

# Anti-Homomorphism in Fuzzy Ideals

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**Abstract**—The anti-homomorphic image of fuzzy ideals, fuzzy ideals of near-rings and anti ideals are discussed in this note. A necessary and sufficient condition has been established for near-ring anti ideal to be characteristic.

**Keywords**—Fuzzy Ideals, Anti fuzzy subgroup, Anti fuzzy ideals, Anti homomorphism, Lower  $\alpha$  level cut.

## I. INTRODUCTION

IN 1971, Rosenfeld [11] constituted the elementary concepts of fuzzy subgroupoid, fuzzy ideals and fuzzy subgroups. Biswas [3] introduced the notion of anti fuzzy subgroups. Fuzzy subnear-rings are introduced by Abou-Zaid [1]. He studied fuzzy left (resp. right) ideals of a near-ring and gave some properties of fuzzy prime ideals of a near-ring. In [7], it has been established that homomorphic image of a fuzzy left (resp. right) ideal which has "sup property" is a fuzzy left (resp. right) ideal. In the year 1998, Sung M.H. et al. [12] proved the same result using the level fuzzy subsets and obtained some properties based on near-ring homomorphism. Properties of anti-homomorphic images of near-rings are discussed in [5]. Homomorphic images and pre images of anti fuzzy ideals are investigated by K.H. Kim et al. [9]. The notion of anti homomorphic image and pre image of fuzzy and anti fuzzy ideals are investigated in this paper. Also, near-ring anti homomorphic image and pre image of ideals are obtained.

### A. Preliminaries

In this section, review of fuzzy set theoretic concepts are given briefly (for details one can refer [4], [11] and [10]). A fuzzy set  $\mu$  of a set  $N$  is a function  $\mu : N \rightarrow [0, 1]$ .

$\mu$  will be called a fuzzy left ideal [11], if  $\mu(xy) \geq \mu(y)$ ; a fuzzy right ideal, if  $\mu(xy) \geq \mu(x)$ ; anti fuzzy left ideal [3] if  $\mu(xy) \leq \mu(y)$ ; anti fuzzy right ideal, if  $\mu(xy) \leq \mu(x)$ .

Let  $f : N \rightarrow N'$  be a function and let  $\mu$  and  $\nu$  be fuzzy sets in  $N$  and  $N'$  respectively. Then  $f(\mu)$  [11], the image of  $\mu$  under  $f$  is a fuzzy set in  $N'$  defined by

$$f(\mu)(y) = \begin{cases} \sup \{ \mu(x) : x \in f^{-1}(y) \} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{Otherwise} \end{cases}$$

for all  $y \in N'$ .  $f^{-1}(\nu)$  [11], the preimage of  $\nu$  under  $f$  is a fuzzy set in  $N$  given by

$$f^{-1}(\nu)(x) = \nu(f(x))$$

for all  $x \in N$ .

Similar to an  $\alpha$  level cut [4], we have lower level cut [9] as follows:

Let  $\mu$  be a fuzzy set in a set  $N$ . For  $\alpha \in [0, 1]$ , the lower  $\alpha$  level cut of  $\mu$  is denoted by  ${}_{\alpha}N_{\mu}$  and is given by

$${}_{\alpha}N_{\mu} = \{n \in N : \mu(n) \leq \alpha\}.$$

**Definition 1.1:** [1], [7] Let  $N$  be a left near-ring and  $\mu$  be a non empty fuzzy sub set of  $N$ .  $\mu$  is said to be a fuzzy left  $N$ -ideal if

- I-1.  $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$ ,
- I-2.  $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ , for all  $x, y \in N$ ,
- I-3.  $\mu(y + x - y) \geq \mu(x)$  and
- I-4.  $\mu(xy) \geq \mu(y)$  where  $x, y \in N$

are satisfied. If axioms (I-1), (I-2), (I-3) with

- I-5.  $\mu((x + z)y - xy) \geq \mu(z)$

holds,  $\mu$  is a fuzzy right  $N$ -ideal.

