

Lagrangian method for solving unsteady gas equation

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Abstract—In this paper we propose, a Lagrangian method to solve unsteady gas equation which is a nonlinear ordinary differential equation on semi-infinite interval. This approach is based on Modified generalized Laguerre functions. This method reduces the solution of this problem to the solution of a system of algebraic equations. We also compare this work with some other numerical results. The findings show that the present solution is highly accurate.

Keywords—Unsteady gas equation, Generalized Laguerre functions, Lagrangian method, Nonlinear ODE.

I. INTRODUCTION

Recently, spectral methods have been successfully applied in the approximation of differential boundary value problems defined in unbounded domains. We can apply different approaches using spectral methods to solve problems in semi-infinite domains.

The first approach is using Laguerre polynomials [1-4]. Guo [1] suggested a Laguerre-Galerkin method for the Burgers equation and Benjamin-Bona-Mahony (BBM) equation on a semi-infinite interval. It is shown that the Laguerre-Galerkin approximations are convergent on a semi-infinite interval with spectral accuracy.

Shen [2] proposed spectral methods using Laguerre functions and analyzed elliptic equations on regular unbounded domains. In [2] is shown that spectral-Galerkin approximations based on Laguerre functions are stable and convergent with spectral accuracy in the Sobolev spaces. Maday, et al. [3] proposed a Laguerre type spectral method for solving partial differential equations. Siyyam [4] applied two numerical methods for solving initial value problem differential equations using the Laguerre Tau method.

The second approach is reformulating the original problem in semi-infinite domain to a singular problem in bounded domain by variable transformation and then using the Jacobi polynomials to approximate the resulting singular problem [5].

The third approach is replacing semi-infinite domain with $[0, K]$ interval by choosing K , sufficiently large. This method is named domain truncation [6].

The fourth approach of spectral method is based on rational orthogonal functions. Boyd [7] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by mapping to the Chebyshev polynomials. Guo et al.

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[8] introduced a new set of rational Legendre functions which is mutually orthogonal in $L^2(0, +\infty)$. They applied a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half line. Boyd et al. [9] applied pseudospectral methods on a semi-infinite interval and compared rational Chebyshev, Laguerre, and mapped Fourier sine.

The authors of [10-12] applied spectral method to solve nonlinear ordinary differential equations on semi-infinite intervals. Their approach was based on a rational Tau method. They obtained the operational matrices of derivative and product of rational Chebyshev, Legendre and Laguerre functions and then applied these matrices together with the Tau method to reduce the solution of these problems to the solution of a system of algebraic equations.

This paper is arranged as follows:

In section 1 we describe Unsteady gas equation. In section 2 we describe the formulation of generalized Laguerre polynomials and modified generalized Laguerre functions required for our subsequent development. Then we obtained the operational matrices of derivative of modified generalized Laguerre functions and then we applied these matrices together with the Lagrangian method to reduce the solution of this problem to the solution of system of algebraic equation. In Section 3 we compare our solutions with some well-known results, comparisons show that the present solutions are highly accurate. The conclusions are described in the final section.

A. Unsteady gas equation

In the study flow of gas through a semi-infinite porous medium [13], [14] initially filled with gas at a uniform pressure $p_0 \geq 0$, at time $t = 0$, the pressure at the outflow face is suddenly reduced from p_0 to $p_1 \geq 0$ ($p_1 = 0$ is the case of diffusion into a vacuum) and is, thereafter, maintained at this lower pressure. The unsteady isothermal flow of gas is described by a nonlinear partial differential equation. The nonlinear partial differential equation that describes the unsteady flow of gas through a semi-infinite porous medium has been derived by Muskat [15] in the form

$$\nabla^2(P^2) = (2\Phi\mu/k) \frac{\partial P}{\partial t} \quad (1)$$

where P is the pressure within porous medium, Φ the porosity, μ the viscosity, k the permeability, and t the time. New variables were introduced by Kidder [13] and Davis [16] to transform the nonlinear partial differential equation (1) to the nonlinear ordinary differential equation. The nonlinear

ordinary differential equation due to Kidder [13] given by (unsteady gas equation)

$$y'' + 2xy' / (1 - \alpha y)^{1/2} = 0, \quad x > 0, \quad 0 < \alpha < 1 \quad (2)$$

The typical boundary conditions imposed by the physical properties are

$$y(0) = 1, \quad y(\infty) = 0. \quad (3)$$

A substantial amount of numerical and analytical work has been invested so far [13], [17] on this model. The main reason of this interest is that the approximation can be used for many engineering purposes. As stated before, the problem (2) was handled by Kidder [13] where a perturbation technique is carried out to include terms of the second order. Recently Wazwaz [18] applied the modified decomposition method for solving this nonlinear equation. The base of his approach is modification of the Adomian decomposition method. The diagonal Pade approximants are effectively used in the analysis to capture the essential behavior of $y(x)$ and to determine the initial slope $y'(0)$.

