Laplace Decomposition Approximation Solution for a System of Multi-Pantograph Equations

M. A. Koroma, C. Zhan, A. F. Kamara, and A. B. Sesay

Abstract**—**In this work we adopt a combination of Laplace transform and the decomposition method to find numerical solutions of a system of multi-pantograph equations. The procedure leads to a rapid convergence of the series to the exact solution after computing a few terms. The effectiveness of the method is demonstrated in some examples by obtaining the exact solution and in others by computing the absolute error which decreases as the number of terms of the series increases.

Keywords**—** Laplace decomposition, pantograph equations, exact solution, numerical solution, approximate solution.

I. Introduction

THE general pantograph equation given by $u^n(t) = f\left(t, u^{n-1}(t), \ldots, u(t), u^{n-1}(qt), \ldots u(qt)\right), t \geq 0$ for $0 < q < 1$ is a very important type of delay differential equation. It arises in many scientific models such as electrodynamics, population studies, number theory and dynamical systems. Ockendon and Tayler, 1971 [8] introduced the concept of pantograph equation in their bid to determine the motion of a pantograph head on an electric locomotive which is collecting current from an overhead trolley wire. Since then, pantograph equations have been the subject of investigation by researchers.

M.Z. Liu and D. Li [7] proved the existence and uniqueness of the analytic solution of the multi-pantograph equation. They constructed the Direchlet series solution and obtained the sufficient condition for the asymptotic stability of the analytic solution obtained.

Derfel and Isreales [4] addressed the existence and uniqueness of solutions of the pantograph equations and their asymptotic behavior.

Abazari and Abazari [1] presented and proved the theorems, in one-dimensional differential transform method, for solving nonlinear higher order multi-pantograph equations.

Buhmann and Isreles [3] investigated the stability of discretized pantograph differential equation by analyzing the numerical solution of the trapezoidal rule discretizations.

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Our interest in this work is to adopt a combination of Laplace transform and the decomposition methods to find numerical solutions of a system of multi-pantograph equations. We shall also seek to demonstrate the effectiveness of the method in approximating solutions of multi-pantograph equations. S. A. Khuri, 2001[6] was the first to introduce this numerical Laplace transform algorithm which is based on the decomposition method. He [5] applied the algorithm to obtain approximate solutions of a class of nonlinear differential equations.

II. Laplace Decomposition Algorithm (LDA)

We illustrate the LDA by considering the following linear system:

$$u^{(m)}(t) = Ru + Nu + f(t) \tag{1}$$

whose initial conditions are

$$u^{(k)}(0) = \lambda_k, \quad k = 0,1,2,..,m-1 \tag{2}$$

R and N are linear operators of order less than m and $f(t)$ is an analytic function.

Taking the Laplace transform (L) of both sides of (1) gives

$$s^m L[u(t)] - \sum_{k=0}^{m-1}\left(s^{m-1-k} u^k(0)\right) = L[Ru + Nu + f(t)] \tag{3}$$

where $u^k(0)$ is the k th derivative of u at $t = 0$ and $u^0(0) = u(0)$.

Substituting the initial conditions (2) in (3) and rearranging terms we have

$$s^m L[u(t)] = \sum_{k=0}^{m-1}\left(s^{m-1-k}\lambda_k\right) + L[Ru + Nu + f(t)]$$

Thus

$$u(t) = H(t) + L^{-1} s^{-m} L[Ru + Nu] \tag{4}$$

where L^{-1} is the inverse Laplace transform and

$$H(t)=L^{-1}\left[\sum_{k=0}^{m-1}(s^{-1-k}\lambda_k)\right]+ \quad L^{-1}s^{-m}L[f(t)]$$

Khuri [6] proposed that $u(t)$ be decomposed (as in the Adomian decomposition algorithm) as

$$u(t)=\sum_{n=0}^{\infty}u_n(t) \tag{5}$$

Ru and Nu are also decomposed as

$$Ru(t)=\sum_{n=0}^{\infty}A_n \text{ and } Nu(t)=\sum_{n=0}^{\infty}B_n \tag{6}$$

where A_n and B_n are the Adomain polynomials.

