International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:12, 2011

An Agent Based Simulation for Network Formation with Heterogeneous Agents

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Abstract—We investigate an asymmetric connections model with a dynamic network formation process, using an agent based simulation. We permit heterogeneity of agents' value. Valuable persons seem to have many links on real social networks. We focus on this point of view, and examine whether valuable agents change the structures of the terminal networks. Simulation reveals that valuable agents diversify the terminal networks. We can not find evidence that valuable agents increase the possibility that star networks survive the dynamic process. We find that valuable agents disperse the degrees of agents in each terminal network on an average.

Keywords—network formation, agent based simulation, connections model.

I. Introduction

NETWORKS play a significant role in social or economic activities. For example, many people get their jobs having help from their acquaintance or friend network. Their friends may give them information on vacancy. The information may originated from the friends or friends of the friends. Social networks deliver information and bring members in utility either directly or indirectly. A research in Massachusetts reported that many of surveyed residents got their jobs through social network [4], [5].

Agents - including individuals, firms, countries, and so on - creates or sever social links on their own initiatives. The connections model is a helpful device to seek properties social network structures have [6]. Consider a situation that a number of agents can create and can sever links to other agents. Agents obtain benefits from other agents through paths on the network to which they belong. Agents obtain benefits from not only directly linked agents but also indirectly connected agents, however, the benefits diminishes to the distance of the path. Since maintaining links is costly for involved agents, agents use their own discretion in creating and severing links to maximize their net benefit. It is known that if agents are homogeneous and parameters are in some range then star networks (say in other words, hub and spoke network) can be stable [6]. The form of star networks is characteristic. One hub agent connects all of other agents directly, although all of agents on the periphery do not have direct links which connect each other. The hub agent has many links and each of other agents has only a link which connect the agent to the hub agent. It is remarkable that the symmetric connections model shows a possibility that this characteristic structure is realizable in society.

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Stability means that no agent has incentive to create new direct links or sever existing direct links, and this is a static concept.[1] Consider a simple dynamic network formation process. Two agents are randomly picked up in every period and they make a decision on having the link between them. All agents are so myopic that they consider whether the link increases their current utility. In the symmetric connections model with the dynamic network formation process, it is known that the probability that process converges to star networks goes to 0, as the number of agents goes to infinity [7]. A simulation research showed that it is hard that star networks survive the dynamic process [1]. These studies revealed that star networks can hardly realize in reality even if the probability is not zero.

We explore dynamic results of network formation process using an agent based simulation. We permit heterogeneity of values of agents which spill to other agents over network. VIPs, for example, ministers or secretaries, executives of major companies and so on, seem to have many links in real society. They might be so valuable for other persons that many people wish to link to VIPs. We focus on this point of view. If there exist VIPs, the dynamic process might converge to star networks more frequently.

An operation of simulation is started from the initial state where no one has any link. After the dynamic network formation process converges, the network should arrive at an stable network, i.e., no agent has incentive to make any new links and to sever any existing links. We repeat sufficiently many operations and explore properties of terminal networks. The shape of terminal network is not unique generally as well as stable network is not unique. Simulation data reveals the frequency of networks which realize as a terminal network. We show several effects that VIPs affect terminal networks. In the next section, we formulate an asymmetric connections model with a dynamic network formation process and established basic results are shown [6], [7]. Section 3 provides the results of simulation. Section 4 provides concluding remarks.

II. Model

A. The connections model

Consider a set of agents $N = \{1, 2, \ldots, n\}$. Let $ij := \{i, j\}$ represents a link between agents i and j and it means that i and j is directly connected. A network g is a list of links. We consider only non-directed graphs. Each agent $i \in N$ receives a payoff $u_i(g)$ from other agents over network g. The payoff

[1] Exact definition of stable networks are detailed later.

