delta^2(N_A + Le)}{\omega + Le^{-1}\delta^2}R_N. \quad (49)$$

Setting $\omega = i\omega_i$ in Eq. (49) and clearing the complex quantities from the denominator, one obtains

$$Ra = \Delta_1 + i\omega_i \Delta_2; \quad (50)$$

where

$$\Delta_1 = \frac{\delta^2}{a^2}\left[\delta^4 \frac{1 + \lambda_1 \lambda_2 \omega_i^2}{1 + \lambda_1^2 \omega_i^2} + \delta^2 \frac{\omega_i^2(\lambda_1 - \lambda_2)}{1 + \lambda_1^2 \omega_i^2} - \frac{\omega_i^2}{\Pr}\right]$$
$$- \frac{\omega_i^2 + Le^{-1}\delta^4(1 + N_A Le^{-1})}{\omega_i^2 + Le^{-2}\delta^4}R_N \quad \text{and} \quad (51)$$

$$\Delta_2 = \frac{\delta^4}{a^2}\left(\frac{1 + \lambda_1 \lambda_2 \omega_i^2}{1 + \lambda_1^2 \omega_i^2} + \frac{1}{\Pr} + \frac{\delta^2(\lambda_2 - \lambda_1)}{1 + \lambda_1^2 \omega_i^2}\right)$$
$$+ \frac{\delta^2[1 + N_A Le^{-1} - Le^{-1}]}{\omega_i^2 + Le^{-2}\delta^4}R_N. \quad (52)$$

Since Ra is a physical quantity, it must be a real value. Hence, it follows from Eq. (50) that either $\omega_i = 0$ (exchange stability, steady onset) or $\Delta_2 = 0$ ($\omega_i \neq 0$ overstability, oscillatory onset).

Stationary convection

Steady onset corresponds to $\omega_i = 0$, and steady convection occurs at

$$Ra^S = \frac{\delta^6}{a^2} - (N_A + Le)R_N. \quad (53)$$

The critical wave number obtained by minimizing Ra with respect to a, i.e., satisfying $\partial Ra/\partial a = 0$, is

$$a_c = \pi/\sqrt{2}. \quad (54)$$

The corresponding critical thermal Rayleigh number for steady onset is

$$Ra_C^S = \frac{27\pi^4}{4} - (N_A + Le)R_N . \qquad (55)$$

It should be noted that the expression for Ra_C^S is independent of viscoelastic parameters and coincides with those of a Newtonian nanofluid problem.

Oscillatory convection

For oscillatory onset, $\Delta_2 = 0$ and $\omega_i \neq 0$, which gives a dispersion relation of the form

$$a_1(\omega_i^2)^2 + a_2(\omega_i^2) + a_3 = 0 ; \qquad (56)$$

where

$$a_1 = \lambda_1(\lambda_2 \, \text{Pr} + \lambda_1)\delta^2 , \qquad (57)$$

$$a_2 = \delta^2[\text{Pr}(1 + \lambda_1\lambda_2 Le^{-2}\delta^4) + (1 + \lambda_1^2 Le^{-2}\delta^4) + \text{Pr}(\lambda_2 - \lambda_1)\delta^2] \\ + a^2 \text{Pr} \lambda_1^2(1 + N_A Le^{-1} - Le^{-1})R_N \qquad , \text{ and} (58)$$

$$a_3 = \delta^6(1 + \text{Pr})Le^{-2} + \text{Pr}(\lambda_2 - \lambda_1)Le^{-2}\delta^8 \\ + a^2 \text{Pr}(1 + N_A Le^{-1} - Le^{-1})R_N \qquad . \qquad (59)$$

Then Eq. (53) with $\Delta_2 = 0$ gives

$$Ra^{OSC} = \frac{\delta^2}{a^2}\left[\delta^4 \frac{1 + \lambda_1\lambda_2\omega_i^2}{1 + \lambda_1^2\omega_i^2} + \delta^2 \frac{\omega_i^2(\lambda_1 - \lambda_2)}{1 + \lambda_1^2\omega_i^2} - \frac{\omega_i^2}{\text{Pr}}\right] \\ - \frac{\omega_i^2 + Le^{-1}\delta^4(1 + N_A Le^{-1})}{\omega_i^2 + Le^{-2}\delta^4}R_N \qquad . \qquad (60)$$

