

# A Multi-period Profit Maximization Policy for a Stochastic Demand Inventory System with Upward Substitution

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**Abstract**—This paper deals with a periodic-review substitutable inventory system for a finite and an infinite number of periods. Here an upward substitution structure, a substitution of a more costly item by a less costly one, is assumed, with two products. At the beginning of each period, a stochastic demand comes for the first item only, which is quality-wise better and hence costlier. Whenever an arriving demand finds zero inventory of this product, a fraction of unsatisfied customers goes for its substitutable second item. An optimal ordering policy has been derived for each period. The results are illustrated with numerical examples. A sensitivity analysis has been done to examine how sensitive the optimal solution and the maximum profit are to the values of the discount factor, when there is a large number of periods.

**Keywords**—Multi-period model, inventory, random demand, upward substitution.

**AMS 2000 Subject Classification**—Primary 62L99, 90B05; Secondary 90B50

## I. INTRODUCTION

**I**N this paper we present a multi-period inventory model where two products of different qualities are considered.

The products are priced separately. They are two qualities of the same type of goods so that substitution is possible. Here an upward substitution is assumed, that is, the lower-quality, cheaper product substitutes the higher quality, costlier product. The basic purpose of any inventory policy is to control production or distribution in such a way that the supply is matched to demand to the greatest possible extent in order to maximize the expected profit. In most of the practical situations, the demand is not deterministic, and hence a decision about an optimal order quantity (production quantity, in case of production inventory model) is very much required. For expensive products, a huge cost may incur due to an excess stock. An inventory manager must maintain a balance between having too much stock (and thereby incurring cost due to excess stock) and an insufficient stock (and thereby incurring lost sales and goodwill costs). To protect business from these situations, the manager may wish to carry some cheaper, lower quality product as well to prevent the entire loss, in case of stock-out of the costlier item. This paper

discusses such a model where there are two products, one of which is better in quality and costlier than the other. A random demand arrives for the better product (product 1) only. Though it is more expensive, the demanding customers prefer to buy it, as they consider it to be worth the price. The substitute can only fulfill the basic need for the item, in case the better one is unavailable. A fraction of unsatisfied customers leaves without buying anything, when they do not find their first choice product, and the remaining proportion of unmet demand goes for buying the substitute (product 2). The profit levels are also assumed to be different for the two products. The costlier item is more profitable than its substitute. The model is solved here for a number of planning periods.

It has been found in literature that much works have been done on substitution of commodities, but mostly they have been on full substitution or downward substitution. A one-way product substitution in downward direction has been assumed in a production inventory model in Duenyas and Tsai [1] where customers who require a given product are satisfied by a higher quality product at the same price. Bassok et al. [2] formulated an optimal policy analyzing the one-way substitution model for the manufacturers and discussed the effects of substitution and the significant gain that can be achieved when substitution is considered. A downward substitution has been discussed in the context of semiconductor chips in Hsu and Bassok[3], where a faster processor can be substituted for a slower processor, in the context of memory chips in Leachman [4], where a higher capacity (4 MB) chip can be used to satisfy demands for lower capacity memory chip (2 MB), and in the context of steel industry where Wagner and Whitin [5] considered that the steel beams of greater strength can substitute for beams of lesser strength. Our model considers an upward substitution in contrast to those models.

We observe a two-way substitution in blood inventory models. Civelek et al. [6] in their model, included upward and downward substitution, in which old blood platelets could be given to new blood platelet demand in upward substitution, and vice versa in downward substitution. According to the industry example in Haijema et al. [7] from a Dutch blood bank, upward substitution is very common in practice. An upward substitution occurs in many real-world decision problems. A better quality product is more desirable than a lesser quality one for luxury items, fashion goods, some

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electronic and microelectronic goods, or some electrical components where reliability is the most important criterion, automotive paints and many more. The automotive industries need the alkyd resin-based paint, a high-premium paint that is used for painting ships, railway coaches etc., but if it is unavailable, they might opt for a second grade paint, which has less shelf life. If a professional photographer wishes to buy the best camera, or accessories, he will probably have to buy a lower-quality one in case of stock-out, in spite of the fact that he will not be happy to do so. Stuer et al. [8] discussed the substitutable grid resources where they introduced 'slow' and 'fast' CPUs where the model was coupled with potentially strong shifts in demand. The substitutability of electrical components (batteries, resistors, transistors etc.) and microelectronics in the Canadian Forces Supply System (CFSS) was studied in Ng and Lam [9]. The importance of substitutability between foreign and domestic farm and non-farm commodities was discussed in Sumner and Alston[10] in the context of the demand for US tobacco in cigarette manufacturing, treating the demand for US farm goods as a derived demand by the processing industries. Pasternack and Drezner [11] developed a model in which they assumed that both the items can be used as substitute for each other, and substitution will occur with probability one. In this paper, the probability of substitution is assumed to be less than one, which means that there is also a non-zero chance of not buying the substitute, even when the product 1 is unavailable. They discussed a single-period model only. Parlar and Goyal [12], Mukherjee and Roychowdhury [13] also modeled a two substitutable product problem in which a single-period was considered. This paper discusses the optimal ordering policy for a multi-period model with a finite and an infinite number of periods. For determining the optimal stocking levels of the two products in each period we maximize the expected profit function.

## II. ASSUMPTIONS AND NOTATION

An  $N$ -period ( $N < \infty$ ) inventory model is considered here with two substitutable commodities. In each period, the demand comes only for product 1. If it is out of stock, only then a fraction of unsatisfied customers goes for its substitute, the product 2. The results derived here hold under the following set of assumptions and notation:

1. The demand  $X_j$  at period  $j$ ,  $j=1,2,\dots,N$ , is a continuous random variable, and  $X_1, X_2, \dots, X_N$  are *i.i.d.* with common

c.d.f.  $F(\cdot)$ .