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:12, 2011

function of i is defined as:

$$u_i(g) = \sum_{j \neq i} \delta^{s(ij)} v_j - d_i(g)c. \quad (1)$$

The first term of the right hand side of equation (1) represents benefits that agent i receives from other agents over network g. Let $v_j > 0$ represent agent j's value which benefits others who linked to j. Let $\delta \in (0,1)$ be a discount factor. Let $s(ij)$ be the length of the shortest paths between agents i and j.[2] Since $\delta < 1$, agent i receives more benefit from a closer agent than a distant agent. For example, if $\delta = \delta$ and agents i and j are linked directly, i receives $0.8v_j$. If j is reachable from i by just two steps, i receives $\delta^2 v_j$. For the convenience, if agents i and j are not connected (neither directly nor indirectly), then $\delta^{s(ij)} = 0$. Specially, if $v_i = v_j$ for all i and j, we say that the model is symmetric, otherwise asymmetric.

The second term of the right hand side of equation (1) represents costs of maintaining each links which i has. Note that all agents faces same link costs. Let $c > 0$ be link costs. Each agent pays c per involved direct link. Let $d_i(g)$ be the degree of i in network g, that is the number of links i has. For example, agent i on the left network g in Fig. 1 links to agent j, k and l. We say that i has three links, or $d_i(g) = 3$. The degree $d_j(g)$ of agent j on the right network g in Fig. 1 is two. Agent i receives the payoff of $u_i(g) = 3\delta$, since i links to all of other agents on g. Agent j on g receives the payoff of $u_j(g) = 2\delta + \delta^2$, since j links to l by just two steps. Agent l on the right network g' receives the payoff of $u_j(g') = 0$, since l does not connected to any agents (neither directly nor indirectly).

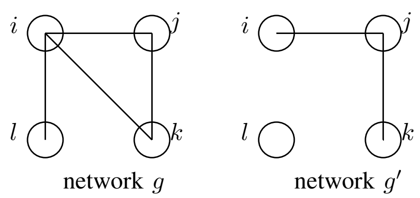

Fig. 1. Examples of networks when $n = 4$.

B. Pairwise stability

Generally speaking, a network is stable if and only if there is no pressure to change the structure. In social networks, the vertexes are agents face decision makings about linkages to other agents. A very plausible formulation for network stability is as follows [6]. A network g is *pairwise stable* if and only if for all agents i and j, (i) $u_i(g) \geq u_i(g - ij)$ and (ii) $u_i(g + ij) > u_i(g) \rightarrow u_j(g + ij) < u_j(g)$. This means that no agents in network g have incentives to sever existing links (condition (i)) or to create new links (condition (ii)). Condition (i) presumes that agents can sever involved links on their own initiative, however, condition (ii) presume that agents cannot create new links without agreements with their opponents.

A well-known static property of stable networks in the symmetric connections model is as follows [6].

Theorem 1 (Jackson and Wolinsky (1996))**.** Suppose a symmetric connections model. For all n, $\delta \in (0,1)$ and $c > 0$, there exists a pairwise stable network such that:

(i) if $\delta \leq c$ then the empty network is pairwise stable,[3]
(ii) if $\delta - \delta^2 \leq c \leq \delta$ then a star network is pairwise stable,
(iii) if $c < \delta$ then the complete network is pairwise stable uniquely.[4]

The structure of star networks is remarkable. A hub agent links to all of other agents and peripheral agents do not link each other. The hub agent connects from a peripheral agent to another by just two steps, and star networks is a class of the least linked network within connected networks.[5] In practice, many social linkage seem to have such characteristics of structure. Theorem 1 showed a possibility that hub and spoke structure is realizable as a stable network in society.

C. Dynamic process

Consider discrete periods $t = 1, 2, \ldots$. In each period t, nature chooses a pair (i, j) of agents with uniform probability, and matched agents make decisions against severing existing links or creating a new link or staying status quo. They make decisions independently. Let $g(t-1)$ be the network decided in period $t-1$ and the network $g(t)$ in current period results from their current decision. If i and j are already linked directly, they decide to sever the link or to stay status quo. If one of them want to sever the link then the link vanishes, and in this case, the network in the period is $g(t) = g(t-1) - ij$.[6] If i and j are not linked directly, they decide to have new link or stay status quo. If both of them want to have the link then the link is created, and in this case, the network in the period is $g(t) = g(t-1) + ij$. Shortly, each agent can sever links on her own authority although she cannot create new links without the partners' agreement. We assume initial state of network is the empty network, $g(0) = \emptyset$.

