

# Lower bound of time span product for a general class of signals in Fractional Fourier domain

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**Abstract**—Fractional Fourier Transform is a generalization of the classical Fourier Transform which is often symbolized as the rotation in time- frequency plane. Similar to the product of time and frequency span which provides the Uncertainty Principle for the classical Fourier domain, there has not been till date an Uncertainty Principle for the Fractional Fourier domain for a generalized class of finite energy signals. Though the lower bound for the product of time and Fractional Fourier span is derived for the real signals, a tighter lower bound for a general class of signals is of practical importance, especially for the analysis of signals containing chirps. We hence formulate a mathematical derivation that gives the lower bound of time and Fractional Fourier span product. The relation proves to be utmost importance in taking the Fractional Fourier Transform with adaptive time and Fractional span resolutions for a varied class of complex signals.

**Keywords**— Fractional Fourier Transform, Uncertainty Principle, Fractional Fourier span, amplitude, phase.

## I. INTRODUCTION

THE traditional Fourier transform decomposes the signal in terms of sinusoids, which are perfectly localized in frequency, but are not at all localized in time [1]. FrFT expresses the signal in terms of an orthonormal basis formed by linear chirps. Linear chirps are complex signals, whose instantaneous frequency varies linearly with time. The Kernel for continuous Fractional Fourier Transform is given by [2]

$$K_{\alpha}(t, u) = \sqrt{\frac{1 - j \cot \alpha}{2\pi}} e^{j \frac{t^2 + u^2}{2} \cot \alpha - j u t \csc \alpha}$$

Using this kernel of FrFT, the FRFT of signal  $x(t)$  with transform order ( $\alpha$ ) is computed as

$$X_{\alpha}(u) = \int_{-\infty}^{\infty} x(t) K_{\alpha}(t, u) dt$$

And  $x(t)$  can be recovered from the following equation,

$$x(t) = \int_{-\infty}^{\infty} X_{\alpha}(u) K_{-\alpha}(u, t) du$$

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## II. MATHEMATICAL DERIVATION FOR THE LOWER BOUND

It is known that the lower bound for the time frequency product for all classes of signals is given by the following:

$$\text{If } \sqrt{t} x(t) \longrightarrow 0 \text{ for } |t| \rightarrow \infty, \text{ then} \\ \Delta_t \Delta_{\omega} \geq \frac{1}{2} \text{ for a signal } x(t).$$

We start with such an assumption for deriving the relation which in other words reiterates that we perform the entire mathematical derivation for a general class of finite energy signals [3,4].

Since, we relate only the spans in time and fractional Fourier domain, which are about the mean, for simplicity, we take the mean value in both the domains as zero, i.e.

$$\langle t \rangle = \langle \omega \rangle = 0$$

Then, we have

$$\Delta_t^2 = \frac{1}{E} \int t^2 |x(t)|^2 dt \\ \Delta_u^2 = \frac{1}{E} \int u^2 |X(u)|^2 du$$

And thus,

$$\Delta_t^2 \Delta_u^2 = \frac{1}{E^2} \int t^2 |x(t)|^2 dt \cdot \int u^2 |X(u)|^2 du$$

We now want to represent everything in terms of  $x(t)$ . It is easy to establish that,

$$\int u^2 |X(u)|^2 du = \int u X(u) \cdot u X^*(u) du \\ = \int h(t) h^*(t) dt$$

With  $h(t) = F_{\alpha}^{-1} \{u X(u)\}$

Using the differentiation property in the Fractional Fourier domain, we can get

$$h(t) = \sin \alpha \left[ -j \frac{d}{dt} x(t) + t \cot \alpha \cdot x(t) \right]$$

Thus,

$$h^*(t) = \sin \alpha \left[ -j \frac{d}{dt} x^*(t) + t \cot \alpha \cdot x^*(t) \right]$$

Hence,

$$\begin{aligned} h(t) h^*(t) &= \sin^2 \alpha [x'(t)(x^*(t))' - jx'(t)t \cot \alpha x^*(t)] \\ &= \sin^2 \alpha \left[ \left| \frac{d}{dt} x(t) \right|^2 + t^2 \cot^2 \alpha |x(t)|^2 + \right. \\ &\quad \left. jt \cot \alpha [-2j\phi'(t)A^2(t)] \right] \end{aligned}$$

Where  $x(t) = A(t) \exp(j\phi(t))$ , with the  $A(t)$  being the real amplitude part and  $\phi(t)$  being the real phase of the signal.

$$\begin{aligned} &= \sin^2 \alpha \left[ \left| \frac{d}{dt} x(t) \right|^2 + |x(t)|^2 \{ t^2 \cot^2 \alpha + 2t \cot \alpha \phi'(t) \} \right] \\ &= \sin^2 \alpha \left| \frac{d}{dt} x(t) \right|^2 + t^2 \cos^2 \alpha |x(t)|^2 + \\ &\quad t \sin(2\alpha) \phi'(t) |x(t)|^2 \end{aligned}$$

Calling

$$\begin{aligned} \sin^2 \alpha \left| \frac{d}{dt} x(t) \right|^2 &\text{--- } x_1(t) \\ t^2 \cos^2 \alpha |x(t)|^2 &\text{--- } x_2(t) \\ t \sin(2\alpha) \phi'(t) |x(t)|^2 &\text{--- } x_3(t) \end{aligned}$$

We have,

$$h(t) h^*(t) = x_1(t) + x_2(t) + x_3(t)$$

Without any loss of generality, we proceed with the derivation of the lower bound for normalized energy signals, i.e.  $E = 1$ .