IV. RESULT

Expression of the stationary critical thermal Rayleigh number is given by Eq. (55). The expression of the oscillatory thermal Rayleigh number is obtained analytically using Eq. (60). It should be noted that parameter N_B affects neither stationary nor oscillatory instability. The effect of N_B on instability in Eq. (42) is the first derivative of the temperature mode which is cancelled due to integration of orthogonal functions. As a result, the contributions of Brownian motion and thermophoresis in the thermal energy equation of instability disappear. Rather, Brownian motion and thermophoresis directly enter the equation expressing the conservation of nanoparticles to produce their effects. In this way, the temperature and particle density are coupled in a particular way in which the instability is almost purely a phenomenon due to buoyancy coupled with the conservation of nanoparticle motion. It is worth discussing the limiting case, $N_A = 0$, which indicates the absence of a thermophoretic effect. At this limit, the problem reduces to double diffusive convection instability. The resultant equations are similar to those of double diffusive convection instability in viscoelastic fluid layers [21] and viscoelastic fluid-saturated porous media [24, 25]. A careful comparison between Eqs. (53)~(60) and the corresponding equations in the work of Malashetty and Swamy [21], which examined the onset of double diffusive convection in a viscoelastic base fluid, shows that they are exactly the same if we let $N_A = 0$ and $R_N = -Ra_S$, where Ra_S is the solute Rayleigh number defined in their work. The stationary critical thermal Rayleigh number was also found to be independent of the viscoelastic parameters and to be identical to those of a Newtonian nanofluid problem. The

critical thermal Rayleigh number for oscillatory convection can be derived by numerically minimizing Eq. (60) with respect to the wavenumber, after substituting various values of physical parameters for ω_i^2 of Eq. (56) to determine their effects on the onset of oscillatory convection. According to Buongiorno [4] and Nield and Kuznetsov [7], for most nanofluids investigated, Lewis number, Le, is large of the order $10^2 - 10^3$, while the modified diffusivity ratio, N_A, is no greater than about 10. In the following, we consider instability by taking values of Le and N_A within these ranges.
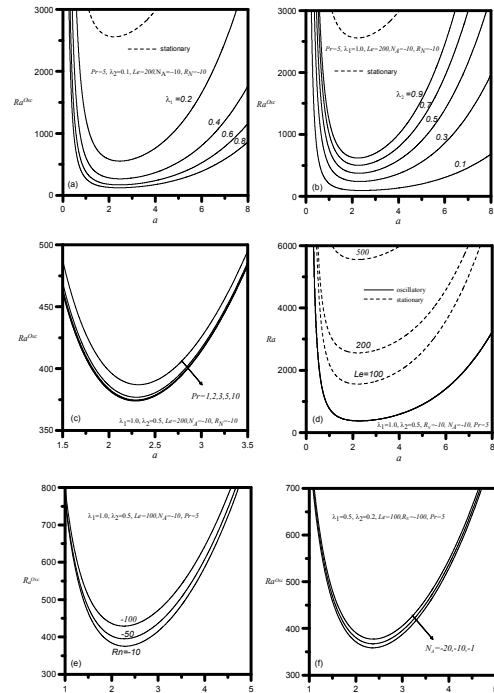

Fig. 1 Neutral curves for different values of (a) the Deborah number (λ_1), (b) retardation parameter (λ_2), (c) Prandtl number (Pr), (d) Lewis number (Le), (e) concentration Rayleigh number (R_N), and (f) modified diffusivity ratio (N_A).

Figure 1 shows the neutral curves for different values of the Deborah number, stress retardation parameter, Prandtl number, Lewis number, concentration Rayleigh number, and modified diffusivity ratio. The effect of the Deborah number, λ_1, is shown in Fig. 1a. The Deborah number is used in rheology to characterize how fluid a material is. It physically represents the ratio of the relaxation time to the thermal diffusion time. The smaller the Deborah number is, the more fluid the material appears. A large Deborah number means that the fluid cannot keep up with the deformation rate. In other words, the material will look more elastic than viscous, and responds more like a Hookian spring. Figure 1a shows that the oscillatory thermal Rayleigh number decreases with an increase in the Deborah number which indicates that the effect of the Deborah number is to advance the onset of convection in a viscoelastic nanofluid layer. Figure 1b shows the effect of the retardation parameter, λ_2, on the neutral curves. It was found that an increase in the value of the retardation parameter increases the minimum