Pairwise stability is a static concept and Theorem 1 is a static result. From a dynamic point of view, there is a negative result against Theorem 1 [7].

Theorem 2 (Watts (2001))**.** Suppose $3 < n < \infty$ and $\delta - \delta^2 \leq c \leq \delta$ in a symmetric connections model. The probability that the network formation process will converge to a star goes to 0, as n goes to infinity.

A simulation research revealed that even if the size of agent set is small, we cannot virtually expect that star networks realize as a terminal of network formation process [1]. The processes converge only three times in 6000 trials if $n = 7$ and none in 6000 trials if $n = 8$. Theoretically, it is possible

[2] The length of a path is the number of links included in the path.

[3] All agents have no link in the empty network.

[4] all agents are linked directly each other in the complete network.

[5] A network is connected if there exists a path between all pair of agents. A path between agents i and j in network g is a sequence $k_1 k_2 \ldots k_l$ of agents such that $k_1 = i$, $k_l = j$ and $k_h k_{h+1} \in g$ for $h = 1, 2, \ldots, l-1$.

[6] For a notational convenience, $g - ij := g \setminus \{i, j\}$ and $g + ij := g \cup \{i, j\}$.

that star networks realize as a terminal of network formation process, however, it is not possible practically.

VIPs, for example, ministers or secretaries, executives of major companies and so on, seem to have many links in real society. We focus on this point of view. Does the terminal networks tend to be a star form if there exist VIPs? We explore an asymmetric connections model with a dynamic network formation process by making use of simulation.[7]

D. Simulation algorithm

The network structure is decided by agents' decision makings in each period. In period t, the network $g(t-1)$ which is decided in period $t-1$ is given and is the status quo in the current period. If agents want to stay status quo then $g(t) = g(t-1)$, otherwise $g(t) = g(t-1) + ij$ or $g(t) = g(t-1) - ij$. The simulation algorithm in the tth period is as follows.

- Step 1 A pair (i, j) of agents is picked up randomly by nature. All pairs are chosen with same probability.
- Step 2 If $ij \in g(t-1)$, i and j decides independently whether to sever the link. If at least one of them want to sever the link, $g(t) = g(t-1) - ij$.
- Step 3 If $ij \notin g(t-1)$, i and j decides whether to create new link between them. If both of them want to create the link, $g(t) = g(t-1) + ij$.

If a network is maintained consecutively over previous many periods, an operation is terminated. We are unable to escape from the error that the terminal network is not pairwise stable in simulation. However, we can reduce the probability of the error to almost 0. The condition a operation ends is shown in Table I. For example, an operation is terminated when a network maintained over consecutive 260 periods, if $n = 8$. When 1000 times of operations are finished, the expected value of the number of errors is less than 0.1, since the probability of the error is at most 0.00783%.[8]

TABLE I

n	termination condition	probability of error
4	60	0.00177%
5	90	0.00762%
6	140	0.00638%
7	190	0.00942%
8	260	0.00783%

1000 times of operations are carried out for each set of the values of parameters. Parameters we manipulate is summarized in Table II. We fix δ to 0.8, the value of a agent who is not a VIP to 1, and the value of a VIP to 5. 45000 times of operations are carried out in all, since the number of cases is 45 (=$5 \times 3 \times 3$). 9000 times of operations are carried out for each n.

[7]There is a theoretical work that extends the symmetric connections model by allowing heterogeneities of both agents' values and link costs [3].

[8]If $n = 8$, the number of combinations of a pair is 28, i.e., ${}_8C_2 = 28$. If the number of pairs which members have incentives to cerate or to sever the link between them is unfortunately only one, the probability the pair is not picked up in a random matching is 27/28. The probability that the pair is not picked up over consecutive 260 periods is $(\frac{27}{28})^{260} = 0.0000783$. At worst, the probability that the terminal network is not pairwise stable is 0.00783%.

