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Hamiltonian Factors in Hamiltonian Graphs

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Abstract—Let G be a Hamiltonian graph. A factor F of G is called a Hamiltonian factor if F contains a Hamiltonian cycle. In this paper, two sufficient conditions are given, which are two neighborhood conditions for a Hamiltonian graph G to have a Hamiltonian factor.

Keywords—graph, neighborhood, factor, Hamiltonian factor.

I. INTRODUCTION

Many physical structures can conveniently be modelled by networks. Examples include a communication network with the nodes and links modelling cities and communication channels, respectively, or a railroad network with nodes and links representing railroad stations and railways between two stations, respectively. Factors and factorizations in networks are very useful in combinatorial design, network design, circuit layout, and so on [1]. It is well known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth we use the term ”graph” instead of ”network”.

The graphs considered in this paper will be finite undirected graphs without loops or multiple edges. In particular, a graph is said to be a Hamiltonian graph if it contains a Hamiltonian cycle. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the neighborhood $N_G(x)$ of x is the set vertices of G adjacent to x, and the degree $d_G(x)$ of x is $|N_G(x)|$. We denote the minimum degree of G by $\delta(G)$. For $S \subseteq V(G)$, $N_G(S) = \cup_{x\in S} N_G(x)$ and $G[S]$ is the subgraph of G induced by S. We write $G-S$ for $G[V(G)\setminus S]$. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. Let r be a real number. Recall that $\lfloor r\rfloor$ is the greatest integer such that $\lfloor r\rfloor \le r$.

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) \le f(x)$ for each $x \in V(G)$. A spanning subgraph F of G is called a (g,f)-factor if it satisfies $g(x) \le d_F(x) \le f(x)$ for each $x \in V(G)$. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then a (g,f)-factor is called an $[a,b]$-factor. A (g,f)-factor F of G is called a Hamiltonian (g,f)-factor if F contains a Hamiltonian cycle. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then we say a Hamiltonian (g,f)-factor to be a Hamiltonian $[a,b]$-factor. The other terminologies and notations not given here can be found in [2].

Many authors have investigated factors [3–8], connected factors [9–11] and Hamiltonian factors [12,13].

The following results on Hamiltonian factors are known.

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Theorem 1. ([12]). Let $2 \le a < b$ be integers and let G be a Hamiltonian graph of order $n \ge \frac{(a+b-4)(2a+b-6)}{b-2}$. Suppose that $\delta(G) \ge a$ and

$$\max\{d_G(x), d_G(y)\} \ge \frac{(a-2)n}{a+b-4} + 2$$

for each pair of nonadjacent vertices x and y of $V(G)$. Then G has a Hamiltonian $[a,b]$-factor.

Theorem 2. ([13]). Let G be a connected graph of order n, a and b be integers such that $4 \le a < b$. Let g and f be positive integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for each $x \in V(G)$. Suppose that $n \ge \frac{(a+b-5)^2}{a-2}$. If $bind(G) \ge \frac{(a+b-5)(n-1)}{(a-2)n-3(a+b-5)}$, and for any nonempty independent subset X of $V(G)$, $|N_G(X)| \ge \frac{(b-3)n+(2a+2b-9)|X|}{a+b-5}$, then G has a Hamiltonian (g,f)-factor.

Liu and Zhang [14] proposed the following problem.

Problem. Find sufficient conditions for graphs to have connected $[a,b]$-factors related to other parameters in graphs such as binding number, neighborhood and connectivity.

We now show our main theorems which partially solve the above problem.

Theorem 3. Let $2 \le a < b$ be nonnegative integers, and let G be a Hamiltonian graph of order n with $n \ge \frac{(a+b-3)(2a+b-6)-a+2}{b-2}$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G) \qquad if \qquad |X| \ge \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor; \quad or$$

$$|N_G(X)| \ge \frac{a+b-3}{b-2}|X| \qquad if \qquad |X| < \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor.$$

Then G has a Hamiltonian $[a,b]$-factor.

Theorem 4. Let $2 \le a < b$ be nonnegative integers, and let G be a Hamiltonian graph of order n with $n \ge \frac{(a+b-3)(a+2b-7)-b+3}{a-1}$. Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for each $x \in V(G)$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G) \qquad if \qquad |X| \ge \left\lfloor \frac{(a-1)n}{a+b-3} \right\rfloor; \quad or$$

$$|N_G(X)| \ge \frac{a+b-3}{a-1}|X| \qquad if \qquad |X| < \left\lfloor \frac{(a-1)n}{a+b-3} \right\rfloor.$$

Then G has a Hamiltonian (g,f)-factor.

II. PROOF OF MAIN THEOREMS

The proofs of our main Theorems relies heavily on the following lemmas. Lemma 2.1 is a well-known necessary and sufficient for a graph to have a (g,f)-factor which was given by Lovasz. The following result is the special case which we use to prove our main theorems.