Putting (5) and (6) in (4) we obtain

$$\sum_{n=0}^{\infty}u_n(t)=H(t)+L^{-1}s^m L\sum_{n=0}^{\infty}[A_n+B_n] \tag{7}$$

From (7) we take

$$u_0(t)=H(t) \tag{8}$$

as our first approximation and obtained higher iterates from the recurrence relation

$$u_{n+1}(t)=L^{-1}s^m L\sum_{n=0}^{\infty}[A_n+B_n] \text{, for } n\geq 0 \tag{9}$$

Substituting (8) and (9) in (5) gives the solution to (1) as

$$u(t)=u_0+u_1+u_2+u_3+.......... \tag{10}$$

III. Examples

A. Example 1

Consider the first order multi-pantograph equation

$$u'(t)=-u\left(\frac{4}{5}t\right)-u(t), \quad u(0)=1 \ , \ 0\leq t\leq 1 \tag{11}$$

Taking the Laplace transform gives

$$sL[u'(t)]-u(0)=-\ L\left[u\left(\frac{4}{5}t\right)+u(t)\right]$$

Substituting the boundary condition in (11) and taking the inverse Laplace transform of both sides, after dividing through by s, we obtain

$$u(t)=1-L^{-1}s^{-1}L\left[u\left(\frac{4}{5}t\right)+u(t)\right] \tag{12}$$

Using (5) in (12) gives

$$u(t)=\sum_{n=0}^{\infty}u_n(t)=1-L^{-1}s^{-1}L\sum_{n=0}^{\infty}\left[u_n\left(\frac{4}{5}t\right)+u_n(t)\right] \tag{13}$$

Thus, our first approximation to $u(t)$ is given by $u_0(t)=1$. And, from (9), higher iterates can be obtained from the recurrence relation

$$u_{n+1}(t)=L^{-1}s^{-1}L\left[u_n\left(\frac{4}{5}t\right)+u_n(t)\right], \quad n\geq 0 \tag{14}$$

We thus obtain the first few iterates of (14) as

$$u_1=-2t;\ u_2=\frac{9}{5}t^2;\ u_3=-\frac{123}{125}t^3;\ u_4=\frac{23247}{62500}t^4;$$
$$u_5=-\frac{20480607}{195312500}t^5;u_6=\frac{28324679481}{1220703125000}t^6;$$
$$u_7=-\frac{79798714863543}{19073486328125000}t^7;\ .... \tag{15}$$

Now $u(t)=\sum_{n=0}^{\infty}u_n(t)$ yields

$$u(t)=$$
$$1-2t+\frac{9}{5}t^2-\frac{123}{125}t^3+\frac{23247}{62500}t^4-\frac{20480607}{195312500}t^5$$
$$+\frac{28324679481}{1220703125000}t^6-\frac{79798714863543}{19073486328125000}t^7$$
$$+\frac{754169674303858 5387}{11920928955078125000000}t^8$$
$$-\frac{3822475475556915 72079923}{4656612873077392578125000000}t^9$$
$$+\frac{84678114242614931318991 8944287}{909494701772928237915039062500000000}t^{10}$$
$$+..... \tag{16}$$

It is interesting to note that an Adomian decomposition solution of (11) produces the same series as in (16). However, the advantage of our method is that it does not require separate calculation of the Adomian polynomials.

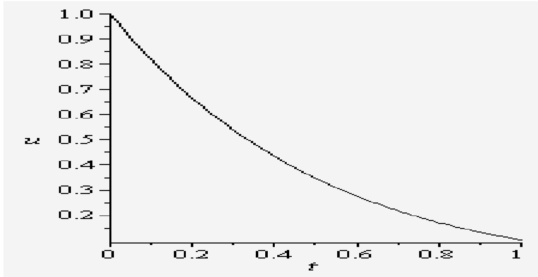

Fig. 1 LDA solution of (11)

TABLE I
COMPARISON OF ABSOLUTE ERRORS BETWEEN SOLUTIONS OF (17)

t	Exact Solution e^t	Adomian Decomposition ($n=14$)	Taylor Series Method [9] ($n=8$)	LDA Method $n=3$	LDA Method $n=4$	LDA Method $n=8$
0.0	1.00	0.00	0.00	0.00	0.00	0.00
0.2	1.221402758	0.00	1.440E-12	1.8677E-5	3.93E-7	1.3560E-14
0.4	1.491824698	2.22E-6	7.524E-10	3.18037E-4	1.3334E-5	1.19713E-12
0.6	1.822118800	2.22E-16	2.953E-8	1.716236E-3	1.07176E-4	2.87045E-10
0.8	2.225540928	1.33E-15	4.018E-7	5.79128E-3	4.78701E-4	3.969202E-9
1.0	2.718281828	4.88E-15	3.059E-6	1.5120803E-2	1.550635E-3	3.072836E-8