TABLE II

parameters	values
n	4 or 5or 6 or 7 or 8
δ	0.8
c	0.3 or 0.5 or 0.7
the number of VIPs	0 or 1 or 2
value of a agent	1
value of a VIP	5

III. Results

A. The case of $n = 4$

A network is characterized by corresponding degree sequence in this paper.[9] The degree d_i of agent i is the number of links that i has.[10] A network is represented by ascending ordered degrees of all agents. For example, network g in Fig. 1 is represented by 1223, and g' is represented by 0112.

Table III shows the results of simulation. When $n = 4$, $c = 0.3$ and agents are homogeneous (see the column of "VIPs = 0" in Table III), the frequency that the terminal network has degree sequence 1122 is 737 times in 1000 times of operations. The exact shape of the network with degree sequence 1122 is described in FIg. 2. The degree sequence 1113 represents a star network. Table III reveals that VIPs do not change the form of terminal networks. For example, suppose $c = 0.7$. If there is no VIP then the probability that star networks realize as a terminal network is 27.1%. If there is a VIP, the probability is 25.8%. If there are two VIPs, the probability is 26.6%. There is no apparent tendency to realize star networks as the number of VIPs increases.

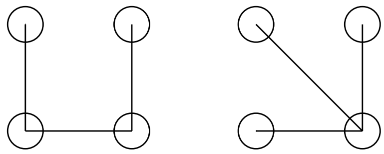

Fig. 2. The terminal networks when $n = 4$.

B. The case of $n \geq 5$

Table III reveals that VIPs diversify the terminal networks. For example, suppose that $n = 8$ and $c = 0.3$. The cumulative probability of the seven terminal degree sequences with the highest occurrence probability is 80.2% if there are two VIPs. If there is one VIP, the cumulative probability is 96.0%, and if there is no VIP then the cumulative probability is 99.5%. However, there is no apparent tendency to realize star networks as the number of VIPs increases.

The trend concerning which degree sequence tends to be terminals is almost same in spite of the number of VIPs. For example, suppose that $n = 6$ and $c = 0.5$. The three terminal degree sequences with the highest occurrence probability are same in spite of the number of VIPs.

[9]The shapes of networks are not corresponds with degree sequence exactly. Generally, a degree sequence corresponds with many networks.

[10]Large degree means that the agent plays central role in the network. Degree centrality and other centrality concepts are discussed in [2]

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:12, 2011

TABLE III

n=4	VIPs=0			VIPs=1			VIPs=2		
	degrees	probability	cumulative prob.	degrees	probability	cumulative prob.	degrees	probability	cumulative prob.
c=0.3	1122	0.7370	0.7370	1122	0.7060	0.7060	1122	0.7190	0.7190
	1113	0.2630	1.0000	1113	0.2940	1.0000	1113	0.2810	1.0000
c=0.5	1122	0.7330	0.7330	1122	0.7420	0.7420	1122	0.7120	0.7120
	1113	0.2670	1.0000	1113	0.2580	1.0000	1113	0.2880	1.0000
c=0.7	1122	0.7290	0.7290	1122	0.4720	0.4720	1122	0.7340	0.7340
	1113	0.2710	1.0000	1113	0.2580	0.7300	1113	0.2660	1.0000

n=5	VIPs=0			VIPs=1			VIPs=2		
	degrees	probability	cumulative prob.	degrees	probability	cumulative prob.	degrees	probability	cumulative prob.
c=0.3	11123	0.4970	0.4970	11123	0.5160	0.5160	11123	0.3950	0.3950
	22222	0.4690	0.9660	22222	0.3840	0.9000	22222	0.3290	0.7240
	11114	0.0340	1.0000	11114	0.1000	1.0000	11224	0.1250	0.8490
							12223	0.0950	0.9440
							11114	0.0560	1.0000
c=0.5	11123	0.5160	0.5160	11123	0.5730	0.5730	11123	0.5560	0.5560
	22222	0.4380	0.9540	22222	0.3820	0.9550	22222	0.3890	0.9450
	11114	0.0460	1.0000	11114	0.0450	1.0000	11114	0.0550	1.0000
c=0.7	11123	0.5230	0.5230	11123	0.4790	0.4790	11123	0.5180	0.5180
	11222	0.4290	0.9520	11222	0.4780	0.9570	11222	0.4320	0.9500
	11114	0.0480	1.0000	11114	0.0430	1.0000	11114	0.0500	1.0000