II. PROPERTIES OF MODIFIED GENERALIZED LAGUERRE FUNCTIONS

This section is devoted to the introduction of the basic notions and working tools concerning orthogonal Modified generalized Laguerre functions. More specifically, we presented some properties of Modified generalized Laguerre functions.

A. Modified generalized Laguerre functions

The Laguerre approximation has been widely used for numerical solutions of differential equations on infinite intervals. $L_n^\alpha(x)$ (generalized Laguerre polynomial) is the n th eigenfunction of the Sturm-Liouville problem [19], [20], [21]:

$$x \frac{d^2}{dx^2} L_n^\alpha(x) + (\alpha + 1 - x) \frac{d}{dx} L_n^\alpha(x) + n L_n^\alpha(x) = 0, \quad x \in I = [0, \infty), \quad n = 0, 1, 2, \dots$$

The generalized Laguerre polynomials are defined with the following recurrence formula:

$$\begin{aligned} L_0^\alpha(x) &= 1, \quad L_1^\alpha(x) = 1 + \alpha - x, \\ n L_n^\alpha(x) &= (2n - 1 + \alpha - x) L_{n-1}^\alpha(x) \\ &- (n + \alpha - 1) L_{n-2}^\alpha(x), \quad n \geq 2. \end{aligned}$$

these are orthogonal polynomials for the weight function $w_\alpha = x^\alpha e^{-x}$. The generalized Laguerre polynomials satisfy the following relation:

$$\partial_x L_n^\alpha(x) = - \sum_{k=0}^{n-1} L_k^\alpha(x). \quad (4)$$

We define Modified generalized Laguerre functions (which we denote (MGL) functions) ϕ_j as follows:

$$\phi_j(x) = \exp(-x/(2L)) L_j^1(x/L), \quad L > 0. \quad (5)$$

This system is an orthogonal basis [22], [23] with weight function $w(x) = \frac{x}{L}$ and orthogonality property:

$$\langle \phi_n, \phi_m \rangle_w = \int_0^\infty \phi_n(x) \phi_m(x) w(x) dx = \left(\frac{\Gamma(n+2)}{L^2 n!} \right) \delta_{nm},$$

where δ_{nm} is the Kronecker function. Boyd [6], [7], [9] offered guidelines for optimizing the map parameter L where $L > 0$ is the scaling parameter.

On a semi-infinite domain, there is always a parameter that must be determined experimentally. For example, if domain truncation to a domain $x \in [0, K]$ is employed, then one must choose K . If one knows the rate at which $y(x)$ decays for large x , one can choose K so that $y(K) < \delta$ where δ is some user-chosen tolerance. But then one is still faced with choosing the grid spacing h so that the error in solving the differential equation on $x \in [0, K]$ is small. If h is small, the error in solving the ODE may be very much less than δ , in which case one has made a bad choice of K because the domain truncation error is dominant, and one would have been better choosing a larger K .

Numerical results depend smoothly on constant parameter L , and therefore are not very sensitive to L because the $\frac{dError}{dL} = 0$ at the minimum itself, so the error varies very slowly with L around the minimum. A little trial and error is usually sufficient to find a value that is nearly optimum. In general, there is no way to avoid a small amount of trial and error in choosing L when solving problems on an unbounded domain. Experience and the asymptotic approximations of Boyd [7] can help, but some experimentation is always necessary as he explains in his book [6].