oscillatory Rayleigh number, indicating that it delays the onset of convection in a viscoelastic nanofluid layer. The effect of the Prandtl number on the oscillatory thermal Rayleigh number is shown in Fig. 1c. One can see that the oscillatory thermal Rayleigh number decreases with the increase in the Prandtl number, indicating that the Prandtl number advances the oscillatory onset of viscoelastic nanofluids. In Fig. 1d, the effect of the Lewis number on neutral curves is shown. It should be noted that the effect of the Lewis number on the oscillatory thermal Rayleigh number is very slight, while its effect on the stationary mode is substantial. The effect of the concentration Rayleigh number is shown in Fig. 1e. It can be seen in Fig. 1e that the oscillatory thermal Rayleigh number decreases with an increase in R_N, which means that R_N enhances oscillatory convection. Figure 1f depicts the effect of a modified diffusivity ratio, N_A, on the neutral curves. The modified diffusivity ratio represents the ratio of thermophoresis to Brownian diffusion of nanoparticles. It can be seen from Fig. 1f that the critical oscillatory thermal Rayleigh number decreases with an increase in N_A, indicating that N_A advances oscillatory onset.
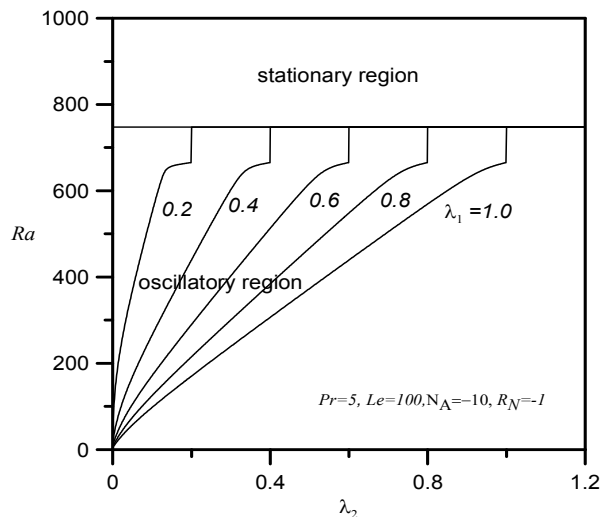

Fig. 2 Variations in the critical thermal Rayleigh number with the strain retardation parameter (λ_2) for different values of the Deborah number (λ_1) and negative values of R_N and N_A

Variations in the critical thermal Rayleigh number with the strain retardation parameter for various parameters are shown in Figures 2~7. Figure 2 shows the effect of the Deborah number (λ_1) on the critical thermal Rayleigh number. It can be seen in this figure that the oscillatory mode sets in before the stationary mode as $\lambda_2 \leq \lambda_1$ for negative values of R_N and N_A at the values given for other parameters. Note that a negative value of R_N represents a bottom-heavy nanoparticle distribution. Note also that for a Oldroyd B type of viscoelastic fluid, values of the relaxation time and retardation time satisfy $\lambda_2 \leq \lambda_1$. As a result, for a typical viscoelastic nanofluid (with a large Lewis number), the oscillatory mode always sets in before the stationary mode for a bottom-heavy nanoparticle distribution at the given parameters values. Figure 3 shows the effect of the Deborah number on the critical thermal Rayleigh number for positive values of R_N and N_A. Note that a positive value of R_N

represents a top-heavy nanoparticle distribution. For each λ_1, it is indicated in Fig. 3 that a critical strain retardation parameter (say λ_2^C) exists which divides the boundary of regimes between oscillatory and stationary convection. Initially convection begins in the form of the oscillatory mode. As the value of λ_2 reaches λ_2^C, convection ceases to be oscillatory, and stationary convection occurs as the first bifurcation. The value of λ_2^C for each case depends on λ_1 and the other parameters. As shown in Eq. (55), the critical stationary thermal Rayleigh number, Ra_C^S, is independent of the viscoelasticity parameters. Hence, as the value of λ_2 exceeds λ_2^C, the curve is horizontal, and the critical thermal Rayleigh number is a constant value, which depends on N_A, Le, and R_N.
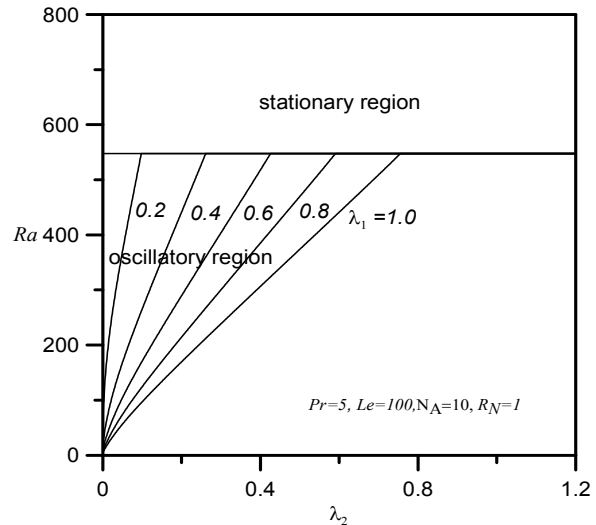

