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Convergence analysis of the generalized alternating two-stage method

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Abstract—In this paper, we give the generalized alternating two-stage method in which the inner iterations are accomplished by a generalized alternating method. And we present convergence results of the method for solving nonsingular linear systems when the coefficient matrix of the linear system is a monotone matrix or an H-matrix.

Keywords—generalized alternating two-stage method, linear system, convergence.

I. Introduction

We consider the $n \times n$ linear system

$$Ax = b, \quad (1)$$

where A is a a nonsingular matrix, x and b are n-dimensional vectors.

Given splittings $A = M - N$ (M nonsingular) and $M = F - G$, a two-stage iterative method is

$$x^{(l+1)} = (F^{-1}G)^{s(l)}x^{(l)} + \sum_{j=0}^{s(l)-1}(F^{-1}G)^j F^{-1}(Nx^{(l)} + b), \quad (2)$$

$$l = 0, 1 \cdots$$

Let $M = P - Q = R - S$ be two splittings of the matrix M, then the alternating two-stage method ([1]) is

$$x^{(l+1)} = (R^{-1}SP^{-1}Q)^{s(l)}x^{(l)} + \sum_{j=0}^{s(l)-1}(R^{-1}SP^{-1}Q)^j R^{-1}(SP^{-1} + I)(Nx^{(l)} + b),$$

$$l = 0, 1 \cdots \quad (3)$$

Then let $M = P_i - Q_i,\ i = 1, \cdots, m$, we can get a generalized alternating two-stage method which can be written as

$$\begin{aligned} x^{(l+1)} = & (\prod_{k=1}^{m} P_{m-k+1}^{-1}Q_{m-k+1})^{s(l)}x^{(l)} \\ & + \sum_{j=0}^{s(l)-1}(\prod_{k=1}^{m} P_{m-k+1}^{-1}Q_{m-k+1})^j P_m^{-1} \\ & [\sum_{i=1}^{m-1}(\prod_{k=1}^{m-i} Q_{m-k+1}P_{m-k}^{-1}) + I](Nx^{(l)} + b), \\ & l = 0, 1 \cdots \end{aligned} \quad (4)$$

In a similar manner as the alternating two-stage method, we say that a generalized alternating two-stage method is stationary when $s(l) = s$, for all l, while an generalized alternating two-stage method is non-stationary if the number of inner iterations $s(l)$ changes with the outer iteration l.

Clearly given an initial vector $x^{(0)}$, the generalized alternating two-stage method (4) produces the sequence of vectors

$$x^{(l+1)} = T^{(l)}x^{(l)} + c_{s(l)},\ l = 0, 1, \cdots, \quad (5)$$

where

$$\begin{aligned} T^{(l)} = & (\prod_{k=1}^{m} P_{m-k+1}^{-1}Q_{m-k+1})^{s(l)} \\ & + \sum_{j=0}^{s(l)-1}(\prod_{k=1}^{m} P_{m-k+1}^{-1}Q_{m-k+1})^j \\ & P_m^{-1}[\sum_{i=1}^{m-1}(\prod_{k=1}^{m-i} Q_{m-k+1}P_{m-k}^{-1}) + I]N \end{aligned} \quad (6)$$

and

$$\begin{aligned} c_{s(l)} = & \sum_{j=0}^{s(l)-1}(\prod_{k=1}^{m} P_{m-k+1}^{-1}Q_{m-k+1})^j P_m^{-1} \\ & [\sum_{i=1}^{m-1}(\prod_{k=1}^{m-i} Q_{m-k+1}P_{m-k}^{-1}) + I]b \end{aligned}$$

In order to analyze the convergence of the generalized alternating two-stage method (5) and considering the splittings $A = M - N$ and $M = P_i - Q_i, i = 1, \cdots, m$, we can write the iteration matrix $T^{(l)}$ defined in (6) as follows

$$\begin{aligned} T^{(l)} &= (\prod_{k=1}^{m} P_{m-k+1}^{-1}Q_{m-k+1})^{s(l)} \\ &\quad +[\sum_{i=1}^{m-1}(\prod_{k=1}^{m-i} Q_{m-k+1}P_{m-k}^{-1}) + I]N \\ &= (\prod_{k=1}^{m} P_{m-k+1}^{-1}Q_{m-k+1})^{s(l)} \\ &\quad +[\sum_{i=1}^{m-1}(\prod_{k=1}^{m-i} Q_{m-k+1}P_{m-k}^{-1}) + I](P_i - Q_i)M^{-1}N \\ &= (\prod_{k=1}^{m} P_{m-k+1}^{-1}Q_{m-k+1})^{s(l)} \\ &\quad +(I - (\prod_{k=1}^{m} P_{m-k+1}^{-1}Q_{m-k+1})^{s(l)})M^{-1}N \end{aligned} \quad (7)$$

In this paper, our study concentrates on convergence analysis of the generalized alternating two-stage method. In section 2, we present some definitions and preliminaries that are used later in the paper. In section 3, we give convergence results of the method for nonsingular linear system when the coefficient matrix A of the linear system is a monotone matrix or an H-matrix.