2. No backlogging of demand is allowed.

3. There is no lead time.

4. Substitution is only performed in a period if there is a stock-out of product 1.

5.  $\alpha$  is the probability (proportion) of unsatisfied customers buying product 2, the substitute, when product 1 is out of stock.  $0 < \alpha < 1$ .  $\bar{\alpha} = 1 - \alpha$ .

6.  $C_i$  is the unit cost of product  $i$ ,  $i = 1, 2$ ,  $C_1 > C_2$ .

7.  $S$  is the shortage (lost sales) cost per unit short.

8.  $P_i$  is selling price per unit of product  $i$ ,  $i = 1, 2$ ,  $P_1 > P_2$ . The substitute is sold at its own price, in the event of product 2 substituting product 1.

9. Profit per unit of product 1 is more than the profit per unit of product 2, *i.e.*,  $(P_1 - C_1) > (P_2 - C_2)$ .

## III. OPTIMAL ORDERING POLICY

Let  $\Delta_n(q_1, q_2)$  be the expected profit from an optimal procedure in a process when  $n$  periods remain ( $n = 0, 1, 2, \dots, N$ ), where  $q_i$  be the number of units of item  $i$  ( $i = 1, 2$ ) in stock before ordering more units. Let  $\tau_i$ ,  $i = 1, 2$ , be the order quantities for product  $i$  required to bring the stock level of the product  $i$  to  $(q_i + \tau_i)$ . First, we establish a recursion relation for  $\Delta_n(q_1, q_2)$ , and then prove the concavity of the expected profit function, which helps determine the optimal stock-heights for the last period first and finally for each of the periods.

The following recursion relation holds in relation to  $\Delta_n(q_1, q_2)$ , where  $X$  denotes the random demand.

$$\begin{aligned} \Delta_n(q_1, q_2) &= \sup_{\tau_1, \tau_2 \geq 0} [-C_1\tau_1 - C_2\tau_2 \\ &+ \int_0^{q_1 + \tau_1} \{P_1x + \Delta_{n-1}(q_1 + \tau_1 - x, q_2 + \tau_2)\} dF(x) \\ &+ \int_{(q_1 + \tau_1) + \frac{(q_2 + \tau_2)}{\alpha}}^{\frac{(q_2 + \tau_2)}{\alpha}} \{P_1(q_1 + \tau_1) + P_2\alpha(x - q_1 - \tau_1) - S\bar{\alpha}(x - q_1 - \tau_1) \\ &+ \Delta_{n-1}(0, (q_2 + \tau_2) - \alpha(x - q_1 - \tau_1))\} dF(x) \\ &+ \int_{(q_1 + \tau_1) + \frac{(q_2 + \tau_2)}{\alpha}}^{\infty} \{P_1(q_1 + \tau_1) + P_2(q_2 + \tau_2) \\ &- S(x - q_1 - \tau_1 - q_2 - \tau_2) + \Delta_{n-1}(0, 0)\} dF(x)] \\ &= \sup_{\tau_1, \tau_2 \geq 0} \phi_n(\tau_1, \tau_2), \text{ say.} \end{aligned} \quad (1)$$

Let  $\Delta_0(q_1, q_2) = 0$ , for all  $q_1, q_2 > 0$ . For  $n = 1$ , equation (1) reduces to

$$\begin{aligned} \Delta_1(q_1, q_2) &= \sup_{\tau_1, \tau_2 \geq 0} \phi_1(\tau_1, \tau_2) \\ &= \sup_{\tau_1, \tau_2 \geq 0} [-C_1\tau_1 - C_2\tau_2 \\ &+ P_1 \int_0^{q_1+\tau_1} x dF(x) + \int_{(q_1+\tau_1)}^{(q_1+\tau_1)+\frac{(q_2+\tau_2)}{\alpha}} \{P_1(q_1+\tau_1) + (P_2\alpha - S\bar{\alpha})(x - q_1 - \tau_1)\} dF(x) \\ &+ \int_{(q_1+\tau_1)+\frac{(q_2+\tau_2)}{\alpha}}^{\infty} \{(P_1+S)(q_1+\tau_1) + (P_2+S)(q_2+\tau_2) - Sx\} dF(x)]. \end{aligned}$$

Now we prove that the expected profit function  $\phi_1(\tau_1, \tau_2)$  is a concave function of  $\tau_1, \tau_2$ .

Write  $q_1^* = q_1 + \tau_1$ ,  $q_2^* = q_2 + \tau_2$  and  $q_\alpha^* = q_1^* + \frac{q_2^*}{\alpha}$ .