Fig. 3 Variations in the critical thermal Rayleigh number with the strain retardation parameter (λ_2) for different values of the Deborah number (λ_1) and positive values of R_N and N_A
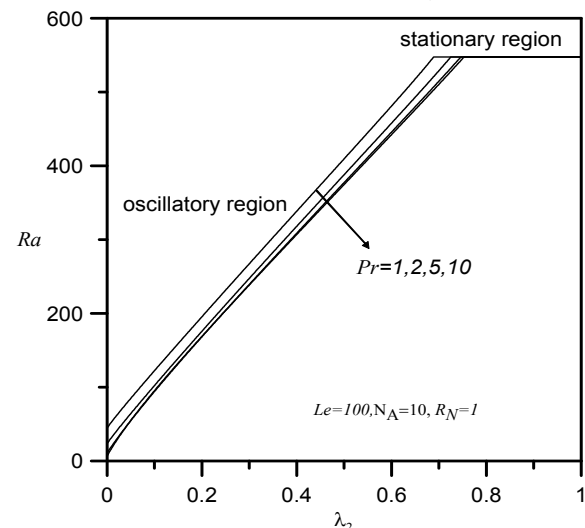

Fig. 4 Variations in the critical thermal Rayleigh number with the strain retardation parameter (λ_2) for different values of the Prandtl number
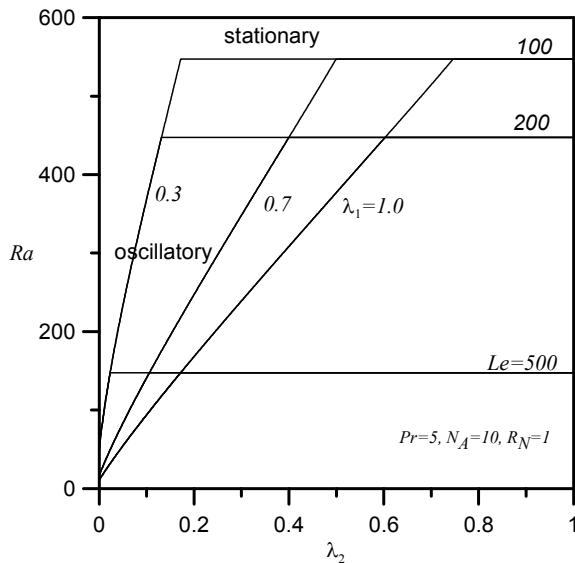
Fig. 5 Variations in the critical thermal Rayleigh number with the strain retardation parameter (λ_2) for different values of the Lewis number (Le) and Deborah number (λ_1)

Figure 4 shows the effect of the Prandtl number on the critical thermal Rayleigh number and λ_2^C for fixed values of other parameters. It was found that λ_2^C increases with the Prandtl number, indicating an increase in the region of the oscillatory mode. The critical oscillatory thermal Rayleigh number, Ra_C^{Osc}, decreases with an increase in the Prandtl number, revealing that the oscillatory mode is more unstable as the Prandtl number increases. As discussed in Fig. 1d, the effect of the Lewis number on the critical oscillatory thermal Rayleigh number is very slight. However, the Lewis number does affect the critical stationary thermal Rayleigh number and λ_2^C. The effects of the Lewis number on the critical thermal Rayleigh number and λ_2^C for positive R_N and various values of λ_1 are shown in Figure 5. An increase in the Lewis number decreases λ_2^C and contracts the region of the oscillatory mode. In the oscillatory region, the critical thermal Rayleigh number depends only on λ_1 and λ_2, as the other parameters are specified. However, in the stationary region, the critical Rayleigh number decreases with an increase in the Lewis number when values of R_N are positive.
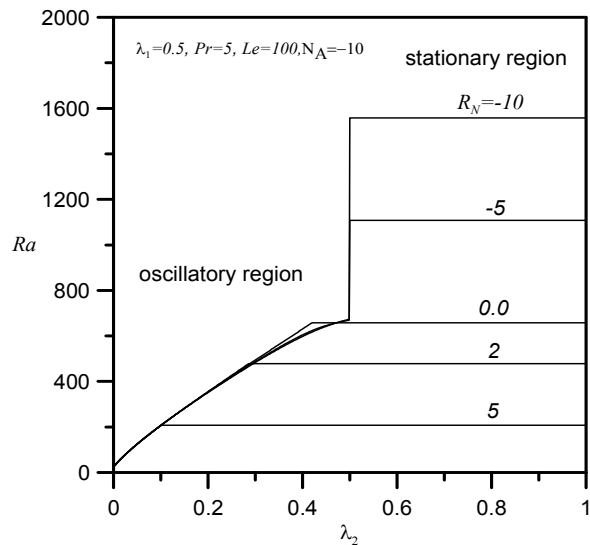