n=6	VIPs=0			VIPs=1			VIPs=2		
	degrees	probability	cumulative prob.	degrees	probability	cumulative prob.	degrees	probability	cumulative prob.
c=0.3	122223	0.7320	0.7320	122223	0.6770	0.6770	122223	0.5300	0.5300
	111124	0.1030	0.8350	111124	0.2050	0.8820	111234	0.1410	0.6710
	111133	0.0920	0.9270	222222	0.0560	0.9380	111124	0.0990	0.7700
	222222	0.0600	0.9870	111133	0.0450	0.9830	222222	0.0730	0.8430
c=0.5	122223	0.7460	0.7460	122223	0.6920	0.6920	122223	0.6070	0.6070
	111124	0.1000	0.8460	111124	0.1270	0.8190	111124	0.0940	0.7010
	111133	0.0980	0.9440	111133	0.1130	0.9320	111133	0.0910	0.7920
	222222	0.0510	0.9950	222222	0.0620	0.9940	222222	0.0830	0.8750
c=0.7	111223	0.5700	0.5700	111223	0.6000	0.6000	111223	0.5930	0.5930
	222222	0.2400	0.8100	222222	0.1680	0.7680	111124	0.1650	0.7580
	111124	0.1140	0.9240	111124	0.1390	0.9070	222222	0.1510	0.9090

n=7	VIPs=0			VIPs=1			VIPs=2		
	degrees	probability	cumulative prob.	degrees	probability	cumulative prob.	degrees	probability	cumulative prob.
c=0.3	1122233	0.4020	0.4020	2222233	0.3650	0.3650	2222233	0.2660	0.2660
	2222233	0.3490	0.7510	1122233	0.2940	0.6590	1122233	0.2130	0.4790
	1122224	0.1780	0.9290	1122224	0.1750	0.8340	1222234	0.1920	0.6710
	1111134	0.0490	0.9780	1222333	0.0660	0.9000	1122224	0.0870	0.7580
	1111125	0.0220	1.0000	1111125	0.0450	0.9450	1222333	0.0720	0.8300
c=0.5	2222233	0.4440	0.4440	2222233	0.3830	0.3830	2222233	0.3850	0.3850
	1122233	0.2950	0.7390	1122233	0.3050	0.6880	1122233	0.2800	0.6650
	1122224	0.1940	0.9330	1122224	0.2140	0.9020	1122224	0.1570	0.8220
c=0.7	1222223	0.5230	0.5230	1222223	0.4100	0.4100	1222223	0.3900	0.3900
	1111233	0.2280	0.7510	1111233	0.2170	0.6270	1111233	0.2260	0.6160
	1111224	0.1370	0.8880	1111224	0.1870	0.8140	1111224	0.1830	0.7990
	1112223	0.0490	0.9370	1112223	0.0830	0.8970	1112223	0.0800	0.8790