Lagrangian interpolants of generalized Laguerre polynomials (we denoted GLP) of order p at the Gauss-Radau-Laguerre quadrature points in R^+ is [24]:

$$\ell_j(x) = \frac{x L_N^\alpha(x)}{\eta_j \frac{d}{dx} L_N^\alpha(\eta_j)} \frac{1}{x - \eta_j}, \quad j = 1, \dots, N, \quad (6)$$

and

$$\ell_0(x) = \frac{L_N^\alpha(x)}{L_N^\alpha(0)},$$

$\eta_j, j=1,2,\dots,N$, are the N GLP-Radau points. derivative operator of GLP is:

$$d_{ij} = \ell_j'(\eta_i),$$

moreover for any polynomial p of degree at most N , one gets:

$$p'(\eta_i) = \sum_{j=0}^N d_{ij} p(\eta_j).$$

Funaro [24] obtained derivative matrix of $GLP(D_N)$:

$$d_{ij}(x) = \begin{cases} \frac{\eta_i}{\eta_j} \frac{d}{dx} L_N^\alpha(\eta_i) \frac{1}{\eta_i - \eta_j} & i, j = 1, \dots, N, i \neq j, \\ \frac{1}{1 - \alpha + \eta_i} & i = j = 1, \dots, N, \\ \frac{d}{dx} L_N^\alpha(\eta_i) & i = 1, \dots, N, j = 0, \\ -\frac{L_N^\alpha(0)}{\eta_j^2 \frac{d}{dx} L_N^\alpha(\eta_j)} & j = 1, \dots, N, i = 0, \\ -\frac{N}{\alpha + 1} & i = j = 0, \end{cases} \quad (7)$$

The second derivative operator is obtained either by squaring D_N either by evaluating $\ell_j''(\eta_i)$:

$$\ell_{ij}''(x) = \begin{cases} \frac{((1-\alpha+\eta_i)(\eta_i-\eta_j)-2\eta_i)}{\frac{d}{dx} L_N^\alpha(\eta_i)(\eta_i-\eta_j)^2} & i, j = 1, \dots, N, i \neq j, \\ \frac{1}{\eta_j} \frac{d}{dx} L_N^\alpha(\eta_i) & i = j = 1, \dots, N, \\ \frac{(\eta_i - \alpha)^2}{3\eta_i^2} - \frac{N-1}{3\eta_i} & i = j = 1, \dots, N, \\ -\frac{\alpha+1-\eta_i L_N^\alpha(0)}{\frac{d}{dx} L_N^\alpha(\eta_i)\eta_i} & i = 1, \dots, N, j = 0, \\ -\frac{2(N+\alpha+1) \frac{d}{dx} L_N^\alpha(x)}{L_N^\alpha(0)\eta_i^3(\alpha+1)} & i = 1, \dots, N, j = 0, \\ \frac{N(N-1)}{(\alpha+1)(\alpha+2)} & i = j = 0, \end{cases} \quad (8)$$

Laguerre polynomials are not suitable for computations [24], and also Lagrangian interpolation of Laguerre polynomials is not suitable for solving some differential equations, such as Lane-Emden equations because of their boundary conditions. So we use Lagrangian interpolation of (MGL) functions. At first we must find Lagrangian interpolants and derivative operators of (MGL) functions.

Lemma 1: Let η_i be a root of $L_N^\alpha(x)$ and $\Gamma_N^\alpha(x) = e^{-x/2} L_N^\alpha(x)$, then

$$\frac{d}{dx} \Gamma_N^\alpha(\eta_i) = e^{-\eta_i/2} \frac{d}{dx} L_N^\alpha(\eta_i).$$

Proof.

$$\begin{aligned} \frac{d}{dx} \Gamma_N^\alpha(x) &= \frac{d}{dx} e^{-x/2} L_N^\alpha(x) = \\ &= \frac{-1}{2} e^{-x/2} L_N^\alpha(x) + e^{-x/2} \frac{d}{dx} L_N^\alpha(x). \end{aligned}$$

And with substitution $x = \eta_i$ Lemma 1 is proved.

Lemma 2: Lagrangian interpolant of $\Gamma_N^\alpha(x) = e^{-x/2} L_N^\alpha(x)$ is,

$$\widehat{\ell}_i(x) = \ell_i(x) \frac{e^{-x/2}}{e^{-\eta_i/2}}, \quad (9)$$

which $\ell_j(x)$ are Lagrangian interpolant of Laguerre polynomials.