B. Example 2

Consider the first order multi-pantograph equation

$$u'(t)=\frac{1}{2}e^{\frac{t}{2}}u\left(\frac{t}{2}\right)+\frac{1}{2}u(t)\ ,\ 0\le t\le 1 \qquad (17)$$

With the boundary condition

$$u(0)=1 \qquad (18)$$

By the procedure outlined in Section II we have

$$u_0=1\ ,\ u_{n+1}=\frac{1}{2}L^{-1}s^{-1}L\left[e^{\frac{1}{2}t}u_n\left(\frac{t}{2}\right)+u_n(t)\right],$$

from which we obtain

$$u_1=-1+\frac{1}{2}t+e^{\frac{1}{2}t};$$

$$u_2=-\frac{1}{6}-\frac{1}{2}t+\frac{1}{8}t^2+\frac{2}{3}e^{\frac{3}{4}t}+\left(\frac{t}{4}-\frac{1}{2}\right)e^{\frac{1}{2}t}+\left(\frac{-5}{12}-\frac{1}{8}t+\frac{1}{32}t^2\right)e^{\frac{1}{2}t};$$

$$u_3=\frac{1}{28}-\frac{1}{12}t-\frac{1}{8}t^2+\frac{1}{48}t^3+\frac{8}{21}e^{\frac{7}{8}t}+\frac{1}{12}te^{\frac{3}{4}t};$$

$$u_4=\frac{143}{2520}-\frac{1}{56}t-\frac{1}{48}t^2-\frac{1}{48}t^3+\frac{1}{384}t^4-\frac{5}{8}e^{\frac{3}{4}t}+\frac{4}{21}e^{\frac{7}{8}t}$$
$$+\frac{1}{42}te^{\frac{7}{8}t}+\frac{64}{315}e^{\frac{15}{16}t}+\frac{1}{192}t^2e^{\frac{3}{4}t}$$
$$+\frac{1}{2688}\left(-464-280t-42t^2+7t^3\right)e^{\frac{1}{2}t};..............$$

Now

$$u(t)=\left(\frac{-187}{2520}-\frac{17}{168}t-\frac{1}{48}t^2+\frac{1}{384}t^3\right)$$
$$+\left(\frac{-5}{56}+\frac{1}{48}t+\frac{41}{1472}t^2+\frac{1}{384}t^3\right)e^{\frac{1}{2}t}$$
$$+\left(\frac{1}{24}+\frac{1}{12}t+\frac{1}{192}t^2\right)e^{\frac{3}{4}t}+\left(\frac{4}{7}+\frac{1}{42}t\right)e^{\frac{7}{8}t}+\frac{64}{315}e^{\frac{15}{16}t}+.............$$
$$(19)$$

C. Example 3

Consider the first order multi-pantograph equation

$$u'(t)=-u(t)+\frac{1}{8}u\left(\frac{1}{4}t\right)-\frac{1}{8}e^{-\frac{1}{4}t}\ ,\ 0\le t\le 1 \qquad (20)$$

with boundary condition

$$u(0)=1 \qquad (21)$$

After computing the first seven iterates of $u(t)$ by the outlined procedure in Section II we have

$$u(t)=-\frac{173061673561}{2}-\frac{192913}{16}t+\frac{4619279}{1024}t^2$$
$$+\frac{56203}{1024}+\frac{5640047}{393216}t^3+\frac{97035239}{805306368}t^4+\frac{28630025417}{8246337208320}t^5$$

$$+\frac{9830069233523}{16888498602639360}t^6+\frac{5461}{2}e^{-\frac{1}{4}t}$$
$$-1498021e^{-\frac{1}{16}t}+198557480e^{-\frac{1}{64}t}$$
$$-6862251264e^{-\frac{1}{256}t}+70127353856e^{-\frac{1}{1024}t}$$
$$-251809234944e^{-\frac{1}{4096}t}+274877906944e^{-\frac{1}{16384}t} \tag{22}$$

Equation (23) and its boundary conditions have the exact solution $u(t)=e^{-t}$. In Fig. 2 below we present the absolute error $E=\left|u-\sum_{i=0}^{n}u_n\right|$ for various values of n.