From the definition of right near-ring [6], ideals can be defined as follows:

**Definition 1.2:** Let  $N$  be a right near-ring and  $\mu$  be a non empty fuzzy sub set of  $N$ .  $\mu$  is said to be a fuzzy left  $N$ -ideal if (I-1), (I-2), (I-3) and

- I-6.  $\mu(xy) \geq \mu(y)$  where  $x, y \in N$

are satisfied. If (I-7) is postulated instead of (I-5) of fuzzy  $N$ -ideal of left near-ring,  $\mu$  is a fuzzy right  $N$ -ideal where

- I-7.  $\mu(y(x + z) - yx) \geq \mu(z)$ .

**Definition 1.3:** [9] Let  $N$  be a left near-ring and  $\mu$  be a non empty fuzzy sub set of  $N$ .  $\mu$  is said to be an anti fuzzy left  $N$ -ideal if

- AI-1.  $\mu(x - y) \leq \max\{\mu(x), \mu(y)\}$ ,

- AI-2.  $\mu(xy) \leq \max\{\mu(x), \mu(y)\}$ ,

- AI-3.  $\mu(y + x - y) \leq \mu(x)$  and

- AI-4.  $\mu(xy) \leq \mu(x)$  where  $x, y \in N$ .

If axioms (AI-1), (AI-2), (AI-3) with the following (AI-5) are satisfied then  $\mu$  is an anti fuzzy right  $N$ -ideal;

- AI-5.  $\mu((x + z)y - xy) \leq \mu(z)$ .

**Definition 1.4:** Let  $N$  be a right near-ring and  $\mu$  be a non empty fuzzy sub set of  $N$ .  $\mu$  is said to be an anti fuzzy left  $N$ -ideal when AI-1 to AI-3 along with

- AI-6.  $\mu(xy) \leq \mu(y)$  where  $x, y \in N$

are postulated. If axioms (AI-1), (AI-2), (AI-3) with (AI-7) are satisfied then  $\mu$  is an anti fuzzy right  $N$ -ideal;

- AI-7.  $\mu(y(x + z) - yx) \leq \mu(z)$ .

Recall that, a function  $f : N \rightarrow N'$  of near-rings is called an **anti-homomorphism** [5] when

1.  $f(n + m) = f(m) + f(n)$

2.  $f(nm) = f(m)f(n)$ , for all  $n, m \in N$ .

A surjective anti-homomorphism is called an **anti-epimorphism** ( $\cong$ ).

II. MAIN RESULTS

A. Fuzzy Ideals

It is to be noted that the anti homomorphic image pre-image of a fuzzy groupoid is again a fuzzy groupoid. Where as,

*Result 2.1:* An anti homomorphic pre-image of a right (left) ideal is a left (right) ideal.

**Proof:** Let  $\nu$  be a fuzzy left ideal. Then

$$\begin{aligned} \mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq \nu(f(x)) \\ &= \mu(x), \end{aligned}$$

a right ideal. When  $\nu$  is a fuzzy right ideal,

$$\begin{aligned} \mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq \nu(f(y)) \\ &= \mu(y), \end{aligned}$$

a left ideal. □

*Result 2.2:* Anti-homomorphic image of a fuzzy left (right) ideal, with spermium property, is a fuzzy right (left) ideal.

**Proof:** Let  $\mu$  be a fuzzy left ideal with sup property. Given  $f(x), f(y)$  in  $f(N)$ , let  $x_0 \in f^{-1}[f(x)], y_0 \in f^{-1}[f(y)]$  be such that

$$\mu(x_0) = \sup_{t \in f^{-1}[f(x)]} \mu(t), \quad \mu(y_0) = \sup_{t \in f^{-1}[f(y)]} \mu(t)$$

respectively. Then

$$\begin{aligned} \nu[f(x)f(y)] &= \nu[f(yx)] \\ &= \sup_{t \in f^{-1}[f(yx)]} \mu(t) \\ &\geq \mu(y_0x_0) \\ &\geq \mu(x_0) \\ &= \sup_{t \in f^{-1}[f(x)]} \mu(t) \\ &= \nu[f(x)], \end{aligned}$$

implies  $\nu$  is a fuzzy right ideal. Similarly, when  $\mu$  is a fuzzy right ideal with above property, we have  $\nu$  is a fuzzy left ideal. □

In sequel to the above results, the following also can be established for the case of anti fuzzy left (right) ideals.

