

# THE WAVELET-BASED DFT: A NEW INTERPRETATION, EXTENSIONS AND APPLICATIONS

Abdulnasir Hossen and Ulrich Heute

**Abstract**— In 1990 [1] the subband-DFT (SB-DFT) technique was proposed. This technique used the Hadamard filters in the decomposition step to split the input sequence into low- and high-pass sequences. In the next step, either two DFTs are needed on both bands to compute the full-band DFT or one DFT on one of the two bands to compute an approximate DFT. A combination network with correction factors was to be applied after the DFTs. Another approach was proposed in 1997 [2] for using a special discrete wavelet transform (DWT) to compute the discrete Fourier transform (DFT). In the first step of the algorithm, the input sequence is decomposed in a similar manner to the SB-DFT into two sequences using wavelet decomposition with Haar filters. The second step is to perform DFTs on both bands to obtain the full-band DFT or to obtain a fast approximate DFT by implementing pruning at both input and output sides.

In this paper, the wavelet-based DFT (W-DFT) with Haar filters is interpreted as SB-DFT with Hadamard filters. The only difference is in a constant factor in the combination network. This result is very important to complete the analysis of the W-DFT, since all the results concerning the accuracy and approximation errors in the SB-DFT are applicable. An application example in spectral analysis is given for both SB-DFT and W-DFT (with different filters). The adaptive capability of the SB-DFT is included in the W-DFT algorithm to select the band of most energy as the band to be computed. Finally, the W-DFT is extended to the two-dimensional case. An application in image transformation is given using two different types of wavelet filters.

**Keywords:** Image Transform, Spectral Analysis, Sub-Band DFT, Wavelet DFT.

## I. INTRODUCTION

In many applications, the computational speed of the transformation of a finite-length sequence is very important. In such cases one may be able to pay some accuracy in order to save execution time. In [1], a method called SB-DFT is introduced, which can be used to obtain a fast approximation

of the transform coefficients of a finite-length sequence (partial-band transform) with a relatively small error. Beyond, in all cases the computation can be completed to yield the exact transform (full-band transform), if required.

The fundamental principle that the FFT is based upon is that of decomposing the computation of the discrete Fourier transform of a sequence of length  $N$  into successively smaller discrete Fourier transforms of the even and odd parts [3]. The underlying idea of the SB-FFT is decomposition, too. But, the decomposition in this case has a physical meaning, since it is done by splitting the input signal into (low and high) frequency subbands and then processing them separately after the down-sampling.

The W-DFT is proposed in [2]. This transform uses the DWT (with Haar filters) to compute the FFT. The principle of the DWT is a pair of filters (low-pass and high-pass) and down-sampling. An exact computation of the DFT can be obtained by finding the DFTs of both bands. An approximate DFT can be obtained for certain signals by implementing pruning at both input and output sides. Input pruning can be done by dropping the insignificant data. Since the twiddle factors (for certain wavelets) have decreasing magnitudes, output pruning is possible for the computations related to the insignificant factors. If, however, the DFT is applied only to one of the two bands, a faster computation can be achieved, but the results are less accurate. So, basically, both SB-DFT and W-DFT are similar by having a physical meaning of the decomposition. Both input pruning and output pruning proposed in [2] can still be applied with the SB-DFT. The SB-DFT has been extended and investigated in detail [4]-[5]. The W-DFT was considered as a fast approximate FFT, but with no deep investigation of its speed and its accuracy. In this paper it is shown that both FFTs are the same under certain condition (when the wavelet filters are of Haar type), and so all the analysis of the SB-FFT can be applied to the W-DFT. Besides that this paper opens the possibility of using the same analysis of the SB-FFT with other wavelet filters to improve the SB-FFT by making it a better approximation but of course by losing some of its speed advantage.

The paper is organized as follows: In section 2, a review of the SB-DFT is given. Section 3 introduces the W-DFT with a new interpretation. A brief investigation on the complexity and accuracy of the W-DFT is included in this section with an

This manuscript submitted for review on: November, 1, 2004.