Then we have

$$\frac{\partial \phi_1(\tau_1, \tau_2)}{\partial \tau_1} = (P_1 + S - C_1) - \alpha(P_2 + S)F(q_\alpha^*) - [(P_1 + S) - \alpha(P_2 + S)]F(q_1^*),$$

$$\frac{\partial \phi_1(\tau_1, \tau_2)}{\partial \tau_2} = (P_2 + S)(1 - F(q_\alpha^*)) - C_2,$$

$$\frac{\partial^2 \phi_1(\tau_1, \tau_2)}{\partial \tau_1^2} = -\alpha(P_2 + S)f(q_\alpha^*) - [(P_1 + S) - \alpha(P_2 + S)]f(q_1^*),$$

$$\frac{\partial^2 \phi_1(\tau_1, \tau_2)}{\partial \tau_2^2} = -\frac{(P_2 + S)}{\alpha} f(q_\alpha^*).$$

It is clear that  $\frac{\partial^2 \phi_1(\tau_1, \tau_2)}{\partial \tau_1^2}$  and  $\frac{\partial^2 \phi_1(\tau_1, \tau_2)}{\partial \tau_2^2}$  are both negative and the determinant of the Hessian matrix  $\Delta$  of  $\phi_1(\tau_1, \tau_2)$  is positive, where the determinant of  $\Delta$  is as follows:

$$\begin{aligned} |\Delta| &= \begin{vmatrix} \frac{\partial^2 \phi_1(\tau_1, \tau_2)}{\partial \tau_1^2} & \frac{\partial^2 \phi_1(\tau_1, \tau_2)}{\partial \tau_1 \partial \tau_2} \\ \frac{\partial^2 \phi_1(\tau_1, \tau_2)}{\partial \tau_1 \partial \tau_2} & \frac{\partial^2 \phi_1(\tau_1, \tau_2)}{\partial \tau_2^2} \end{vmatrix} \\ &= \frac{(P_2 + S)}{\alpha} [(P_1 + S) - \alpha(P_2 + S)] f(q_1^*) f(q_\alpha^*), \end{aligned}$$

which is positive. Hence  $\Delta$  is negative definite and the concavity of  $\phi_1(\tau_1, \tau_2)$  is established, vide Roberts and Varberg [14].

Because of the concavity of  $\phi_1(\tau_1, \tau_2)$ ,  $\frac{\partial \phi_1(\tau_1, \tau_2)}{\partial \tau_1} = 0$  and

$$\frac{\partial \phi_1(\tau_1, \tau_2)}{\partial \tau_2} = 0$$

provide the optimal values of the order

quantities,  $\tau_1$  and  $\tau_2$ , which maximize the expected profit. Thus we can write the following: Given  $q_i$ , the stock of product  $i$ , at the beginning of the last period, the order-quantities of product  $i$ , is given by

$$q_{iN} = \max(\tau_i, 0), i = 1, 2,$$

$$\text{or, } q_{iN} = \max(q_i^* - q_i, 0), i = 1, 2,$$

where  $q_i^*$ ,  $i = 1, 2$ , can be obtained by solving the following equations:

$$F(q_1^*) = 1 - \frac{C_1 - \alpha C_2}{(P_1 + S) - \alpha(P_2 + S)} \tag{2}$$

$$F(q_\alpha^*) = 1 - \frac{C_2}{(P_2 + S)}, \tag{3}$$

where  $q_\alpha^* = q_1^* + \frac{q_2^*}{\alpha}$ .

#### IV. MAIN RESULT

Now the question may arise as to how the values of the optimal stock-heights  $(q_1^*, q_2^*)$  would change, if the initial stock levels  $(q_1, q_2)$  change. If  $(q_1, q_2)$  change, the optimal values of  $(\tau_1, \tau_2)$  will obviously change. From the following theorem it is evident that there will be no change in the values of optimal stock-heights  $(q_1^*, q_2^*)$  whatever the values of the initial stock levels  $(q_1, q_2)$  be. Accordingly we can determine the optimal order quantities as a difference between optimal stock-height and the initial stock of the period under consideration.

**Theorem 1.** For an  $N$ -period inventory problem defined in the earlier sections, given  $q_{in}$ , the initial stock level of product  $i$  when  $n$  periods remain,  $i = 1, 2$ , the optimal order-quantities are given by

$$\max[(q_{in}^* - q_{in}), 0], n = 1, 2, \dots, N, i = 1, 2,$$

where  $0 \leq q_{i1}^* \leq q_{i2}^* \leq \dots \leq q_{i,(N-1)}^* \leq q_{i,N}^*$ , and  $q_{in}^*$  is the optimal stock-height of product  $i$ , when  $n$  periods remain,  $i = 1, 2$ .

**Proof.** Following the line of De Groot [15] the theorem is proved.

Let us write

$$\Delta_n(q_1, q_2) = \sup_{\tau_1, \tau_2 \geq 0} [-C_1\tau_1 - C_2\tau_2 + \Delta_n(q_1 + \tau_1, q_2 + \tau_2)] \tag{4}$$

For each  $(q_1, q_2)$ , let  $\tau_i = \tau_{in}(q_i)$ ,  $i = 1, 2$ , be a value of  $\tau_i$  for which the supremum in equation (4) is attained. Suppose,

$\tau_{in}(q_i) > 0$ , for some initial stock level  $q_i = q_{i0}$ ,  $i = 1, 2$ . The optimal stock-height will be  $q_{in}^* = q_{i0} + \tau_{in}(q_{i0})$ .

The possible ranges of values of initial stock-levels  $(q_1, q_2)$  are as under:

- (i)  $q_{10} \leq q_1 \leq q_{1n}^*, q_{20} \leq q_2 \leq q_{2n}^*$
- (ii)  $q_{10} - \tau_{1n}(q_{10}) \leq q_1 \leq q_{10}, q_{20} - \tau_{2n}(q_{20}) \leq q_2 \leq q_{20}$
- (iii)  $q_{10} - \tau_{1n}(q_{10}) \leq q_1 \leq q_{10}, q_{20} \leq q_2 \leq q_{2n}^*$
- (iv)  $q_{10} \leq q_1 \leq q_{1n}^*, q_{20} - \tau_{2n}(q_{20}) \leq q_2 \leq q_{20}$ .