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II. Notation and preliminaries

The notation and terminology adopted in this paper are along the lines of those used in paper [4].We say that a vector x is nonnegative(positive), denoted $x \geq 0 (x > 0)$, if all of its entries are nonnegative(positive).Similarly, a matrix B is said to be nonnegative, denoted $B \geq 0$(where 0 is the zero matrix), if all of its entries are nonnegative.Given a matrix $A = (a_{ij})$, we define $|A| = (|a_{ij}|)$.It follows that $|A| \geq 0$ and that $|AB| \leq |A||B|$ for any two matrices A and B of compatible size.By $\rho(A)$ we denote the spectral radius of the square matrix A. For any matrix $A = (a_{ij}) \in R^{n\times n}$, we define its comparison matrix $\langle A\rangle = (\alpha_{ij})$ by $\alpha_{ii} = |a_{ii}|, \alpha_{ij} = -|a_{ij}|, i \neq j$.

Definition 2.1[4]Let $A \in R^{n\times n}$, if $A^{-1} \geq 0$, then the matrix A is called a monotone matrix.

Definition 2.2[4] A nonsingular matrix A is said to be an H-matrix if $\langle A\rangle$ is an M-matrix.

Lemma 2.1[2, 3] Let $A, B \in R^{n\times n}$.

(a)if A is an H-matrix, then $|A^{-1}| \leq \langle A\rangle^{-1}$,

(b)if $|A| \leq B$, then $\rho(A) \leq \rho(B)$.

Definition 2.3[4, 5, 6] Let $AB \in R^{n\times n}$.A splitting $A = M - N$ is called

(a)regular if $M^{-1} \geq 0$ and $N \geq 0$,

(b)weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$,

(c)H-splitting if $\langle M\rangle - |N|$ is a nonsingular M-matrix, and

(d)H-compatible splitting if $\langle A\rangle = \langle M\rangle - |N|$.

Definition 2.4[7] Let A be an $n \times n$ matrix, we say that the splitting $A = M - N$ is weak nonnegative of the first type if $M \geq 0$ and $M^{-1}N \geq 0$; weak nonnegative of the second type if $M \geq 0$ and $NM^{-1} \geq 0$.

Obviously, a weak nonnegative regular splitting is a weak nonnegative splitting of the first and of the second type.

Lemma 2.2[4,5] Let $A = M - N$ be a splitting

(a)If the splitting is weak regular, then $\rho(M^{-1}N) < 1$ if and only if $A^{-1} \geq 0$.

(b)If the splitting is an H-splitting, then A and M are H-matrices and $\rho(M^{-1}N) \leq \rho(\langle M\rangle^{-1}|N|) < 1$.

(c)If the splitting is an H-compatible splitting and A is an H-matrix,then it is an H-splitting and thus convergent.

III. Convergence results for nonsingular linear systems

Lemma 3.1[8] Let $A^{(l)}(l = 0, 1, \cdots)$ be a sequence of nonnegative matrices in $R^{n\times n}$. If there exist a real number $0 \leq \theta < 1$,and a vector $\upsilon > 0$ in R^n, such that $A^{(l)}\upsilon \leq \theta\upsilon$, for all $l = 0, 1, \cdots$, then $\rho(\nu^j) \leq \theta^j < 1$, where $\nu^j = A^{(j)} \cdots A^{(l)}A^{(0)}$ and $\lim_{j\to 0} \nu^j = 0$.

Lemma 3.2[7] Let A be a nonsingular and monotone matrix. Assume that the splittings

$$A = M_l - N_l (l = 1, 2, \cdots, p)$$

are weak nonnegative of the first(respectively,second) type. If we consider $\mu(l, k) = \mu_l$, then the generalized alternating iterative method converges to the unique solution of system (1). Furthermore, the unique splitting $A = B - C$ induced by matrix $\Pi_{l=1}^{p}(M_{p-l+1}^{-1}N_{p-l+1})^{\mu_l}$ is also weak nonnegative of the first(respectively,second) type.