n=8	VIPs=0			VIPs=1			VIPs=2		
	degrees	probability	cumulative prob.	degrees	probability	cumulative prob.	degrees	probability	cumulative prob.
c=0.3	22223333	0.3980	0.3980	22223333	0.3360	0.3360	22223333	0.2680	0.2680
	11122234	0.2190	0.6170	12222333	0.1870	0.5230	12222333	0.1420	0.4100
	12222234	0.1800	0.7970	12222234	0.1840	0.7070	11222235	0.1010	0.5110
	12222333	0.1470	0.9440	11122234	0.1290	0.8360	12222234	0.1000	0.6110
	11122225	0.0360	0.9800	11122225	0.0560	0.8920	12223334	0.0790	0.6900
	11111135	0.0090	0.9890	11222334	0.0370	0.9290	11122234	0.0610	0.7510
	22222233	0.0060	0.9950	11222244	0.0310	0.9600	11222334	0.0510	0.8020
c=0.5	22223333	0.3760	0.3760	12222234	0.2630	0.2630	22223333	0.2550	0.2550
	12222234	0.1680	0.5440	22223333	0.2570	0.5200	22222233	0.1880	0.4430
	12222333	0.1530	0.6970	11122234	0.1500	0.6700	12222234	0.1620	0.6050
	11122234	0.1450	0.8420	12222333	0.1470	0.8170	11122234	0.1310	0.7360
	22222233	0.1130	0.9550	22222233	0.1190	0.9360	12222333	0.0970	0.8330
c=0.7	11222233	0.4010	0.4010	11222233	0.3110	0.3110	11222233	0.2720	0.2720
	22222233	0.1680	0.5690	11111234	0.1590	0.4700	11111234	0.1160	0.3880
	11222224	0.1310	0.7000	11222224	0.1290	0.5990	22222233	0.1010	0.4890
	11111234	0.0900	0.7900	22222233	0.1180	0.7170	12222223	0.0970	0.5860
	12222223	0.0730	0.8630	12222223	0.1080	0.8250	11222224	0.0940	0.6800
	11112233	0.0410	0.9040	11112233	0.0740	0.8990	11122234	0.0840	0.7640
	11111333	0.0330	0.9370	11111225	0.0340	0.9330	11112233	0.0570	0.8210

Tables IV-VIII shows the expected values of variances of degrees in a terminal network. For example, suppose $n = 5$, $c = 0.3$ and there is no VIP. The terminal degree sequences are 11123 with probability 0.497, 22222 with probability 0.469 and 11114 with probability 0.034. The expected value of variance is 0.46 which is the weighted sum of the variance of degrees of agents belongs to each terminal network. Tables IV-VIII reveals that VIPs increase the expected variance. Large expected variance means that the distribution of agents' degree is biased on an average in a terminal network. VIPs lead the structures of terminal networks to be biased.

TABLE IV
EXPECTED VARIANCES WHEN $n = 4$

c	VIPs=0	VIPs=1	VIP=2
0.3	0.51	0.53	0.52
0.5	0.51	0.51	0.53
0.7	0.51	0.51	0.51

TABLE V
EXPECTED VARIANCES WHEN $n = 5$

c	VIPs=0	VIPs=1	VIP=2
0.3	0.46	0.59	0.65
0.5	0.50	0.54	0.54
0.7	0.63	0.60	0.63

TABLE VI
EXPECTED VARIANCES WHEN $n = 6$

c	VIPs=0	VIPs=1	VIP=2
0.3	0.58	0.66	0.80
0.5	0.56	0.60	0.66
0.7	0.64	0.71	0.74

TABLE VII
EXPECTED VARIANCES WHEN $n = 7$

c	VIPs=0	VIPs=1	VIP=2
0.3	0.66	0.70	0.87
0.5	0.59	0.67	0.68
0.7	0.66	0.75	0.74

TABLE VIII
EXPECTED VARIANCES WHEN $n = 8$

c	VIPs=0	VIPs=1	VIP=2
0.3	0.66	0.76	0.89
0.5	0.59	0.68	0.72
0.7	0.68	0.77	0.85

IV. CONCLUDING REMARKS

We investigate asymmetric connections model with a dynamic process by making use of simulation. Obtained data reveals that VIPs do not change overall trend of terminal networks dramatically, however, diversify the terminal networks. When there are VIPs, more networks can survive a dynamic network formation process.

VIPs also disperse the degrees of agents in a network. Agents wish to link VIPs, since VIPs are valuable to another agents. The agents who link to VIPs is valuable to another agents, since the value of VIPs spills to the linked agents. Agents wish to link to not only VIPs but also the agents who link to VIPs. This increases the possibility of realization of the biased distribution of agents' links.

ACKNOWLEDGMENT

The authors would like to thank Kozo Keikaku Engineering Inc. for providing the simulator "artisoc".

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