We proceed until  $q_{i0} - k_i \tau_{in}(q_{i0})$  becomes negative,  $q_{i0} - k_i \tau_{in}(q_{i0}) \leq q_i \leq q_{i0}$ ,  $i = 1, 2$ ,  $k_i = 1, 2, \dots$

Then consider  $0 \leq q_i \leq q_{i0}$ ,  $i = 1, 2$ . Now

(i) let  $\tau_i = \tau'_i - (q_i - q_{i0})$ ,  $i = 1, 2$ .

The equation (4) then can be rewritten as

$$\Delta_n(q_1, q_2) = \sup_{\substack{\tau'_1 \geq (q_1 - q_{10}) \\ \tau'_2 \geq (q_2 - q_{20})}} [\Delta_n(q_{10} + \tau'_1, q_{20} + \tau'_2) - C_1 \tau'_1 - C_2 \tau'_2 + C_1(q_1 - q_{10}) + C_2(q_2 - q_{20})]. \quad (5)$$

The supremum in equation (5) is attained when  $\tau'_i = \tau_{in}(q_{i0})$ , by definition.

Hence  $\tau_i = \tau_{in}(q_{i0}) - (q_i - q_{i0})$ .

So,  $q_i + \tau_{in}(q_i) = q_{in}^*$  for  $q_{i0} \leq q_i \leq q_{in}^*$ ,  $i = 1, 2$ .

Now we widen the interval of the initial values  $(q_1, q_2)$ .

(ii) Consider any value  $q_i$  such that  $q_{i0} - \tau_{in}(q_{i0}) \leq q_i \leq q_{i0}$ ,  $i = 1, 2$ , or,  $0 \leq q_i \leq q_{i0}$  (if  $q_{i0} - \tau_{in}(q_{i0}) < 0$ ).

Since  $\tau_{in}(q_i)$  can be chosen as a non-increasing function of  $q_i$ , equation (4) can be rewritten as

$$\begin{aligned} \Delta_n(q_1, q_2) &= \sup_{\substack{\tau_1 \geq \tau_{1n}(q_{10}) \\ \tau_2 \geq \tau_{2n}(q_{20})}} [-C_1 \tau_1 - C_2 \tau_2 + \Delta_n(q_1 + \tau_1, q_2 + \tau_2)] \\ &= \sup_{\substack{\tau'_1 \geq q_1 - [q_{10} - \tau_{1n}(q_{10})] \\ \tau'_2 \geq q_2 - [q_{20} - \tau_{2n}(q_{20})]}} [\Delta_n(q_{10} + \tau'_1, q_{20} + \tau'_2) - C_1 \tau'_1 - C_2 \tau'_2 - C_1(q_{10} - q_1) - C_2(q_{20} - q_2)]. \end{aligned}$$

Here the supremum is attained when  $\tau'_i = \tau_{in}(q_i)$ , i.e., when  $\tau_i = \tau_{in}(q_{i0}) - (q_i - q_{i0})$ . Hence  $q_i + \tau_{in}(q_i) = q_{in}^*$  holds also for this interval of  $q_i$ . Thus the result  $q_i + \tau_{in}(q_i) = q_{in}^*$  is established for all initial values of  $q_i$  satisfying  $q_{i0} - \tau_{in}(q_{i0}) \leq q_i \leq q_{i0}$ ,  $i = 1, 2$ .

Next we consider the initial values  $q_i$  in the interval  $q_{i0} - 2\tau_{in}(q_{i0}) \leq q_i \leq q_{i0} - \tau_{in}(q_{i0})$  and proceed similarly until  $q_i + \tau_{in}(q_i) = q_{in}^*$  is correct for all values of  $q_i$  in the interval  $0 \leq q_i \leq q_{in}^*$ ,  $i = 1, 2$ .

(iii) Let  $\tau_1 = \tau'_1 - (q_1 - q_{10}), \tau_2 = \tau'_2 - (q_2 - q_{20})$  (here  $\tau_1$  will be at least  $\tau_{1n}(q_{10})$ ).

Thus

$$\Delta_n(q_1, q_2) = \sup_{\substack{\tau'_1 \geq q_1 - [q_{10} - \tau_{1n}(q_{10})] \\ \tau'_2 \geq (q_2 - q_{20})}} [\Delta_n(q_{10} + \tau'_1, q_{20} + \tau'_2) - C_1 \tau'_1 - C_2 \tau'_2 - C_1(q_{10} - q_1) - C_2(q_{20} - q_2)]. \quad (6)$$

The supremum in equation (6) is attained when  $\tau'_i = \tau_{in}(q_{i0})$ ,  $i = 1, 2$ , i.e., when  $\tau_1 = \tau_{1n}(q_{10}) - (q_1 - q_{10})$  and  $\tau_2 = \tau_{2n}(q_{20}) - (q_2 - q_{20})$ .

Hence  $q_1 + \tau_{1n}(q_1) = q_{1n}^*$  for  $q_{10} - \tau_{1n}(q_{10}) \leq q_1 \leq q_{10}$  and  $q_2 + \tau_{2n}(q_2) = q_{2n}^*$  for  $q_{20} \leq q_2 \leq q_{20}$ .

Similarly this can be proved for case (iv).