Lemma 2.1. ([15]). Let G be a graph, and let g and f be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) < f(x)$ for each $x \in V(G)$. Then G has a (g, f)-factor if and only if

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \geq 0$$

for any disjoint subsets S and T of $V(G)$.

Lemma 2.2. Let G be a graph of order n which satisfies the assumption of Theorem 3. Then $\delta(G) \geq \frac{(a-1)n+b-2}{a+b-3}$.

Proof. Let t be a vertex of G with degree $\delta(G)$. Let $X = V(G) \setminus N_G(t)$. Clearly, $t \notin N_G(X)$, then we have

$$(a+b-3)|X| \leq (b-2)|N_G(X)| \leq (b-2)(n-1).$$

Since $|X| = n - \delta(G)$, we obtain

$$(a+b-3)(n-\delta(G)) \leq (b-2)(n-1).$$

Thus, we get

$$\delta(G) \geq \frac{(a-1)n+b-2}{a+b-3}.$$

Completing the proof of Lemma 2.2.

Lemma 2.3. Let G be a graph of order n which satisfies the assumption of Theorem 4. Then $\delta(G) \geq \frac{(b-2)n+a-1}{a+b-3}$.

Proof. The proof is similar to that of Lemma 2.2.

Proof of Theorem 3. By assumption, G has a Hamiltonian cycle C. Let $G' = G - E(C)$. Note that $V(G') = V(G)$.

Clearly, G has the desired factor if and only if G' has an $[a-2, b-2]$-factor. By way of contradiction, we assume that G' has no $[a-2, b-2]$-factor. Then, by Lemma 2.1, there exist disjoint subsets S and T of $V(G')$ satisfying

$$\delta_{G'}(S,T) = (b-2)|S| + d_{G'-S}(T) - (a-2)|T| \leq -1. \quad (1)$$

We choose such subsets S and T so that $|T|$ is as small as possible.

The following claims hold.

Claim 1. $T \neq \emptyset$.

Proof. If $T = \emptyset$, then by (1) we have $-1 \geq \delta_{G'}(S,T) = (b-2)|S| \geq 0$. It is a contradiction.

Claim 2. $d_{G'-S}(x) \leq a-3$ for each $x \in T$.

Proof. If $d_{G'-S}(x) \geq a-2$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (1). This contradicts the choice of S and T.

Claim 3. $d_{G-S}(x) \leq d_{G'-S}(x) + 2 \leq a-1$ for each $x \in T$.

Proof. Note that $G' = G - E(C)$. Thus, we obtain by Claim 2

$$d_{G-S}(x) \leq d_{G'-S}(x) + 2 \leq a-1$$

for each $x \in T$.

Since $T \neq \emptyset$ by Claim 1, let

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

According to Claim 3, we get

$$0 \leq h \leq a-1.$$

We shall consider three cases according to the value of h and derive contradictions.

Case 1. $h = 0$.

Set $l = |\{t : t \in T,\ d_{G-S}(t) = 0\}|$. Clearly, $l \geq 1$. Let $X = V(G) \setminus S$. Then $N_G(X) \neq V(G)$ since $h = 0$. In terms of the assumption of the theorem, we obtain

$$n - l \geq |N_G(X)| \geq \frac{a+b-3}{b-2}|X| = \frac{a+b-3}{b-2}(n-|S|),$$

that is,

$$|S| \geq n - \frac{(b-2)(n-l)}{a+b-3}. \quad (2)$$

In view of Claim 3, $|S| + |T| \leq n$, $2 \leq a < b$, (1) and (2), we get

$$\begin{aligned}
-1 \quad &\geq \quad \delta_{G'}(S,T) = (b-2)|S| + d_{G'-S}(T) - (a-2)|T| \\
&\geq \quad (b-2)|S| + d_{G-S}(T) - a|T| \\
&\geq \quad (b-2)|S| + |T| - l - a|T| \\
&= \quad (b-2)|S| - (a-1)|T| - l \\
&\geq \quad (b-2)|S| - (a-1)(n-|S|) - l \\
&= \quad (a+b-3)|S| - (a-1)n - l \\
&\geq \quad (a+b-3)(n - \frac{(b-2)(n-l)}{a+b-3}) - (a-1)n - l \\
&= \quad (b-3)l \geq 0,
\end{aligned}$$

which is a contradiction.

Case 2. $h = 1$.

Subcase 2.1. $|T| > \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor$.