*Result 2.3:* An anti homomorphic pre-image of an anti fuzzy right (left) ideal is an anti fuzzy left (right) ideal.

*Result 2.4:* An anti homomorphic image of an anti fuzzy right (left) ideal with sup property, is an anti fuzzy left (right) ideal.

B. Fuzzy Ideals in Near-rings

*Result 2.5:* ([5] Theorem 2.2) Anti homomorphic image of a right near-ring (left near-ring) is a left near-ring (right near-ring).

*Result 2.6:* Let  $f : N \rightarrow N'$  be an anti-epimorphism of near-rings. If  $\nu$  is a fuzzy (left/right) ideal in the right (left) near-ring  $N'$ , then  $\mu$ , which is  $f^{-1}(\nu)$  is a fuzzy (left/right) ideal in the left (right) near-ring  $N$ .

**Proof:**

Let  $\nu$  be a fuzzy left ideal of right near-ring  $N'$ . The proof of conditions (I-1), (I-2) and (I-3) of definition 1.1 are similar to that of proof of [7] Theorem 2.12. For any  $x, y \in N$ , we have

$$\begin{aligned} \mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq \nu(f(y)) \\ &= \mu(y). \end{aligned}$$

Thus  $\mu$  is a fuzzy left ideal of the left near-ring  $N$ . When  $\nu$  is a right ideal of right near-ring  $N'$ , for any  $x, y, z \in N$  we have,

$$\begin{aligned} \mu((x+z)y - xy) &= \nu(f((x+z)y - xy)) \\ &= \nu(f(y)f(x+z) - f(xy)) \\ &= \nu(f(y)(f(x) + f(z)) - f(y)f(x)) \\ &\geq \nu(f(z)) \\ &= \mu(z), \end{aligned}$$

$\mu$  is a fuzzy right ideal of left near-ring  $N$ .

Let  $\nu$  be a right ideal of left near-ring  $N'$ . For any  $x, y, z \in N$ , we have

$$\begin{aligned} \mu(y(x+z) - yx) &= \nu(f((x+z)y - xy)) \\ &= \nu(f(x+z)f(y) - f(yx)) \\ &= \nu((f(x) + f(z))f(y) - f(x)f(y)) \\ &\geq \nu(f(z)) \\ &= \mu(z). \end{aligned}$$

Thus  $\mu$  is a fuzzy left ideal of right near-ring  $N$ . Let  $\nu$  is a fuzzy left ideal of left near-ring  $N'$ . Then

$$\begin{aligned} \mu(xy) &= \nu(f(xy)) \\ &= \nu(f(y)f(x)) \\ &\geq \nu(f(x)) \\ &= \mu(x) \end{aligned}$$

for all  $x, y \in N$ , implies  $\mu$  is a left ideal of right near ring  $N$ . □

*Result 2.7:* Let  $f : N \rightarrow N'$  be an anti-epimorphism of near-rings. If  $\mu$  is a fuzzy (left/right) ideal in the left (right) near-ring  $N$  with sup property, then  $\nu = f(\mu)$  is a fuzzy (left/right) ideal in the right (left) near-ring  $N'$ .

**Proof:** Let  $\mu$  be a fuzzy left ideal of the left near-ring  $N$  with sup property and  $\nu$  be the image of  $\mu$  under  $f$ . Let  $x_0 \in f^{-1}[f(x)]$ ,  $y_0 \in f^{-1}[f(y)]$  such that

$$\begin{aligned} \mu(x_0) &= \sup_{t \in f^{-1}[f(x)]} \mu(t), \quad \mu(y_0) = \sup_{t \in f^{-1}[f(y)]} \mu(t). \\ \nu[f(x)f(y)] &= \sup_{t \in f^{-1}[f(yx)]} \mu(t) \\ &\geq \mu(y_0x_0) \\ &\geq \mu(x_0) \\ &= \sup_{t \in f^{-1}[f(x)]} \mu(t) \\ &= \nu[f(x)]. \end{aligned}$$

That is  $\nu$  is a fuzzy left ideal of right near-ring  $N'$ .