Further, clearly  $q_{in}^* \leq q_{i(n+1)}^*$ ,  $n = 1(1)N-1$ ,  $i = 1, 2$ , the larger the number of remaining periods, the larger is the stock which is likely to be sold out in subsequent periods. Hence we may write  $0 \leq q_{i1}^* \leq q_{i2}^* \leq \dots \leq q_{i(N-1)}^* \leq q_{iN}^*$ . ■

Here we illustrate the method for obtaining the optimal stock-heights, as well as the optimal order quantities, for both the products for a two-period model, i.e., for  $N = 2$ . First we determine the optimal stock-heights  $(q_1^*, q_2^*)$  of the two items for the last period (second period, here). As shown in equations (2) and (3), the optimal values  $(q_1^*, q_2^*)$  of  $(q_1, q_2)$  can be obtained from:

$$F(q_1^*) = 1 - \frac{C_1 - \alpha C_2}{(P_1 + S) - \alpha(P_2 + S)}$$

and

$$F(q_2^*) = 1 - \frac{C_2}{(P_2 + S)}, \quad q_2^* = q_1^* + \frac{q_2^*}{\alpha}.$$

Hence at the beginning of second period, if initial stock of product  $i$  be  $q_{i2}$ , we have to order  $\max[(q_i^* - q_{i2}), 0]$  units of product  $i$ ,  $i = 1, 2$ , by Theorem 1.

Now, for the first period, let us denote the optimal stock-height of product  $i$  by  $Q_i$ ,  $i = 1, 2$ . Next we determine  $Q_1$  and  $Q_2$  as follows:

We may write  $Q_i = q_i + \tau_i$ ,  $i = 1, 2$  and  $Q_\alpha = Q_1 + \frac{Q_2}{\alpha}$ .

Then

$$\begin{aligned} \Delta_2(q_1, q_2) &= \sup_{Q_1, Q_2 \geq 0} \phi_2(Q_1, Q_2) \\ &= \sup_{Q_1, Q_2 \geq 0} [-C_1(Q_1 - q_1) - C_2(Q_2 - q_2)] \\ &+ \int_0^{Q_1 - q_1^*} \{P_1 x + \Delta_1(Q_1 - x, Q_2)\} dF(x) \\ &+ \int_{Q_1 - q_1^*}^{Q_1} \{P_1 x + \Delta_1(Q_1 - x, Q_2)\} dF(x) \\ &+ \int_{Q_1}^{Q_\alpha - \frac{q_2^*}{\alpha}} \{P_1 Q_1 + \alpha P_2(x - Q_1) - S\bar{\alpha}(x - Q_1) \\ &\quad + \Delta_1(0, Q_2 + \alpha(x - Q_1))\} dF(x) \\ &+ \int_{Q_\alpha - \frac{q_2^*}{\alpha}}^{Q_\alpha} \{P_1 Q_1 + \alpha P_2(x - Q_1) \\ &\quad - S\bar{\alpha}(x - Q_1) + \Delta_1(0, Q_2 + \alpha(x - Q_1))\} dF(x) + \\ &+ \int_{Q_\alpha}^{\infty} \{P_1 Q_1 + P_2 Q_2 - S(x - Q_1 - Q_2) + \Delta_1(0, 0)\} dF(x), \end{aligned}$$

where

$$\begin{aligned} \Delta_1(Q_1 - x, Q_2) &= P_1 \left[ \int_0^{Q_1 - x} y dF(y) + (Q_1 - x)(1 - F(Q_1 - x)) \right] \\ &+ P_2 \left[ \alpha \int_{Q_1 - x}^{Q_2 - x} (y - Q_1 + x) dF(y) + Q_2(1 - F(Q_\alpha - x)) \right] \\ &- S[\bar{\alpha} \int_{Q_1 - x}^{Q_\alpha - x} (y - Q_1 + x) dF(y) + \int_{Q_\alpha - x}^{\infty} (y - Q_1 - Q_2 + x) dF(y)] \\ &\quad \text{if } 0 < x < (Q_1 - q_1^*). \end{aligned}$$

$$\begin{aligned} \Delta_1(Q_1 - x, Q_2) &= -C_1(q_1^* - Q_1 + x) + P_1 \left[ \int_0^{q_1^*} y dF(y) + q_1^*(1 - F(q_1^*)) \right] \\ &+ P_2 \left[ \alpha \int_{q_1^*}^{q_1^* + \frac{Q_2}{\alpha}} (y - q_1^*) dF(y) + Q_2(1 - F(q_1^* + \frac{Q_2}{\alpha})) \right] \\ &- S[\bar{\alpha} \int_{q_1^*}^{q_1^* + \frac{Q_2}{\alpha}} (y - q_1^*) dF(y) \\ &+ \int_{q_1^* + \frac{Q_2}{\alpha}}^{\infty} (y - q_1^* - Q_2) dF(y)], \quad \text{if } (Q_1 - q_1^*) < x < Q_1. \\ \Delta_1(0, Q_2 - \alpha(x - Q_1)) &= -C_1 q_1^* + P_1 \left[ \int_0^{q_1^*} y dF(y) + q_1^*(1 - F(q_1^*)) \right] \\ &+ P_2 \left[ \alpha \int_{q_1^*}^{q_1^* + Q_\alpha - x} (y - q_1^*) dF(y) + \alpha(Q_\alpha - x)(1 - F(q_1^* + Q_\alpha - x)) \right] \\ &- S[\bar{\alpha} \int_{q_1^*}^{q_1^* + Q_\alpha - x} (y - q_1^*) dF(y) + \int_{q_1^* + Q_\alpha - x}^{\infty} (y - q_1^* - \alpha(Q_\alpha - x)) dF(y)], \\ &\quad \text{if } Q_1 < x < Q_\alpha - \frac{q_2^*}{\alpha}. \\ \Delta_1(0, Q_2 - \alpha(x - Q_1)) &= -C_1 q_1^* - C_2 \{q_2^* - Q_2 + \alpha(x - Q_1)\} \\ &+ P_1 \left[ \int_0^{q_1^*} y dF(y) + q_1^*(1 - F(q_1^*)) \right] \\ &+ P_2 \left[ \alpha \int_{q_1^*}^{q_\alpha^*} (y - q_1^*) dF(y) + q_2^*(1 - F(q_\alpha^*)) - S[\bar{\alpha} \int_{q_1^*}^{q_\alpha^*} (y - q_1^*) dF(y)] \right] \\ &+ \int_{q_\alpha^*}^{\infty} (y - q_1^* - q_2^*) dF(y)], \quad q_\alpha^* = q_1^* + \frac{q_2^*}{\alpha}, \\ &\quad \text{if } Q_\alpha - \frac{q_2^*}{\alpha} < x < Q_\alpha. \end{aligned}$$