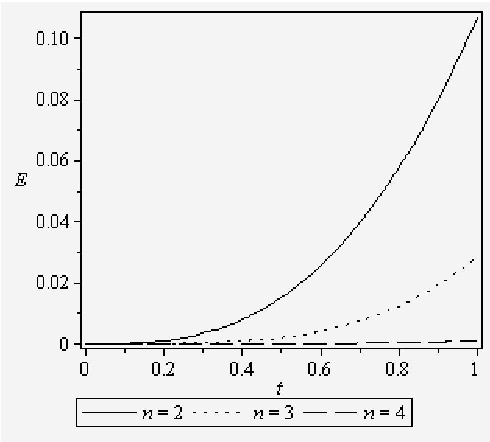

Fig. 2 Absolute error $=\left|u-\sum_{i=0}^{n}u_n\right|$

We observe that the absolute error decreases sharply as we increase the number of iterates. Thus a few iterates guarantee a high degree of accuracy.

D. Example 4

Consider the second order multi-pantograph equation

$$u''(t)=\frac{3}{4}u(t)+u\left(\frac{1}{2}t\right)-t^2+2,\qquad 0\le t\le 1$$
$$u(0)=u'(0)=0, \tag{23}$$

The first few iterates of the series solution of (24) are

$$u_0=t^2-\frac{1}{12}t^4,$$
$$u_1=\frac{1}{12}t^4-\frac{13}{5760}t^6,$$
$$u_2=\frac{13}{5760}t^6-\frac{91}{2949120}t^8,$$
$$u_3=\frac{91}{2949120}t^8-\frac{17563}{67947724800}t^{10},$$
$$u_4=\frac{17563}{67947724800}t^{10}-\frac{13505947}{9184358065766400}t^{12}, \tag{24}$$
$$u_5=\frac{13505947}{9184358065766400}t^{12}$$
$$-\frac{45685441}{75238261274758348800}t^{14},\ldots$$

From (24) we observe the presence of terms of equal magnitude but opposite signs between consecutive iterates. This phenomenon, so called “noise terms” [2], when present in a series solution leads to rapid convergence to the exact solution of the differential equation.
We thus have

$$u(t)=u_0+u_1+u_2+u_3+\ldots\ldots\ldots\ldots.$$

This gives $u(t)=t^2$, the exact solution of (23).

E. Example 5

Consider the third order multi-pantograph equation

$$u'''(t)=-u(t)-u\left(t-\frac{3}{10}\right)+e^{-t+\frac{3}{10}},\qquad 0\le t\le 1$$
$$u(0)=1,u'(0)=-1,u''(0)=1 \tag{25}$$

Equation (25) has an exact solution given by

$$u(t)=e^{-t} \tag{26}$$

In the table that follows we present the absolute error $\left|u(t)-\sum_{i=0}^{n}u_n(t)\right|$ incurred by LDA solution of (25) after computing a few iterates.

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TABLE II

ABSOLUTE ERRORS OF $\sum_{i=0}^{n} u_n(t)$

t	ABSOLUTE ERROR $n=1$	$n=2$	$n=3$
0.0	0.00	0.00	0.00
0.2	9.1268E-6	8.36E-8	5.00E-10
0.4	3.72647E-5	4.099E-7	5.31E-9
0.6	1.82083E-5	8.032E-7	1.11E-8
0.8	5.820844E-4	1.11145E-6	1.88E-9
1.0	2.824549E-3	3.5468E-6	6.607144E-9

Our computations shown in the table above indicate that the error incurred by our approximation method decreases as the number of iterates increases. With n=3 we observe that the largest error is of order 10^{-8}, thus implying a high degree of accuracy.