Lemma 3.3[5] Let $A = M - N$ be a convergent regular splitting and let $M = F - G$ be a convergent weak rugular splitting. Then the two-stage iterative method (2) converges to the solution of the linear system (1) for any initial vector $x^{(0)}$ and for any sequence of the inner iterations $s(l) \geq 1, l = 0, 1, \cdots$.

Theorem 3.1 Let $A^{-1} \geq 0$.Consider the splitting $A = M - N$ is regular and the splittings $M = P_i - Q_i (i = 1, \cdots, m)$ are weak regular. Then the generalized alternating two-stage method (5) converges to the solution of the linear system (1) for any initial vector $x^{(0)}$ and for any sequence of inner iteration $s(l) \geq 1, l = 0, 1, \cdots$.

Proof Since $A = M - N$ is regular splitting, we have $M^{-1} \geq 0$, then from Lemma 3.2 we know that there exists a unique pair of matrices B, C, such that

$$\prod_{k=1}^{m} P_{m-k+1}^{-1} Q_{m-k+1} = B^{-1}C$$

and $M = B-C$ is a weak regular splitting. That is,the iteration matrices defined in(7)can be written as

$$T^{(l)} = (B^{-1}C)^{s(l)} + (I - (B^{-1}C)^{s(l)})M^{-1}N, l = 0, 1, \cdots.$$

Thus,$T^{(l)}, l = 0, 1, \cdots$ are the iteration matrices of a non-stationary two-stage method for the matrix A, with the regular splitting $A = M-N$ and a weak regular splitting $M = B-C$. Therefore,using Lemma 2.2 (a) and Lemma 3.3 the proof is complete.

Now we study the convergence of the generalized alternating two-stage method (5) when A is an H-matrix. In the following theorem the fact that A is an H-matrix follows from Lemma 2.2 (b).

Theorem 3.2 Let $A = M - N$ be an H-splitting and let $M = P_i-Q_i (i = 1, \cdots, m)$ be H-compatible splittings. Then the generalized alternating two-stage method (5) converges to the solution of the linear system (1) for any initial vector $x^{(0)}$ and for any sequence of inner iterations $s(l) \geq 1,\ l = 0, 1, \cdots$.

Proof From $M = P_i - Q_i (i = 1, \cdots, m)$ are H-compatible splittings, we have M is an H-matrix and $< M >=< P_i > -|Q_i|$, therefore the matrices $P_i (i = 1, \cdots, m)$ are H-matrices.

By using Lemma 2.1 (a) and (6), we obtain the following bounds

$$\begin{aligned} |T^{(l)}| &\leq (\prod_{k=1}^{m} |P_{m-k+1}^{-1}||Q_{m-k+1}|)^{s(l)} \\ &\quad +[\sum_{i=1}^{m-1} (\prod_{k=1}^{m-i} |Q_{m-k+1}||P_{m-k}^{-1}|) + I]|N| \\ &\leq (\prod_{k=1}^{m} \langle P_{m-k+1}\rangle^{-1}|Q_{m-k+1}|)^{s(l)} \\ &\quad +[\sum_{i=1}^{m-1} (\prod_{k=1}^{m-i} |Q_{m-k+1}|\langle P_{m-k}\rangle^{-1}) + I]|N| \\ &= \widehat{T}^{(l)} \qquad (8) \end{aligned}$$

Obviously, $\widehat{T}^{(l)} \geq 0$. Moreover, $\widehat{T}^{(l)}$ is the iteration matrix of an generalized alternating two-stage method for the matrix

$D = \langle M\rangle - |N|$ with the regular splittings $D = \langle M\rangle - |N|$ and $\langle M\rangle = \langle P_i\rangle - |Q_i|(i = 1, \cdots, m)$. Therefore, from (7), we obtain

$$\widehat{T}^{(l)} = (\prod_{k=1}^{m} |P_{m-k+1}^{-1}||Q_{m-k+1}|)^{s(l)} + \\ (I - ((\prod_{k=1}^{m} |P_{m-k+1}^{-1}||Q_{m-k+1}|)^{s(l)})^{s(l)})\langle M^{-1}\rangle|N|.$$

From Lemma 3.2, it follows that there is a unique pair of matrices B, C, such that

$$\prod_{k=1}^{m} |P_{m-k+1}^{-1}||Q_{m-k+1}| = B^{-1}C$$

and $\langle M\rangle = B - C$ is a weak regular splitting. Thus,

$$\begin{aligned} \widehat{T}^{(l)} &= (B^{-1}C)^{s(l)} + (I - B^{-1}C)^{-1}B^{-1} \\ &= I - ((I - (B^{-1}C)^{s(l)})\langle M^{-1}\rangle)(\langle M\rangle - |N|) \quad (9) \end{aligned}$$