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application example in spectral analysis. The adaptive W-DFT is introduced in section 4. Extending the idea of the W-DFT to the two-dimensional transformation with an application in image transformation is included in section 5. Concluding remarks are given in section 6.

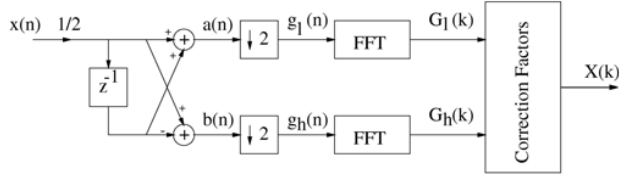


Figure 1: DFT calculation using a two-band decomposition

## II. SUBBAND-DFT

The signal  $x(n)$  is decomposed in Fig.1 into two subsequences corresponding to the low-pass signal  $a(n)$  and the high-pass signal  $b(n)$  in the upper and lower-branch of the figure, respectively. The filters used in this step are Hadamard filters. After down-sampling by a factor of 2,  $g_l(n)$  and  $g_h(n)$  are obtained:

$$\begin{aligned} g_l(n) &= \frac{1}{2}[x(2n) + x(2n+1)] \\ g_h(n) &= \frac{1}{2}[x(2n) - x(2n+1)] \end{aligned} \quad (1)$$

The filter's responses can be described by the matrix:

$$C_{Had} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (2)$$

The exact full-band size- $N$  DFT  $X(k)$  can be obtained by [4], [5]:

$$X(k) = (1 + W_N^k)G_l(k) + (1 - W_N^k)G_h(k) \quad (3)$$

If only the low-pass band sequence is to be followed (depending on a-priori information about the energy distribution of the signal),  $X(k)$  can be approximated as

$$X(k) \approx (1 + W_N^k)G_l(k), \quad k \in (0, 1, \dots, N/4 - 1) \quad (4)$$

In case of following only the high-pass band sequence,  $X(k)$  can be approximated as

$$X(k) \approx (1 - W_N^k)G_h(k), \quad k \in (N/4, \dots, N/2 - 1) \quad (5)$$

The decomposition process in Fig.1 can be applied  $m$  times to obtain  $M = 2^m$  subbands, out of which only one band is to be computed depending on the information (known a priori or

derived from the signal) about the input-signal power distribution [6].

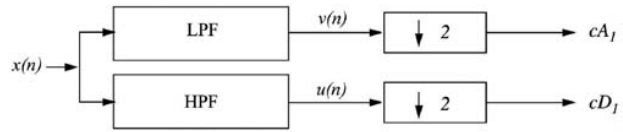


Figure 2: Wavelet decomposition.

## III. THE W-DFT

### A. Basic Idea

The block-diagram of the wavelet decomposition is shown in Fig.2. The input data is first filtered by low-pass (LPF) and high-pass (HPF) filters and then down-sampled to produce both the "approximation"  $cA_1$  and the "details"  $cD_1$ . If Haar filters are used, their impulse responses can be described by the matrix.

$$C_{Haar} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (6)$$

The full-band W-DFT can be computed as described in [2] by computing the DFT of both approximation and details bands. The approximate W-DFT is to be computed by applying pruning at both input and output sides. The wavelet decomposition stage of Fig.2 can be repeated on the approximation band again and again as proposed in [2]. Then also either the exact DFT (applying no pruning) or an approximate DFT (applying pruning on both sides) is obtainable.

### B. A New Interpretation

Comparing the block-diagram of Fig.1 and Fig.2 and Eqs.(2) and (6) shows that both SB-DFT and W-DFT are equivalent with one difference that the division in Eq.(2) is by 2 and in Eq.(6) is by  $\sqrt{2}$ . This means that:

$$\begin{aligned} cA_1 &= \sqrt{2}g_l(n) \\ cD_1 &= \sqrt{2}g_h(n) \\ FFT(cA_1) &= \sqrt{2}G_l(k) \\ FFT(cD_1) &= \sqrt{2}G_h(k) \end{aligned} \quad (7)$$

The missing part in the W-DFT, however, is the correction factor or the combination network corresponding to the terms multiplied by both transforms in Eq.(3).