F. Example 6

Consider the third order multi-pantograph equation

$$u'''(t) = tu''(t) - u'\left(\frac{1}{2}t\right) + \cos(2t) + \cos\left(\frac{1}{2}t\right), 0 \le t \le 1 \quad (27)$$
$$u(0) = 1, u'(0) = 1, u''(0) = -1$$

After computing the first three iterates, using the LDA, we obtain the approximate solution

$$\begin{aligned}\sum_{n=0}^{2} u_n(t) &= -\frac{4529685251}{262144} - \frac{1283944099}{40448}t \\ &+ \frac{51094259}{8192}t^2 + \frac{1895}{24}t^3 + \frac{1195}{384}t^4 - \frac{1083}{1280}t^5 \\ &- \frac{1687}{4608}t^6 + \frac{587}{92160}t^7 - \frac{1}{5160960}t^8 \\ &+ 6528\cos\left(\frac{1}{2}t\right) + (-167 + 1056t)\sin\left(\frac{1}{2}t\right) + \\ &10752\cos\left(\frac{1}{4}t\right) + 262144\sin\left(\frac{1}{8}t\right) \\ &+ \left(\frac{9}{32}t + 48t\right)\cos t - 160\sin t \\ &+ \frac{9}{1024}(11 + 184320t)\cos(2t) \\ &+ \frac{1}{128}\left(-156673 + 8t + 102912t^2\right)\sin(2t) \\ &+ \frac{21}{512}t\cos(4t) - \frac{65}{2048}\sin(4t) \\ &+ \frac{1}{20709376}(237 + 64768t)\cos(8t) \\ &+ \frac{1}{2588672}\left(-9240 + 97t + 1024t^2\right)\sin(8t) \quad (28)\end{aligned}$$

By using inbuilt functions of Maple 13 it is easily verified that (28) satisfies the boundary conditions $u(0) = 1$, $u'(0) = 1$, and $u''(0) = -1$

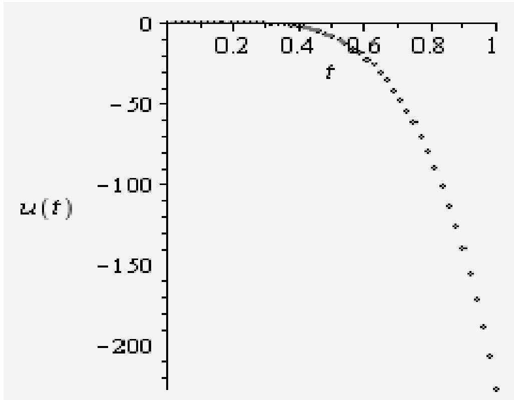

Fig. 3 LDA approximation solution of $u(t)$ for $0 \le t \le 1$

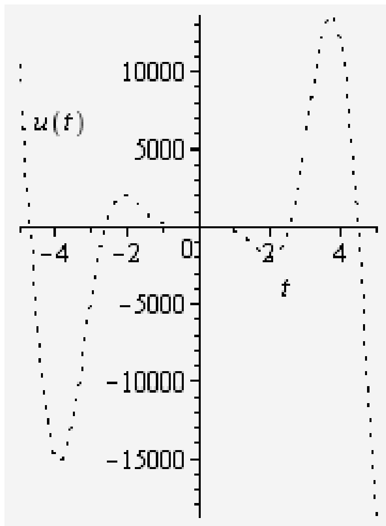

Fig. 4 LDA approximation solution of $u(t)$ for $0 \le t \le 1$

IV. CONCLUSION

In this piece of work we adopted a combination of the Laplace Transform method and the Adomian decomposition algorithm, which do not require separate calculation of Adomian polynomials, to determine the solutions of a system of multi-pantograph equations. We have demonstrated, as shown in Fig. 2, Tables I and II, that only a few terms of the series are needed to get a close approximation to the exact solution. Our accuracy in approximating the solution increases with the number of iterates. The result shows (see example 4) that it is possible to determine the exact solution, by LDA, for multi-pantograph equations that exhibit the "noise term phenomenon". For the same number of series terms LAD gives a better approximation to the solution than Taylor method (see Table I). Our findings have thrown more light to understanding and use of the Adomain decomposition algorithm method. Figs. 1, 3 and 4 show graphical solutions of some of the problems investigated.

The advantages of the LDA are as follows:

- The iterates can be easily calculated using inbuilt functions in any mathematical software;
- The series solution rapidly converges to the exact solution after a few iterates;
- It does not require discretization, perturbation, linearization or any modeling assumptions.
- We thus conclude that the LDA is a useful tool in determining solutions of multi-pantograph equations.

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