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Adaptive functional projective lag synchronization of Lorenz system

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Abstract—This paper addresses functional projective lag synchronization of Lorenz system with four unknown parameters, where the output of the master system lags behind the output of the slave system proportionally. For this purpose, an adaptive control law is proposed to make the states of two identical Lorenz systems asymptotically synchronize up. Based on Lyapunov stability theory, a novel criterion is given for asymptotical stability of the null solution of an error dynamics. Finally, some numerical examples are provided to show the effectiveness of our results.

Keywords—Adaptive function projective synchronization, Chaotic system, Lag synchronization, Lyapunov method

I. Introduction

Chaos synchronization is very hot topic of nonlinear society, which has attracted much interest from scientists and engineers since Pecora and Carroll [1] introduced the concept of synchronization. Up to date, chaos synchronization has been developed extensively due to its various applications [2]-[24]. Originally, chaos synchronization refers to the state in which the master (or drive) and the slave (or response) systems have precisely identical trajectories for time to infinity. We usually regard such a synchronization as complete synchronization or identical synchronization.

Over the last decade, various methods for chaos synchronization have been proposed, which include complete synchronization [12]-[13], phase synchronization [14] , lag synchronization [15], intermittent lag synchronization [16], time scale synchronization [17], intermittent generalized synchronization [18], projective synchronization (PS) [19], generalized synchronization [20], and adaptive modified projective synchronization [21]-[22]. Amongst all kinds of chaos synchronization, FPS is the state of the art subject of synchronization study. Recently, FPS has been reported by Chen et al. [23] and Runzi [24], that is the generalization of PS. As compared with PS, FPS means that the master and slave systems could be synchronized up to a scaling function, but not a constant. In real-world situation, time delay is ubiquitous between communication channels. and its existence is frequently a source of instability and poor performance in systems. Thus it is natural to consider time delay when we deal with synchronization problem among chaotic systems. In this regard, the synchronization concept is called functional projective lag synchronization (FPLS). In addition, if we consider parametric uncertainties of the systems, then we have more general scheme of synchronization. This scheme calls adaptive functional projective lag synchronization (AFPLS). In this paper, we investigate AFPLS for chaotic systems for the first time.

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II. Synchronization of Lorenz system

Consider the following master (drive) and slave (response) chaotic systems

$$\dot{x}(t) = f(t, x), \tag{1}$$
$$\dot{y}(t) = g(t, y) + u(t, x, y), \tag{2}$$

where $x(t) = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ and $y(t) = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n$ are drive and response state vectors respectively, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are continuous nonlinear vector functions and $u(t, x, y) = (u_1, u_2, \ldots, u_n)^T \in \mathbb{R}^n$ is the control input for synchronization between master (1) and slave (2).

Definition 1. It is said that AFPLS occurs between master system (1) and response system (2) if there exist scaling functions $\alpha_i(t)$ such that $\lim_{t\to\infty} \|\alpha_i(t)y_i(t) - x_i(t-\tau)\| = 0$ for given positive scalar τ.

For convenience's sake to illustrate the scheme of AFPLS, we consider the following Lorenz master and slave systems:

$$\begin{aligned}
\text{Master sys. :}\quad & \dot{x}_1(t) = a(x_2(t) - x_1(t)) \\
& \dot{x}_2(t) = bx_1(t) - x_1(t)x_3(t) - cx_2(t) \\
& \dot{x}_3(t) = x_1(t)x_2(t) - dx_3(t), \\
\text{Slave sys. :}\quad & \dot{y}_1(t) = a_1(y_2(t) - y_1(t)) + u_1 \\
& \dot{y}_2(t) = b_1y_1(t) - y_1y_3 - c_1y_2 + u_2 \\
& \dot{y}_3(t) = y_1(t)y_2(t) - d_1y_3(t) + u_3,
\end{aligned} \tag{3}$$

where a, b, c, d are unknown parameters of master system and a_1, b_1, c_1, d_1 are parameters of slave system which need to be estimated.

In order to see chaotic motion of the system (3), let us take initial condition $x(0) = \{0, 1, 1\}^T$ and the parameters $a = 10, b = 28, c = 1, d = 8/3$. Actually, Fig. 1 shows chaotic motion of Lorenz system.

Now, for AFPLS, let us define error signals in the sense of

Definition 1 as

$$\begin{aligned} e_1(t) &= \alpha_1(t)y_1(t) - x_1(t-\tau) \\ e_2(t) &= \alpha_2(t)y_2(t) - x_2(t-\tau) \\ e_3(t) &= \alpha_3(t)y_3(t) - x_3(t-\tau). \end{aligned} \quad (4)$$

The time derivative of error signal (4) is

$$\begin{aligned} \dot{e}_1(t) &= \dot{\alpha}_1(t)y_1(t) + \alpha_1(t)\dot{y}_1(t) - \dot{x}_1(t-\tau) \\ \dot{e}_2(t) &= \dot{\alpha}_2(t)y_2(t) + \alpha_2(t)\dot{y}_2(t) - \dot{x}_2(t-\tau) \\ \dot{e}_3(t) &= \dot{\alpha}_3(t)y_3(t) + \alpha_3(t)\dot{y}_3(t) - \dot{x}_3(t-\tau). \end{aligned} \quad (5)$$