### C. Accuracy

If for example an approximate SB-DFT is computed as in Eq.(4) or an approximate W-DFT is computed by transforming the approximation band  $cA_1$ , two main types of approximation errors are involved in this process [5]:

1. Linear distortions corresponding to non-constant frequency responses in the band of interest (the low-frequency band, i.e. the approximation band). This type of error can be easily compensated.
2. Aliasing due to non-ideal attenuation in the zeros of the original DFT filters. This error affects the accuracy of computation and depends on the non-zero frequency components in the high-frequency band or in the details band.

Assuming small components with equal amplitudes  $e$  outside the band of interest, the normalized aliasing error  $E(k)/e$ , for  $N = 128$  and for different values of  $m$  is plotted in Fig.3. In all cases only the low-pass band (approximation band) is to be followed.

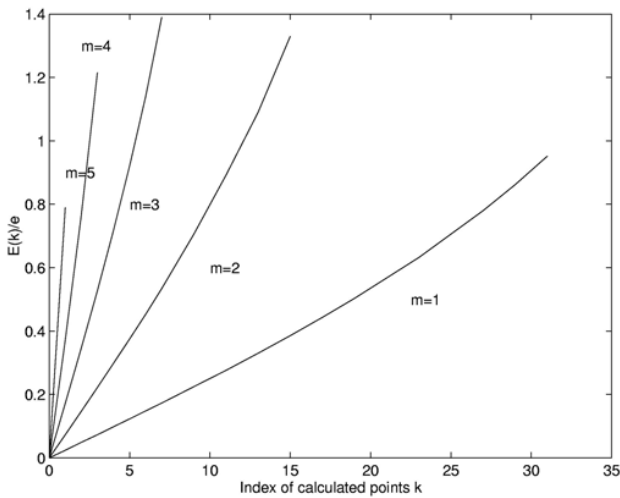


Figure 3: Normalized aliasing error of W-DFT.

#### D. Computational Complexity

In order to define the complexity of the approximate W-DFT algorithm well, let us assume that we deal with a real signal. The frequency transform of such a signal has a complex-conjugate symmetry. So if we have an input data of length  $N$ , the half-band DFT (low-pass DFT or high-pass DFT) will be of length  $N/2$ , but only  $N/4$  points are to be computed. Repeating the wavelet decomposition more and more up to  $m$  stages, a length  $N/2^m$  sequence results and is to be computed. In Fig.4, the execution time versus  $m$  (number of decomposition stages) is shown for four different values of  $N$ . At  $m = 0$ , a "Cooley-Tukey"-type FFT of length  $N$  is considered for comparison. Execution-time saving at  $m = 1$  (half-band case) is 40%, and it increases to 65% at  $m = 2$  (quarter-band case).

#### E. Application Example

The W-DFT is applied in spectral analysis in detecting two adjacent sinusoids in wide-band noise, as shown in Fig.5. The two sinusoids are  $f_1 = 35$  Hz and  $f_2 = 40$  Hz, with unity amplitude. The sampling frequency is 1024 Hz and the SNR is 3db. It is to be noted that using SB-DFT or W-DFT (db2, db4

or db8), the two sinusoids are easily detected. The aliasing errors contained in the spectral are less using db8 filters compared to Hadamard filters or db2 or even db4 filters.

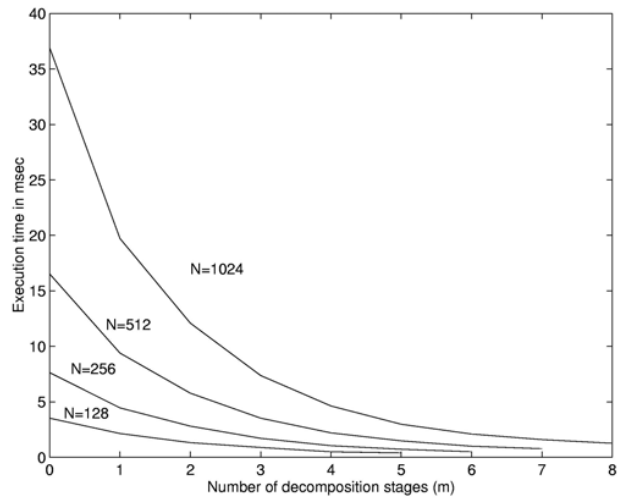


Figure 4: Running-time comparison of W-DFT.