Given any fixed vector $e = (1, 1, \cdots, 1)^T > 0$ and $x = (\langle M\rangle - |N|)^{-1}e > 0$. Since $B^{-1}e > 0$, and $\langle M\rangle^{-1} = (I - B^{-1}C)^{-1}B^{-1} = \sum_{j=0}^{\infty}(B^{-1}C)^j B^{-1}$, from (9) it follows that $\widehat{T}^{(l)}x = x - \sum_{j=0}^{s(l)-1}(B^{-1}C)^j B^{-1}e \le x - B^{-1}e < x$. Therefore,there exists $0 \le \theta < 1$ such that $\widehat{T}^{(l)}x \le \theta x, l = 1, 2, \cdots$. From Lemma 3.1 and Lemma 2.1 (b), we know that the generalized alternating two-stage method (5) converges to the solution of the linear system (1) for any initial vector $x^{(0)}$ and for any sequence of inner iterations $s(l) \ge 1,\ l = 0, 1, \cdots$.

IV. Example

We consider $Ax = b$, where

$$A = \begin{bmatrix} 10 & -5 & -6 \\ -3 & 9 & -3 \\ -5 & 0 & 10 \end{bmatrix}.$$

$$A^{-1} = \begin{bmatrix} \frac{2}{9} & \frac{10}{81} & \frac{23}{135} \\ \frac{1}{9} & \frac{14}{81} & \frac{16}{135} \\ \frac{1}{9} & \frac{5}{81} & \frac{5}{27} \end{bmatrix} > 0,$$

so A is a monotone matrix. Let

$$M = \begin{bmatrix} 10 & -5 & -6 \\ -3 & 9 & -2 \\ -5 & 0 & 10 \end{bmatrix}, N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 10 & -5 & -7 \\ -3 & 12 & -2 \\ -5 & 0 & 12 \end{bmatrix}, Q_1 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 21/2 & -5 & -7 \\ -3 & 12 & -2 \\ -5 & 0 & 12 \end{bmatrix}, Q_2 = \begin{bmatrix} 1/2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 11 & -5 & -7 \\ -3 & 12 & -2 \\ -5 & 0 & 12 \end{bmatrix}, Q_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$N = M - A,\ Q_i = P_i - M(i = 1, 2, 3),\ s(l) = 1$, we know that the splitting $A = M - N$ is regular, the splittings $M = P_i - Q_i(i = 1, 2, 3)$ are weak regular. We choose $m = 3$, from (7) we know that

$$T^{(l)} = \begin{bmatrix} 0 & \frac{608841}{15900883} & \frac{9521809}{79504415} \\ 0 & \frac{5323683}{127207064} & \frac{105511603}{636035320} \\ 0 & \frac{3171285}{127207064} & \frac{8225057}{127207064} \end{bmatrix},$$

$\rho(T^{(l)}) = \frac{4271825441892345}{36028797018963968} < 1.$

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References

[1] Migallon H, Migallon V, Penades J, Alternating two-stage methods for consistent linear system with applications to the parallel solution of Markov chains, *Advances in Engineering Software*, vol.41, pp.13-21, 2010.
[2] Neumaier A. New techniques for the analysis of linear interval equations, *Linear Algebra Appl.*, vol.58, pp.273-325, 1984.
[3] Ortega JM, Rheinboldt WC, *Tterative solution of nonlinear equations in several variables*, New York and London:Academic Press,1970.
[4] Berman A, Plemmons RJ. *Nonnegative matrices in the mathematical sciences* 3rd ed. New York:Academic Press,1979. Reprinted by SIAM, Philadelphia,1994.
[5] Frommer A, Szyld DB. H-splittings and two-stage iterative methods. *Number Math*, vol.63, pp.345-356,1992.
[6] Varga RS. *Matrix iterative analysis*, Englewood Cliffs, New Jersey:PrenticeHall,1962.
[7] Joan-Josep Climent, Carmen Perea. Convergence and comparison theorem for a generalized alternating iterative method. *Applied Mathematics and Computation*, vol.143, pp.1-14, 2003.
[8] Robert F, Charnay M,Musy F.Iterations chaotiques serie-parallele pour desequations non-lineaires de point fixe. *Appl. Math.*, vol.20, pp.1-38, 1975.

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