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Direct Block Backward Differentiation Formulas for Solving Second Order Ordinary Differential Equations

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***Abstract*—**In this paper, a direct method based on variable step size Block Backward Differentiation Formula which is referred as BBDF2 for solving second order Ordinary Differential Equations (ODEs) is developed. The advantages of the BBDF2 method over the corresponding sequential variable step variable order Backward Differentiation Formula (BDFVS) when used to solve the same problem as a first order system are pointed out. Numerical results are given to validate the method.

***Keywords*—**Backward Differentiation Formula, block, second order.

I. Introduction

MANY mathematical problems which arise in real life applications are in the form of Ordinary Differential Equations (ODEs), Delay Differential Equations (DDEs) or Differential Algebraic Equations (DAEs). Many of these problems are of higher order and cannot be solved analytically and hence the use of numerical methods is advocated.

Higher order ODEs are generally reduced to an equivalent system of first order ODEs and then solved numerically using first order ODEs method which is computationally expensive. Among the earliest research on the direct method of solving higher order ODEs is due to Krogh [1]. He proposed the generalized form of the modified form of the modified divided difference version for multistep method, where backvalues of any one of the derivatives are interpolated. Suleiman [2] proposed a direct method for solving higher order nonstiff ODEs called Direct Integration (DI) method where the *d*th derivative of a *d*th order equation are interpolated whilst the case lower derivatives are used is called the generalized backward differentiation (GBDF) method for solving stiff ODEs. The construction of these methods is based on divided difference formulation of the multistep method. The computation of the integration and differentiation coefficients at each step requires a lot of computational effort. The clear advantage of BBDF2 method is that no differentiation coefficients need to be calculated at each step since the coefficients of the y and y' values are stored. Furthermore, in BBDF2 method, two solution i.e. y_{n+1} and y_{n+2} values are computed simultaneously as discussed in the next section. This again will lead to a quicker execution time.

The efficiency of the BBDF2 method is measured in terms of execution time, accuracy and the total number of steps and compared with BDFVS method.

II. The BBDF2 Method

In this paper, we shall consider BBDF2 method for the numerical solution of solving second order ODEs in the form

$$y'' = f(x, y, y') \tag{1}$$

subject to an initial conditions $y(a) = y_0$, $y'(a) = y'_0$ in the interval $x \in [a,b]$.

The step size, h of the computed block is $2h$ and the step size of the previous block is $2rh$ where r is the step size ratio between the current and the previous block (Refer Fig 1). We limit the decrease of the step size to halving and increment of the step size by a factor of 1.6 to ensure zero stability. Therefore, the values considered are $r = 1$, $r = 2$ and $r = 5/8$ which corresponds respectively to constant step size, half the step size and increment of the step size by a factor of 1.6. We do not consider doubling the step size, i.e $r = 0.5$ due to zero instability.

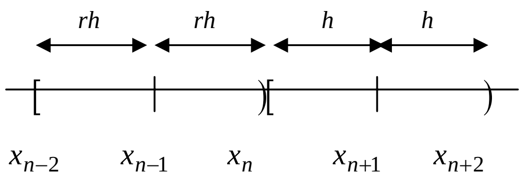

Fig. 1 2-point block method

To derive the coefficients of BBDF2, the equation in (1) is replaced by Lagrange interpolating polynomial $P_k(x)$ which interpolates the values $y_n, y_{n-1}, ..., y_{n-k+1}$ at the interpolating points $x_n, x_{n-1}, ..., x_{n-k+1}$

$$P_k(x) = \sum_{j=0}^{k} L_{k,j}(x) y(x_{n+1-j}) \tag{2}$$

where

$$L_{k,j}(x) = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{(x - x_{n+1-i})}{(x_{n+1-j} - x_{n+1-i})} \text{ ,for each } j = 0,1,...,k.$$

In BBDF2, the backvalues of y_{n+2-i}, $i = 0,1,...,4$, are used to interpolate the Lagrange polynomial. Differentiating twice the interpolating polynomial in (2) and evaluating at $x = x_{n+1}$ and $x = x_{n+2}$ and on substituting with $r = 1$, $r = 2$, and $r = 5/8$ produce the following formulas of the BBDF2 method.

(i) for $r = 1$

$$y'_{n+1} = \frac{1}{h}\left(-\frac{1}{12}y_{n-2} + \frac{1}{2}y_{n-1} - \frac{3}{2}y_n + \frac{5}{6}y_{n+1} + \frac{1}{4}y_{n+2}\right)$$
$$y_{n+1} = -\frac{1}{20}y_{n-2} + \frac{1}{5}y_{n-1} + \frac{3}{10}y_n + \frac{11}{20}y_{n+2} - \frac{3}{5}h^2 f_{n+1}$$
$$y'_{n+2} = \frac{1}{h}\left(\frac{1}{4}y_{n-2} - \frac{4}{3}y_{n-1} + 3y_n - 4y_{n+1} + \frac{25}{12}y_{n+2}\right)$$
$$y_{n+2} = -\frac{11}{35}y_{n-2} + \frac{8}{5}y_{n-1} - \frac{114}{35}y_n + \frac{104}{35}y_{n+1} + \frac{12}{35}h^2 f_{n+2}$$