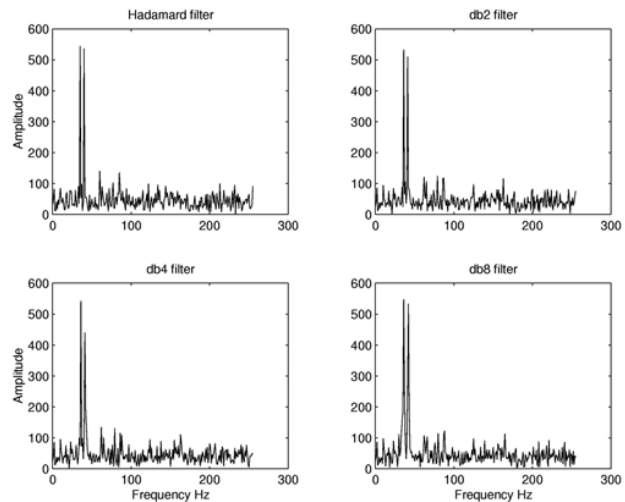


Figure 5: Application example of sinusoid detection

#### IV. ADAPTIVE W-DFT

If there is no information about the energy distribution of the input sequence, a band-selection algorithm identical to that used with the SB-DFT [6] can be used. This method depends on the energy comparison between the approximation and details subsequences after the down sampling in Fig.2:

$$B = \sum_{n=0}^{\frac{N}{2}-1} (cA_1(n))^2 - (cD_1(n))^2 \quad (8)$$

According to the sign of  $B$ , the decision is taken: If  $B$  is positive, the low-frequency band is considered, and if  $B$  is

negative, the high-frequency band is considered. Since only the sign of  $B$  is important, Eq.8 can be simplified to:

$$\text{sgn}(B) = \text{sgn} \sum_{n=0}^{\frac{N}{2}-1} |cA_1(n)| - |cD_1(n)| \quad (9)$$

## V. TWO-DIMENSIONAL W-DFT

The same idea of computing one-out-of-two bands in a one-dimensional W-DFT can be implemented to compute the two-dimensional W-DFT by computing one-out-of-four bands at each stage. Fig.6 shows a single stage of a wavelet decomposition in two dimensions. Firstly the LPF and HPF and down sampling are applied in row direction. Then the LPF and HPF and down sampling are applied in column direction on the resulting two sequences to obtain four subbands. Those subbands are called the approximation  $cA_{j+1}$  and the horizontal details  $(cD_{j+1})^{(h)}$  and vertical details  $(cD_{j+1})^{(v)}$  and diagonal details  $(cD_{j+1})^{(d)}$ . So if we know that the signal is concentrated in only the first approximation band, the other 3 details bands can be ignored and an approximate 2-D W-DFT results. This idea is applied in image transformation as, e.g., in Fig.7. The reconstructed image "Woman" is obtained with the IFFT of the approximation band of Fig.6. This case is considered as half-band case in which one out of four bands is transformed. Also the FFT of this band is shown. In the same figure, the idea is repeated for another decomposition stage and the quarter-band (1-out-of-16 bands) reconstructed image is shown with its transform also. Fig.8 shows a similar example of that of Fig.7 by using "db2" wavelet-filters instead of the "Haar" filters. The reconstructed image with db2 filters is better than that with Haar filters.