and

$$\begin{aligned} \Delta_1(0,0) &= -C_1q_1^* - C_2q_2^* \\ &+ P_1 \left[ \int_0^{q_1^*} y dF(y) + q_1^* (1 - F(q_1^*)) \right] \\ &+ P_2 \left[ \alpha \int_{q_1^*}^{q_\alpha^*} (y - q_1^*) dF(y) + q_2^* (1 - F(q_\alpha^*)) \right] \\ &- S \left[ \bar{\alpha} \int_{q_1^*}^{q_\alpha^*} (y - q_1^*) dF(y) \right. \\ &\quad \left. + \int_{q_\alpha^*}^{\infty} (y - q_1^* - q_2^*) dF(y) \right], \quad q_\alpha^* = q_1^* + \frac{q_2^*}{\alpha}. \end{aligned}$$

In this situation,  $\frac{\partial \phi_2(Q_1, Q_2)}{\partial Q_1} = 0$  and  $\frac{\partial \phi_2(Q_1, Q_2)}{\partial Q_2} = 0$  simultaneously give the optimal solution for  $(Q_1, Q_2)$ . Hence the equations giving optimal solutions are:

$$\begin{aligned} &\int_0^{Q_1 - q_1^*} F(Q_1 - x) dF(x) \\ &+ \frac{(P_1 + S - C_1) - (P_2 + S)F(q_1^* + \frac{Q_2}{\alpha})}{(P_1 + S) - \alpha(P_2 + S)} \times [F(Q_1) - F(Q_1 - q_1^*)] \\ &= \left[ 1 - \frac{C_1 - \alpha C_2}{(P_1 + S) - \alpha(P_2 + S)} \right] \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\int_0^{Q_1 - q_1^*} F(Q_\alpha - x) dF(x) \\ &+ \int_0^{Q_\alpha - q_2^*/\alpha} F(q_1^* + Q_\alpha - x) dF(x) + F(q_1^* + \frac{Q_2}{\alpha}) [F(Q_1) - F(Q_1 - q_1^*)] \\ &+ \left[ 1 - \frac{C_2}{P_2 + S} \{ F(Q_\alpha) - F(Q_\alpha - \frac{q_2^*}{\alpha}) \} \right] \\ &= \left[ 1 - \frac{C_2}{P_2 + S} \right]. \end{aligned} \tag{8}$$

Solving equations (7) and (8) we get the optimal solution for  $Q_1$  and  $Q_2$ , the optimal stock-heights for the first period.

The following numerical example illustrates the method:

*Example 1:* Suppose that the demand distribution is uniform with a minimum of 100 and a maximum of 300. Let  $C_1 = 5$ ,  $C_2 = 2$ ,  $P_1 = 8$ ,  $P_2 = 3$  and  $S = 1$ . If the probability that a customer will purchase the substitute when item 1 is out of stock is 0.5, i.e.,  $\alpha = 0.5$ , then using equations (2) and (3) we get the optimal stock-heights for the two products for the second

period, and from equations (7) and (8) we get the optimal stock-heights for the first period.

The optimal stock-heights for the two products are, respectively, 295 and 35 for the first period, and 186 and 7 for the second period (all quantities are in appropriate units).

### V. OPTIMAL ORDERING POLICY FOR AN INFINITE NUMBER OF PERIODS

In this section we discuss a sequential procedure for determining the optimal order quantities for each period which will maximize the total expected profit when the number of periods is infinite. Here a discount factor  $\beta$  is introduced [De Groot [15], pp. 408-410] to bring the costs or profit to their present values.

The model is such that at the beginning of each period we order that much units which bring the stock-levels of the two products to  $(q_1, q_2)$ . To obtain the optimal stock-heights, we write the total expected profit with the initial stock levels  $(q_1, q_2)$  as

$$\begin{aligned} \Delta(q_1, q_2) &= \sup_{\tau_1, \tau_2 \geq 0} [-C_1\tau_1 - C_2\tau_2 \\ &+ \int_0^{q_1 + \tau_1} \{ P_1x + \beta\Delta(q_1 + \tau_1 - x, q_2 + \tau_2) \} dF(x) \\ &+ \int_{(q_1 + \tau_1) + \frac{(q_2 + \tau_2)}{\alpha}}^{(q_1 + \tau_1) + \frac{(q_2 + \tau_2)}{\alpha}} \{ P_1(q_1 + \tau_1) + P_2\alpha(x - q_1 - \tau_1) - S\bar{\alpha}(x - q_1 - \tau_1) \\ &\quad + \beta\Delta(0, (q_2 + \tau_2) - \alpha(x - q_1 - \tau_1)) \} dF(x) + \\ &+ \int_{(q_1 + \tau_1) + \frac{(q_2 + \tau_2)}{\alpha}}^{\infty} \{ P_1(q_1 + \tau_1) + P_2(q_2 + \tau_2) \\ &\quad - S(x - q_1 - \tau_1 - q_2 - \tau_2) + \beta\Delta(0, 0) \} dF(x)]. \end{aligned}$$

Let  $\Gamma(q_1, q_2)$  = expected profit at the first period -- the discounted cost of bringing the stock levels of the two products back up to  $q_1, q_2$  units at the beginning of the next period,

and  $\Sigma(q_1, q_2)$  = total expected profit when the initial stock levels of the two products are  $q_1, q_2$ , respectively.