In this case, there exists $t \in T$ such that $d_{G-S}(t) = h = 1$. It is easy to see that

$$t \notin N_G(T \setminus N_G(t)). \quad (3)$$

According to $|T| > \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor$ and $d_{G-S}(t) = h = 1$, we obtain

$$|T \setminus N_G(t)| \geq |T| - 1 > \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor - 1.$$

In terms of the integrity of $|T \setminus N_G(t)|$, we get

$$|T \setminus N_G(t)| \geq \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor.$$

Combining this with the condition of the theorem, we have

$$N_G(T \setminus N_G(t)) = V(G),$$

that contradicts (3).

Subcase 2.2. $|T| \leq \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor$.

Let $m = |\{t : t \in T,\ d_{G-S}(t) = 1\}|$. Clearly, $m \geq 1$ and $|T| \geq m$. According to $h = 1$, $\delta(G) \leq |S| + h$ and Lemma 2.2, we obtain

$$|S| \geq \delta(G) - 1 \geq \frac{(a-1)n+b-2}{a+b-3} - 1 = \frac{(a-1)(n-1)}{a+b-3}. \quad (4)$$

Subcase 2.2.1. $|T| > \frac{(b-2)(n-1)}{a+b-3}$.

By (4), we have

$$|S| + |T| > \frac{(a-1)(n-1)}{a+b-3} + \frac{(b-2)(n-1)}{a+b-3} = n-1.$$

Using this and $|S| + |T| \leq n$, we get

$$|S| + |T| = n. \tag{5}$$

From (1), (5), Claim 3 and $|T| \leq \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor \leq \frac{(b-2)n}{a+b-3}$, we obtain

$$\begin{aligned} -1 &\geq \delta_{G'}(S,T) = (b-2)|S| + d_{G'-S}(T) - (a-2)|T| \\ &\geq (b-2)|S| + d_{G-S}(T) - a|T| \\ &\geq (b-2)|S| - (a-1)|T| \\ &= (b-2)(n-|T|) - (a-1)|T| \\ &= (b-2)n - (a+b-3)|T| \\ &\geq (b-2)n - (a+b-3) \cdot \frac{(b-2)n}{a+b-3} \\ &= 0, \end{aligned}$$

it is a contradiction.

Subcase 2.2.2. $|T| \leq \frac{(b-2)(n-1)}{a+b-3}$.

According to Claim 3, (4) and $|T| \geq m$, we have

$$\begin{aligned} \delta_{G'}(S,T) &= (b-2)|S| + d_{G'-S}(T) - (a-2)|T| \\ &\geq (b-2)|S| + d_{G-S}(T) - a|T| \\ &\geq (b-2)|S| + 2|T| - m - a|T| \\ &= (b-2)|S| - (a-2)|T| - m \\ &\geq (b-2) \cdot \frac{(a-1)(n-1)}{a+b-3} \\ &\quad -(a-2) \cdot \frac{(b-2)(n-1)}{a+b-3} - m \\ &= \frac{(b-2)(n-1)}{a+b-3} - m \\ &\geq |T| - m \geq 0. \end{aligned}$$

This contradicts (1).

Case 3. $2 \leq h \leq a-1$.

In terms of (1), Claim 3, $|S| + |T| \leq n$ and $a - h \geq 1$, we get that

$$\begin{aligned} -1 &\geq \delta_{G'}(S,T) = (b-2)|S| + d_{G'-S}(T) - (a-2)|T| \\ &\geq (b-2)|S| + d_{G-S}(T) - a|T| \\ &\geq (b-2)|S| + h|T| - a|T| \\ &= (b-2)|S| - (a-h)|T| \\ &\geq (b-2)|S| - (a-h)(n-|S|) \\ &= (a+b-h-2)|S| - (a-h)n. \end{aligned}$$

This inequality implies

$$|S| \leq \frac{(a-h)n-1}{a+b-h-2}. \tag{6}$$

From Lemma 2.2, $\delta(G) \leq |S| + h$ and (6), we obtain

$$\frac{(a-1)n+b-2}{a+b-3} \leq \delta(G) \leq |S| + h \leq \frac{(a-h)n-1}{a+b-h-2} + h. \tag{7}$$

If the LHS and RHS of (7) are denoted by A and B respectively, then (7) says that $A - B \leq 0$. But, after some rearranging, we find that

$$\begin{aligned} &(a+b-3)(a+b-h-2)(A-B) \\ = \; &(h-1)((b-2)n + a - 1 - (a+b-3)(a+b-h-2)) \\ &-(a+b-3)(a-2). \end{aligned} \tag{8}$$

Since $2 \leq h \leq a-1$ and $n \geq \frac{(a+b-3)(2a+b-6)-a+2}{b-2}$, it is easy to see that the expression in (8) is positive, and this contradicts (7).

From the contradictions we deduce that G' has an $[a-2, b-2]$-factor. This completes the proof of Theorem 3.

The proof of Theorem 4 is quite similar to that of Theorem 3 and is omitted.

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