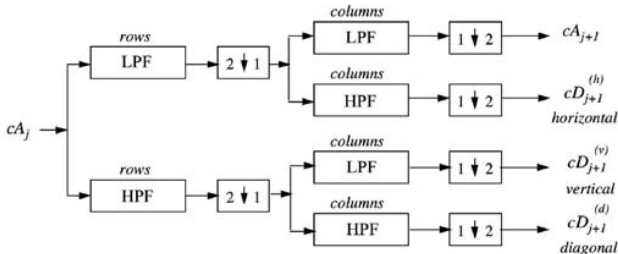


Figure 6: First stage 2-D wavelet decomposition

## VI. CONCLUSIONS

The W-DFT with Haar-filters is interpreted as a SB-DFT (with Hadamard-filters) by introducing a constant factor in the combination network. The more general choice of the "band of interest", as commonly applied in the SB-DFT, can also be transferred to the W-DFT: Not necessarily only the low-pass section has to be followed as proposed in [2]. The accuracy and complexity analysis of the SB-DFT can thus be also applied here. An application example of the W-DFT in spectral analysis is introduced. The idea of the adaptive SB-DFT is also applicable with the W-DFT, so that the W-DFT has an adaptive capability to decide at each stage of decomposition which band is to be followed and which band is to be ignored. Lastly, the 2-D W-DFT is implemented with

an example in image transformation. Two different types of filters are used with this example. Implementing wavelet decomposition with other than Haar filters can result in better approximations but of course on the cost of more complexity, since the filters now are no more additions and subtractions only.

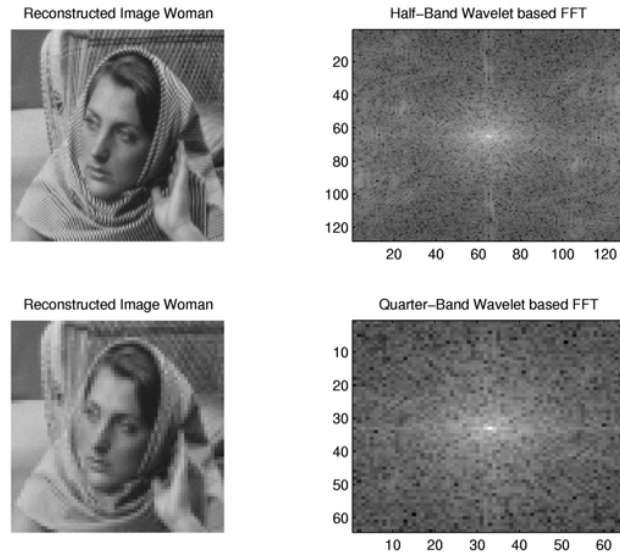


Figure 7: 2-D W-DFT (with Haar filters) and reconstructed image "Woman".

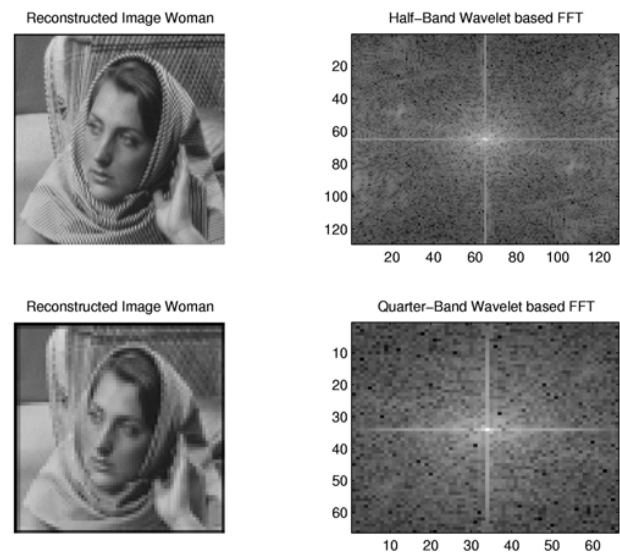


Figure 8: 2-D W-DFT (with db2 filters) and reconstructed image "Woman".

## ACKNOWLEDGEMENT

This work was supported by DAAD (German Academic Exchange Service) through a research scholarship in winter 2004 and Sultan Qaboos University at Muscat, Sultanate of Oman.

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