Proof. Suppose $\gamma \frac{x \ell_N^\alpha(x)}{x - \eta_j}$ is Lagrangian interpolant and using Lemma 1 we can find constant γ ,

$$\begin{aligned} \widehat{\ell}_i(\eta_i) &= 0, \\ \lim_{x \rightarrow \eta_j} \gamma \frac{x \ell_N^\alpha(x)}{x - \eta_j} &= \gamma \left(\lim_{x \rightarrow \eta_j} \ell_N^\alpha(x) + x \frac{d}{dx} \ell_N^\alpha(x) \right) = 1 \\ \Rightarrow \gamma &= \frac{1}{\eta_j e^{-\eta_i/2} \frac{d}{dx} L_N^\alpha(\eta_i)}, \end{aligned}$$

so

$$\widehat{\ell}_i(x) = \frac{1}{\eta_j e^{-\eta_i/2} \frac{d}{dx} L_N^\alpha(\eta_i)} \frac{x \ell_N^\alpha(x)}{x - \eta_j}, \quad (10)$$

and with comparison of (6) and (10) Lemma 2 is proved.

By Eq. (9) derivative operator of $\Gamma_N^\alpha(x) = e^{-x/2} L_N^\alpha(x)$ (we

denote by \widehat{D}_N) is:

$$\widehat{d}_{ij} = d_{ij} \frac{e^{-\eta_j/2}}{e^{-\eta_i/2}} - 1/2 \delta_{ij}, \quad i, j = 1, \dots, N. \quad (11)$$

Where matrix d_{ij} is defined in Eq. (7). As pointed out in before the second derivative operator is obtained either by squaring \widehat{D}_N either by evaluating $\widehat{\ell}_j''(\eta_i)$. For evaluating $\widehat{\ell}_j''(\eta_i)$ we can use the following relation:

$$\widehat{\ell}_{ij}''(\eta_j) = \frac{1}{4} \delta_{ij} - d_{ij} \frac{e^{-\eta_j/2}}{e^{-\eta_i/2}} + \frac{e^{-\eta_j/2}}{e^{-\eta_i/2}} \ell_{ij}''(\eta_j), \quad (12)$$

and $\ell_{ij}''(\eta_j)$ is defined in Eq. (8). It is obvious that (MGL) functions is $\Gamma(x/L)$, so Lagrangian interpolant of (MGL) functions is $\widehat{\ell}_i(x/L)$, and derivative operators can be obtained easily.

B. Function Approximation

We define the interpolant approximation of $y(x)$ by

$$I_N y(x) = \sum_{j=0}^N b_j \widehat{\ell}_j(x/L). \quad (13)$$

Where $\widehat{\ell}_j(x)$ is defined in Eq. (10). The b_j 's are the expansion coefficients associated with the family $\{\widehat{\ell}_j(x/L)\}$. A semilogarithmic plot of $abs(b_j)$ versus j is also useful to determine a good choice of L when the exact solution for $y(x)$ is unknown. One can run the code for several different L and then plot the coefficients from each run on the same graph. The best L is the choice that gives the most rapid decrease of the coefficients.[6] Therefore,

$$I_N y(\mathfrak{S}_j) = b_j, \quad j = 1, \dots, N. \quad (14)$$

where $\mathfrak{S}_j = L\eta_j$ are the zeroes of $\widehat{\ell}_j(x/L)$. So derivative operator of (MGL) functions is (we denote by \widehat{D}_{LN}):

$$\widehat{d}_{Lij} = \frac{1}{L} \widehat{\ell}_j'(\mathfrak{S}_i/L) = \frac{1}{L} \widehat{\ell}_j'(\eta_i) = \frac{1}{L} \widehat{d}_{ij}, \quad (15)$$

and the second derivative operator is

$$\widehat{d}_{Lij}^{(2)} = \frac{1}{L^2} \widehat{\ell}_j''(\mathfrak{S}_i/L) = \frac{1}{L^2} \widehat{\ell}_j''(\eta_i), \quad (16)$$

The relationship between the derivative $\frac{d}{dx} I_N y(x)$ and $I_N y(x)$ at the collocation points $\mathfrak{S}_i, i = 1, \dots, N$ can be obtained by differentiation. the result is as:

$$\frac{d}{dx} I_N y(\mathfrak{S}_k) = \sum_{j=0}^N b_j \widehat{d}_{Lkj}, \quad (17)$$

and

$$\frac{d}{dx^2} I_N y(\mathfrak{S}_k) = \sum_{j=0}^N b_j \widehat{d}_{Lkj}^{(2)}. \quad (18)$$