Fig. 6 Variations in the critical thermal Rayleigh number with the strain retardation parameter (λ_2) for different values of the concentration Rayleigh number (R_N)

Figure 6 shows the effect of the concentration Rayleigh number, R_N, on the critical thermal Rayleigh number for fixed values of parameters. The critical stationary thermal Rayleigh number, Ra_C^S, decreases with an increase in R_N. The effect of decreasing R_N is to stabilize the stationary mode. Although R_N significantly affects the stationary mode, its influence on the oscillatory mode is very slight for a typical viscoelastic nanofluid with a large Lewis number. However, λ_2^C decreases with an increase in the R_N, which implies that an increasing R_N will reduce the region of oscillatory instability. Note that a negative value of the R_N indicates a bottom-heavy nanoparticle distribution. For Newtonian nanofluids, it was found that oscillatory instability is possible only in the case of a bottom-heavy nanoparticle distribution. In Figure 6, cases with positive values of R_N correspond to top-heavy nanoparticle distributions. It can clearly be seen that there are regions of oscillatory instability in these cases. Figure 6 reveals that oscillatory instabilities are possible in both top- and bottom-heavy nanoparticle distributions of viscoelastic nanofluids.
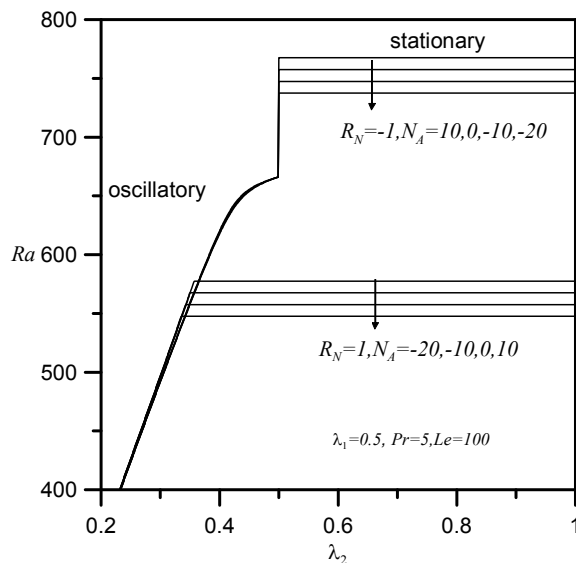
Fig. 7 Variations in the critical thermal Rayleigh number with the strain retardation parameter (λ_2) for different values of the modified diffusivity ratio (N_A)

Figure 7 displays the effect of a modified diffusivity ratio, N_A, on the critical thermal Rayleigh number for both top- and bottom-heavy nanoparticle distributions. It was observed that with increasing values of N_A, the critical stationary thermal Rayleigh number, Ra_C^S, decreases for top-heavy distributions, while it increases for bottom-heavy distributions. This indicates that the effect of increasing N_A on the stationary mode is to stabilize bottom-heavy cases, while destabilizing top-heavy cases. It can also be seen in Fig. 7 that for a typical viscoelastic nanofluid with a large Lewis number, the effect of N_A on the oscillatory mode is very small. For top-heavy distributions, λ_2^C decreases with an increase in N_A.

V. CONCLUSIONS

The onset of convection in a viscoelastic nanofluid layer was studied using a linear instability analysis employing a model that incorporates the effects of Brownian motion, thermophoresis, and viscoelasticity. The onset criterion for stationary and oscillatory convection was derived analytically. Oscillatory instability is possible in both bottom- and top-heavy nanoparticle distributions. For a typical viscoelastic nanofluid with a large Lewis number, results indicated the dependence of Ra_C^{Osc} on Le, R_N, and N_A is very slight. However, Le, R_N, and N_A do affect the region of the oscillatory mode and Ra_C^S.

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