Thus

$$\begin{aligned} \Sigma(q_1, q_2) &= \Gamma(q_1, q_2) + \beta\Gamma(q_1, q_2) + \beta^2\Gamma(q_1, q_2) + \dots \\ &= \frac{\Gamma(q_1, q_2)}{(1 - \beta)}. \end{aligned} \tag{9}$$

Let  $\Delta(q_1, q_2)$  = total expected profit, when initial stock-levels are  $(0, 0)$  and to order  $(q_1, q_2)$  units of the two products to bring the stock levels up to the required heights.

Thus

$$\Delta(q_1, q_2) = \Sigma(q_1, q_2) - C_1q_1 - C_2q_2. \tag{10}$$

Here

$$\begin{aligned} \Gamma(q_1, q_2) &= \int_0^{q_1} (P_1 - \beta C_1) x dF(x) \\ &+ \int_{q_1}^{q_\alpha} [(P_1 - \beta C_1)q_1 + \alpha(P_2 - \beta C_2)(x - q_1) \\ &\quad - S\bar{\alpha}(x - q_1)] dF(x) \\ &+ \int_{q_\alpha}^\infty [(P_1 - \beta C_1)q_1 + (P_2 - \beta C_2)q_2 \\ &\quad - S(x - q_1 - q_2)] dF(x) \\ &= (P_1 + S - \beta C_1) \left[ \int_0^{q_1} x dF(x) + q_1(1 - F(q_1)) \right] \\ &+ \alpha(P_2 + S - \beta C_2) + q_\alpha(1 - F(q_\alpha)) \\ &\quad - q_1(1 - F(q_1)) - SE(X), \end{aligned} \tag{11}$$

where

$$q_\alpha = q_1 + \frac{q_2}{\alpha}.$$

In order to develop the optimal ordering policy, we first prove that the function  $\Delta(q_1, q_2)$ , as stated in equation (10), is a concave function of  $(q_1, q_2)$ .

Considering equations (9), (10) and (11), we find the following:

$$\begin{aligned} \frac{\partial \Delta(q_1, q_2)}{\partial q_1} &= -C_1 + \frac{1}{(1-\beta)} [(P_1 + S - \beta C_1)(1 - F(q_1)) \\ &\quad - \alpha((P_2 + S - \beta C_2)(F(q_\alpha) - F(q_1)))] \\ \frac{\partial \Delta(q_1, q_2)}{\partial q_2} &= -C_2 + \frac{1}{(1-\beta)} [(P_2 + S - \beta C_2)(1 - F(q_\alpha))] \\ \frac{\partial^2 \Delta(q_1, q_2)}{\partial q_1^2} &= -\frac{1}{(1-\beta)} [(P_1 + S - \beta C_1) \\ &\quad - \alpha(P_2 + S - \beta C_2)f(q_1) \\ &\quad + \alpha(P_2 + S - \beta C_2)f(q_\alpha)], \end{aligned}$$

which is negative.

$$\frac{\partial^2 \Delta(q_1, q_2)}{\partial q_2^2} = -\frac{1}{\alpha(1-\beta)} [(P_2 + S - \beta C_2)f(q_\alpha)],$$

is also negative.

$$\frac{\partial^2 \Delta(q_1, q_2)}{\partial q_1 \partial q_2} = -\frac{1}{(1-\beta)} [(P_2 + S - \beta C_2)f(q_\alpha)].$$

The determinant of the Hessian matrix of  $\Delta(q_1, q_2)$  is given by

$$\begin{aligned} &\begin{vmatrix} \frac{\partial^2 \Delta(q_1, q_2)}{\partial q_1^2} & \frac{\partial^2 \Delta(q_1, q_2)}{\partial q_1 \partial q_2} \\ \frac{\partial^2 \Delta(q_1, q_2)}{\partial q_1 \partial q_2} & \frac{\partial^2 \Delta(q_1, q_2)}{\partial q_2^2} \end{vmatrix} \\ &= \frac{1}{\alpha(1-\beta)^2} [(P_1 + S - \beta C_1) \\ &\quad - \alpha(P_2 + S - \beta C_2)](P_2 + S - \beta C_2)f(q_1)f(q_\alpha), \end{aligned}$$

which is positive.

Thus the Hessian matrix of  $\Delta(q_1, q_2)$  is negative definite which proves its concavity.