III. SOLVING UNSTEADY GAS EQUATION

To apply Lagrangian interpolant of (MGL) functions to the Unsteady gas Equation introduced in Eq. (2) with boundary conditions Eq. (3), at first we multiply both side of Eq. (2) in $(1 - \alpha y)^{1/2}$, so we have:

$$y''(1 - \alpha y)^{1/2} + 2xy' = 0, \quad x > 0, \quad 0 < \alpha < 1, \quad (19)$$

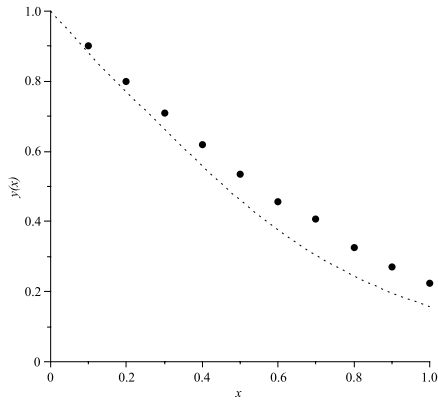


Fig. 1. Unsteady gas equation graph obtained by MGL function Lagrangian method (circles) and Wazwaz method (dotted line)

TABLE I
COMPARISON OF INITIAL SLOPE $y'(0)$ FOR $\alpha = 0.5$.

N	MGL function method	Padé[2,2]	Padé[3,3]
6	-1.37310852	-1.37317809	
7	-1.37317352	-1.37317809	-1.02552970

then with Eq. (13) we expand $y(x)$, as follows:

$$I_N y(x) = \sum_{j=0}^N b_j \hat{\ell}_j(x/L).$$

To find the unknown coefficients b_j 's, we substitute the truncated series into the Eq. (19) and boundary condition in Eq. (3). So we have:

$$\sum_{j=0}^N b_j \hat{d}_{Lij}^{(2)} (1 - \alpha b_i) + 2\mathfrak{S}_i \sum_{j=0}^N b_j \hat{d}_{Lij} = 0, \quad (20)$$

$$i = 1, \dots, N.$$

And for boundary condition:

$$b_0 = 1 \quad (21)$$

we have N equations in Eq. (20), that generates a set of $N + 1$ nonlinear equations with boundary equation in Eq. (21).

Fig. 1 shows comparison of Unsteady gas equation graph obtained by MGL function Lagrangian method and Wazwaz method.

Table 1 shows the comparison of the $y'(0)$, between (MGL) functions for $N = 6, 7$, $L = 0.87$ and Padé approximation used by [18].

Table 2 shows the approximations of $y(x)$ for standard unsteady gas with $\alpha = 0.5$ obtained by the method proposed in this paper for $N = 7$ and $L = 0.87$, the perturbation method used by [13] and Padé approximation by Wazwaz [18].

IV. CONCLUSION

The fundamental goal of this paper has been to construct an approximation to the solution of nonlinear Unsteady gas equation. A set of orthogonal functions are proposed to provide an effective but simple way to improve the convergence of the solution by Lagrangian method. In this method with a few

TABLE II
VALUES OF $y(x)$ FOR $\alpha = 0.5$ FOR $x = 0.1$ TO 1.0

x	MGL function method	Perturbation	Padé[2,2]	Padé[3,3]
0.1	0.90931873	0.88165883	0.86330606	0.89791670
0.2	0.81748763	0.76630768	0.73012623	0.79852282
0.3	0.71522344	0.65653800	0.60330541	0.70411297
0.4	0.60982075	0.55440240	0.48488987	0.61650379
0.5	0.51632348	0.46136503	0.37616039	0.53705338
0.6	0.41932385	0.37831093	0.27773116	0.46656257
0.7	0.40982377	0.30559765	0.18968434	0.40624260
0.8	0.31999068	0.24313255	0.11171052	0.35608017
0.9	0.20820285	0.19046237	0.04323673	0.31799666
1.0	0.21991074	0.15876898	0.01646751	0.29002550

number of points we approximate the solution of unsteady gas equation. With this method we can approximate some well-known differential equations like unsteady gas, blasius and lane-emen with highly accurate.

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