If  $\mu$  is a fuzzy right ideal of left near-ring. For any  $f(z) \in f(N)$ , let  $z_0 \in f^{-1}[f(z)]$  such that

$$\mu(z_0) = \sup_{t \in f^{-1}[f(z)]} \mu(z).$$

Now,

$$\begin{aligned} &\nu[f\{(x+z)y - (xy)\}] \\ &= \nu[f(y)f(x+z) - f(xy)] \\ &= \nu[f(y)\{f(x) + f(z)\} - f(y)f(x)] \\ &= \sup_{t \in f^{-1}\{f[f(y)\{f(x)+f(z)\} - f(y)f(x)]\}} \mu(t) \\ &\geq \mu[y_0\{x_0 + z_0\} - y_0x_0] \\ &\geq \mu[z_0] \\ &= \sup_{t \in f^{-1}[f(z)]} \mu(t) \\ &= \nu[f(z)]. \end{aligned}$$

That is,  $\nu$  is a fuzzy right ideal of right near-ring.

The image and pre-image of the fuzzy ideal of a fuzzy right near-ring  $N$  can be proved to be the fuzzy ideal of a left near-ring  $N'$ . □

### C. Anti Fuzzy Ideals in Near-rings

**Definition 2.8:** A left  $N$ -ideal  $A$  of a near-ring is said to be characteristic [2], [8] if

$$f(A) = A \quad \forall f \in \text{Aut}(N)$$

where  $\text{Aut}(N)$  is set of all automorphism of  $N$ .

Anti fuzzy left  $N$ -ideal of  $\mu$  of a near-ring  $N$  is said to be anti fuzzy characteristic if

$$\mu f(x) = \mu(x) \quad \forall x \in N, f \in \text{Aut}(N).$$

**Lemma 2.9:** Let  $\mu$  be an anti fuzzy left  $N$ -ideal of a near-ring  $N$  and let  $x \in N$ . Then  $\mu(x) = s$  if and only if  $x \in {}_sN_\mu$  and  $x \notin {}_tN_\mu \quad \forall s > t$ .

Proof is obvious.

The proof of the following theorem is analogous to the proof of theorem 3.9 [2], [8].

**Theorem 2.10:** Let  $\mu$  be an anti fuzzy  $N$ -ideal of a near-ring  $N$ . Then each lower  $\alpha$  level left  $N$ -ideal of  $\mu$  is characteristic iff  $\mu$  is an anti fuzzy characteristic of  $N$ .

K. H. Kim et al. [9] proved the following theorems:

**Result 2.11 ([9], Theorem 3.19 (1)):** Let  $f : N \rightarrow N'$  be an epimorphism of near-rings. Let  $\nu$  be an anti-fuzzy left  $N'$ -ideal and  $\mu$  be the pre-image of  $\nu$  under  $f$ . Then  $\mu$  is an anti-fuzzy left  $N$ -ideal.

**Result 2.12 ([9], Theorem 3.19 (2)):** Let  $f : N \rightarrow N'$  be a surjective homomorphism of near-rings. If  $\mu$  is an anti fuzzy left  $N'$ -ideal then  $f^{-1}(\mu)$  is an anti fuzzy left  $N$ -ideal.

**Result 2.13:** Let  $f : N \rightarrow N'$  be an anti-epimorphism of near-rings. Let  $\nu$  be an anti-fuzzy (left/right) ideal of right (left) near-ring  $N'$  then  $\mu$ , the pre-image of  $\nu$  under  $f$ , is an anti-fuzzy (left/right) ideal of left (right) near-ring  $N$ .

The proof of AI-4 and AI-5 are similar to the proof of result 2.6.

**Result 2.14:** Let  $f : N \rightarrow N'$  be an anti epimorphism. If  $\mu$  is an anti fuzzy (left/right) ideal of left (right) near-ring  $N$  with sup property,  $f(\mu)$  is an anti fuzzy (left/right) ideal of right (left) near-ring  $N'$ .

The proof of AI-6 and AI-7 are similar to that of result 2.7.

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