(ii) for $r = 2$

$$y'_{n+1} = \frac{1}{h}\left(-\frac{1}{80}y_{n-2} + \frac{5}{48}y_{n-1} - \frac{15}{16}y_n + \frac{8}{15}y_{n+1} + \frac{5}{16}y_{n+2}\right)$$
$$y_{n+1} = -\frac{1}{224}y_{n-2} + \frac{5}{224}y_{n-1} + \frac{15}{32}y_n + \frac{115}{224}y_{n+2} - \frac{15}{28}h^2 f_{n+1}$$
$$y'_{n+2} = \frac{1}{h}\left(\frac{1}{30}y_{n-2} - \frac{1}{4}y_{n-1} + \frac{3}{2}y_n - \frac{16}{5}y_{n+1} + \frac{23}{12}y_{n+2}\right)$$
$$y_{n+2} = -\frac{1}{20}y_{n-2} + \frac{5}{14}y_{n-1} - \frac{51}{28}y_n + \frac{88}{35}y_{n+1} + \frac{3}{7}h^2 f_{n+2}$$

(iii) for $r = 5/8$

$$y'_{n+1} = \frac{1}{h}\left(-\frac{208}{825}y_{n-2} + \frac{3072}{2275}y_{n-1} - \frac{117}{50}y_n + \frac{279}{52}y_{n+1} + \frac{3}{14}y_{n+2}\right)$$
$$y_{n+1} = -\frac{512}{2125}y_{n-2} + \frac{12288}{14875}y_{n-1} - \frac{819}{4250}y_n + \frac{723}{1190}y_{n+2} - \frac{117}{170}h^2 f_{n+1}$$
$$y'_{n+2} = \frac{1}{h}\left(\frac{224}{275}y_{n-2} - \frac{2048}{525}y_{n-1} + \frac{273}{50}y_n - \frac{14}{3}y_{n+1} + \frac{1195}{546}y_{n+2}\right)$$
$$y_{n+2} = -\frac{70784}{67575}y_{n-2} + \frac{96256}{22525}y_{n-1} - \frac{125853}{22525}y_n + \frac{9086}{2703}y_{n+1} + \frac{273}{901}h^2 f_{n+2}$$

III. Results

We will compare the numerical results obtained using BBDF2 method with the variable step variable order method, BDFVS. See Suleiman [11] for the details of the code. Below are two of the problems tested:

Problem 1:

2nd order ODE: $y'' = -1000y - 70y'$

Initial values: $y(0) = 2, y'(0) = -70$

Interval: $0 \le x \le 10$

Solution: $y(x) = e^{-20x} + e^{-50x}$

Eigenvalues: $\lambda_1 = -50, \lambda_2 = -20$

REDUCTION TO FIRST ORDER

First order systems: $y'_1(x) = y_2$, $y'_2(x) = -1000y_1 - 70y_2$

Initial values: $y_1(0) = 2, y_2(0) = -70$

Solution: $y_1(x) = e^{-50x} + e^{-20x}$, $y_2(x) = -50e^{-50x} - 20e^{-20x}$

Source: Artificial.

Problem 2:

2nd order ODE: $y'' = -8y' - 16y$

Initial values: $y(0) = 1, y'(0) = -12$

Intervals: $0 \le x \le 10$

Solution: $y(x) = e^{-4x}(1 - 8x)$

REDUCTION TO FIRST ORDER

First order systems**:** $y'_1(x) = y_2$, $y'_2(x) = -16y_1 - 8y_2$

Solution: $y_1(x) = e^{-4x}(1 - 8x)$, $y_2(x) = -e^{-4x}(12 - 32x)$

The notations used in the tables take the following meaning:

TOL	:	Tolerance used
TS	:	Total steps used
FA	:	Total number of rejected steps
IST	:	Total number of accepted steps
MAXE	:	Magnitude of the maximum error of the computed solution
BDFVS	:	Method variable step variable order BDF
BBDF2	:	Method variable step BBDF2
TIME	:	The execution time in microseconds

TABLE I
NUMERICAL RESULT FOR PROBLEM 1

TOL	MTD	FA	IST	TS	MAXE	TIME
10^{-2}	BDFVS	11	62	73	7.2354e-01	0.002424
	BBDF2	0	27	27	1.8840e-03	0.001235
10^{-4}	BDFVS	16	103	119	2.9447e-02	0.002852
	BBDF2	0	52	52	1.1381e-04	0.001442
10^{-6}	BDFVS	19	160	179	3.4817e-04	0.003574
	BBDF2	0	120	120	4.5819e-06	0.001999

TABLE II
NUMERICAL RESULT FOR PROBLEM 2

TOL	MTD	FA	IST	TS	MAXE	TIME
10^{-2}	BDFVS	13	64	77	1.1749e+00	0.002357
	BBDF2	0	26	26	1.9115e-03	0.001136
10^{-4}	BDFVS	18	109	127	1.2581e-02	0.003025
	BBDF2	0	51	51	1.1411e-04	0.001310
10^{-6}	BDFVS	19	163	182	1.2791e-04	0.003578
	BBDF2	1	128	129	4.6212e-06	0.001898

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IV. Conclusion

Comparing method BBDF2 and BDFVS, we find that within the required tolerance, method BBDF2 gives better result in terms of execution time, total number of steps and accuracy. In conclusion, solving the equations directly using BBDF2 is more efficient than reduction to first order system.

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