So, for calculating the lower bound [5],

$$\begin{aligned} \Delta_t^2 \Delta_u^2 &= \int t^2 |x(t)|^2 dt \cdot \int [x_1(t) + x_2(t) + x_3(t)] dt \\ &= \int t^2 |x(t)|^2 dt \cdot \int x_1(t) dt \\ &\quad + \int t^2 |x(t)|^2 dt \cdot \int x_2(t) dt \\ &\quad + \int t^2 |x(t)|^2 dt \cdot \int x_3(t) dt \\ &= I_1 + I_2 + I_3 \end{aligned}$$

Now,

$$\begin{aligned} I_1 &= \int t^2 |x(t)|^2 dt \cdot \int x_1(t) dt \\ &+ j[x^*(t)]' t \cot \alpha x(t) + t^2 \cot^2 \alpha x(t) x^*(t) \end{aligned}$$

$$\begin{aligned} &= \sin^2 \alpha \int |tx(t)|^2 dt \cdot \int \left| \frac{d}{dt} x(t) \right|^2 dt \\ &\geq \sin^2 \alpha \int |tx(t) \cdot \frac{d}{dt} x(t)|^2 dt \\ &\geq \sin^2 \alpha \left| -\frac{1}{2} \right|^2 \\ &\geq \frac{\sin^2 \alpha}{4} \end{aligned}$$

(with  $\sqrt{t}x(t) \rightarrow 0$  for  $|t| \rightarrow \infty$ )

$$\begin{aligned} I_2 &= \int t^2 |x(t)|^2 dt \cdot \int x_2(t) dt \\ &= \cos^2 \alpha \cdot \int |tx(t)|^2 dt \cdot \int |tx(t)|^2 dt \\ &\geq \cos^2 \alpha \left| \int tx(t) \cdot tx(t) dt \right|^2 \\ I_3 &= \int t^2 |x(t)|^2 dt \cdot \int x_3(t) dt \\ &= \int t^2 |x(t)|^2 dt \cdot \int \sin(2\alpha) t \phi'(t) |x(t)|^2 dt \\ &= \sin(2\alpha) \cdot \int t^2 A^2(t) dt \cdot \int t \phi'(t) A^2(t) dt \end{aligned}$$

Hence, the lower bound for the time- Fractional Fourier product can be given dependent on the amplitude and phase functions of a unit energy signal as follows:

For  $\sqrt{t}x(t) \rightarrow 0$  for  $|t| \rightarrow \infty$ ,

$$\begin{aligned} \Delta_t^2 \Delta_u^2 &\geq \frac{\sin^2 \alpha}{4} + \cos^2 \alpha \left| \int t^2 A^2(t) dt \right|^2 \\ &+ \sin(2\alpha) \int t^2 A^2(t) dt \cdot \int t \phi'(t) A^2(t) dt \end{aligned}$$

### III. ILLUSTRATIONS

It is easy to see that the derived expression satisfies the Uncertainty Principle for all the signals in the frequency domain keeping  $\alpha = \pi/2$ .

Let us now consider a complex time domain signal, such as

$$x(t) = \left(\frac{1}{\pi}\right)^{\frac{1}{4}} \cdot e^{-\frac{1}{2}t^2} \cdot e^{j(t^2+t+1)}$$

which is a linear chirp with a Gaussian envelope of unit energy. The signal is time limited from -4 to 4 and hence is finite energy and thus satisfies the starting condition for the lower bound of the time-span product.

We have, as functions of the transform order,

$$\begin{aligned} \Delta_t^2 \Delta_u^2 &\geq \\ 0.25 * \sin(\alpha)^2 \end{aligned}$$

$$\begin{aligned}
& + \\
& \cos(a)^2 * (-2.2567 * \exp(16) + 0.282 * \pi^{1/2} * \operatorname{erf}(4)) \\
& + \\
& \sin(2*a) * \\
& [-2.2567 * \exp(-16) + 0.282 * \pi^{1/2} * \operatorname{erf}(4)] * \\
& (-4.513 * \exp(-16) + 0.5641 * \pi^{1/2} * \operatorname{erf}(4))
\end{aligned}$$

Where  $\operatorname{erf}(\cdot)$  is the error function and  $\operatorname{erf}(4)$  has a value of 1.

Thus, we are able to find out the relation between the time span of a damping chirp and the span at any transform order domain.

The expression derived in Section II finds its utmost importance in the case where one needs to define a trade-off between the time resolution and the resolution at any transform order domain. The different resolutions in different domains are important for extracting the maximum amount of information with perfect reconstruction accuracy [6]. Very often, we come across the concept of adaptive window switching in audio coding applications in order to adapt the time and frequency resolutions as per the content of the signal. With a similar requirement in the Fractional Fourier domain, it becomes natural to alter the time resolution of the signal with varying window lengths and shapes and then calculating the lower bound of the time-span product for a certain transform order. Since the derived expression relates the span in time and any transform order domain, such an adaptive process becomes easier to implement for any class of finite energy signals.

#### IV. CONCLUSION

The lower bound for the time-Fractional Fourier span product is derived with dependence on the amplitude and phase function of the signal, for a generalized class of signals. The expression is useful for finding the uncertainty relation for a complex signal between time domain and any transform order domain. Also, the expression is advantageous for adaptive switching between domain resolutions.

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