Because of the concavity of  $\Delta(q_1, q_2)$ , the optimal values of  $q_1, q_2$ , that maximize the total expected profit  $\Delta$ , can be determined from

$$\frac{\partial \Delta(q_1, q_2)}{\partial q_1} = 0 \text{ and } \frac{\partial \Delta(q_1, q_2)}{\partial q_2} = 0,$$

which give the following two equations:

$$F(q_1) = 1 - \frac{(1-\beta)(C_1 - \alpha C_2)}{[(P_1 + S) - \alpha(P_2 + S)] - \beta(C_1 - \alpha C_2)}$$

$$F(q_\alpha) = 1 - \frac{(1-\beta)C_2}{(P_2 + S - \beta C_2)}.$$

Hence  $q_1^*$  and  $q_2^*$ , the respective optimal values of  $q_1$  and  $q_2$ , can be obtained from

$$q_1^* = F^{-1} \left[ \frac{\{(P_1 + S) - \alpha(P_2 + S)\} - (C_1 - \alpha C_2)}{\{(P_1 + S) - \alpha(P_2 + S)\} - \beta(C_1 - \alpha C_2)} \right] \tag{12}$$

$$q_2^* = \alpha \left[ F^{-1} \left\{ \frac{(P_2 + S - C_2)}{(P_2 + S) - \beta C_2} \right\} - q_1^* \right] \tag{13}$$

Finally,

$$\begin{aligned} \max \Delta(q_1^*, q_2^*) &= \frac{1}{(1-\beta)} [SE(X) \\ &+ (P_1 + S - \beta C_1) \int_0^{q_1^*} x dF(x) + \alpha(P_2 + S - \beta C_2) \int_{q_1^*}^{q_\alpha^*} x dF(x), \end{aligned} \tag{14}$$

where  $q_{\alpha}^* = q_1^* + \frac{q_2^*}{\alpha}$ .

and following the similar line of proof of Theorem 1 we have the following theorem:

**Theorem 2.** In the infinite-period inventory model defined in this section, if  $q_i$  is the stock-level of item  $i$  before ordering for further units at any period, then one should order  $\max(q_i^* - q_i, 0)$  units of item  $i$ ,  $i = 1, 2$ , in order to maximize the total expected profit.

**Remark:** In this problem, if the discount factor is zero (*i.e.*,  $\beta = 0$ ), then it reduces to a one-period problem. The expressions for  $q_1^*$  and  $q_2^*$ , in equations (12) and (13), become the same as that is obtained in the previous section.

If  $\beta \rightarrow 1$ , then  $F(q_1) \rightarrow 1$  and  $F(q_{\alpha}) \rightarrow 1$ , and therefore the optimal values,  $q_1^*$  and  $q_{\alpha}^*$ , become very large.

The following numerical example studies the sensitivity of the optimal order quantities as  $\beta$  approaches 1.

*Example 2:* Let the demand ( $X$ ) at any period follow exponential law with parameter  $\lambda$  with

$$F(t) = 1 - \exp(-\lambda t), t > 0, \lambda > 0.$$

Let the cost parameters be  $C_1 = 5, C_2 = 2, P_1 = 8, P_2 = 3, S = 1$ . Let  $\alpha = 0.5$ .

Suppose, the demand-distribution is exponential with mean demand 100, *i.e.*,  $\lambda = 0.01$ . Then using equations (12), (13) and (14), the optimal stock-heights for the two items and the maximum expected profit can be obtained. Table 1 below displays the optimal stock-heights  $q_1^*$  and  $q_2^*$  of products 1 and 2, respectively, and the maximum  $\Delta(q_1^*, q_2^*)$ , corresponding to different values of the discount factor  $\beta$ :

TABLE I  
OPTIMAL STOCK-HEIGHTS  $q_1^*, q_2^*$  AND MAXIMUM EXPECTED PROFIT  
 $\Delta(q_1^*, q_2^*)$  FOR DIFFERENT VALUES OF  $\beta$

$\beta$	$q_1^*$	$q_2^*$	$\Delta(q_1^*, q_2^*)$
0.00	55.96	6.68	6.83
0.10	60.61	7.05	16.14
0.20	66.14	7.48	29.35
0.50	91.63	9.12	123.64
0.70	125.28	10.68	357.88
0.90	214.01	12.89	1904.22
0.95	277.26	13.60	4586.55
0.97	325.81	13.90	8343.15
0.99	433.07	14.22	27806.13

All values are in appropriate units.

## VI. CONCLUSION AND DISCUSSION

In the present paper a multi-period inventory model has been developed for two substitutable products, the main features of which are as follows:

Here an upward substitution is considered. The demand comes only for the first item, and the second item acts as a substitute of the first one. Both are not substitutable for each

other. In case of zero inventory of the first item, there is a non-zero, non-unity probability that an unsatisfied customer will go for the substitute. The substitute, on one hand, is less expensive, but on the other hand, cannot satisfy the customer fully. It cannot fulfill anything beyond the basic need of the item; quality-wise, feature-wise, the first item is much more superior, which drives the customers demanding for it. In the present work, a multi-period model has been solved in which the optimal stock-heights have been determined for each period. The model has been developed for a finite number of periods, and for an infinite number of periods as well, where a discount factor has been considered to bring the total expected profit to the present value. Numerical examples have been provided to illustrate the results. A sensitivity analysis has been done to examine how the optimal order quantities change with the change in the value of the discount factor  $\beta$ . It has been observed that the optimal stock-heights become large, when the discount factor gets large. The optimal stock-height of item 1 is more sensitive to  $\beta$ -value compared to the optimal stock-height of item 2. When  $\beta$  approaches 1, *i.e.*, when future sales are not discounted seriously, the optimal stock level for item 1 becomes very large, and the rate of change is faster as it gets closer to 1. Then practically we would suffer no great loss by having a stock which may remain in inventory for a long period of time. In case  $\beta$  is zero, the infinite-period model reduces to a one-period model. In this paper the demand has not been considered to be time-dependent. The problem can be revisited for time-dependent demand. We can allow partially or fully backlogging of demand, which can compensate the loss of opportunity cost or the shortage cost to some extent. A random lead time can also be assumed to be present in the model. The model can also be examined when the items we are dealing with are decaying.

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