By substituting (3) into (5), we have the following error dynamics

$$\begin{aligned} \dot{e}_1(t) &= \dot{\alpha}_1(t)y_1 + \alpha_1(t)\big(a_1(y_2-y_1) + u_1\big) \\ &\quad -a\big(x_2(t-\tau) - x_1(t-\tau)\big) \\ \dot{e}_2(t) &= \dot{\alpha}_2(t)y_2 + \alpha_2(t)\big(b_1y_1 - y_1y_3 - c_1y_2 + u_2\big) \\ &\quad -\big(bx_1(t-\tau) - x_1(t-\tau)x_3(t-\tau) - cx_2(t-\tau)\big) \\ \dot{e}_3(t) &= \dot{\alpha}_3(t)y_3 + \alpha_3(t)\big(y_1y_2 - d_1y_3 + u_3\big) \\ &\quad -\big(x_1(t-\tau)x_2(t-\tau) - dx_3(t-\tau)\big). \end{aligned} \quad (6)$$

Here, our goal is to achieve synchronization between two Lorenz systems with different initial conditions. For this end, the following control laws are designed:

$$\begin{aligned} u_1(t) &= \frac{1}{\alpha_1(t)}\Big(-\dot{\alpha}_1(t)y_1 - \alpha_1(t)a_1(y_2-y_1) \\ &\quad +a_1(x_2(t-\tau) - x_1(t-\tau)) - k_1e_1\Big) \\ u_2(t) &= \frac{1}{\alpha_2(t)}\Big(-\dot{\alpha}_2(t)y_2 - \alpha_2(t)(b_1y_1 - y_1y_3 - c_1y_2) \\ &\quad +b_1x_1(t-\tau) - x_1(t-\tau)x_3(t-\tau) \\ &\quad -c_1x_2(t-\tau) - k_2e_2(t)\Big) \\ u_3(t) &= \frac{1}{\alpha_3(t)}\Big(-\dot{\alpha}_3(t)y_3 - \alpha_3(t)(y_1y_2 - d_1y_3) \\ &\quad +x_1(t-\tau)x_2(t-\tau) - d_1x_3(t-\tau) - k_3e_3\Big), \end{aligned} \quad (7)$$

where $k_i > 0$ and $\alpha_i(t) \neq 0$ for all $t(i = 1, 2, 3)$.
Substituting the control input (7) into Eq. (6) gives that

$$\begin{aligned} \dot{e}_1(t) &= (a_1 - a)(x_2(t-\tau) - x_1(t-\tau)) - k_1e_1, \\ \dot{e}_2(t) &= (b_1 - b)x_1(t-\tau) - (c_1 - c)x_2(t-\tau) - k_2e_2, \\ \dot{e}_3(t) &= -(d_1 - d)x_3(t-\tau) - k_3e_3. \end{aligned} \quad (8)$$

Then, we have the following theorem.
Theorem 1. For given scaling functions $\alpha_i(t)(i = 1, 2, 3)$ and time delay τ, the AFPLS between master and slave systems given in Eq. (3) will occur by the control law (7) and the update rule for four unknown parameters a_1, b_1, c_1, d_1:

$$\begin{aligned} \dot{a}_1 &= -(x_2(t-\tau) - x_1(t-\tau))e_1(t) \\ \dot{b}_1 &= -x_1(t-\tau)e_2(t) \\ \dot{c}_1 &= x_2(t-\tau)e_2(t) \\ \dot{d}_1 &= x_3(t-\tau)e_3(t). \end{aligned} \quad (9)$$

This implies that the error signals satisfy $\lim_{t\to\infty}\|e_i(t)\| = 0(i = 1, 2, 3)$. Furthermore, the uncertain parameters are well estimated as $\lim_{t\to\infty}\|a_1 - a\| = 0$, $\lim_{t\to\infty}\|b_1 - b\| = 0$, $\lim_{t\to\infty}\|c_1 - c\| = 0$, and $\lim_{t\to\infty}\|d_1 - d\| = 0$.

Proof. Let us define the following Lyapunov function candidate

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2), \quad (10)$$

where $e_a = a_1 - a$, $e_b = b_1 - b$, $e_c = c_1 - c$ and $e_d = d_1 - d$. By differentiating Eq. (10) and using (7) and (9) we obtain

$$\begin{aligned} \dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + e_a\dot{e}_a + e_b\dot{e}_b + e_c\dot{e}_c + e_d\dot{e}_d \\ &= -\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}^T \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \\ &\equiv -e^TPe. \end{aligned} \quad (11)$$

Since $\dot{V}$ is negative semidefinite, we cannot immediately obtain that the origin of error system (5) is asymptotically stable. In fact, as $\dot{V} \leq 0$, then $e_1, e_2, e_3, e_a, e_b, e_c, e_d \in \mathcal{L}_\infty$. From the error system (4), we have $\dot{e}_1, \dot{e}_2, \dot{e}_3 \in \mathcal{L}_\infty$. Since $\dot{V} = -e^TPe$ and P is a positive-definite matrix, then we have

$$\begin{aligned} &\int_0^t \lambda_{min}(P)\|e\|^2dt \leq \int_0^t e^TPe\ dt \leq \int_0^t -\dot{V}dt \\ &= V(0) - V(t) \leq V(0), \end{aligned}$$

where $\lambda_{min}(P)$ is the minimum eigenvalue of positive-definite matrix P. Thus $e_1, e_2, e_3 \in \mathcal{L}_2$. According to the Barbalat's lemma, we have $e_1(t), e_2(t), e_3(t) \to 0$ as $t \to \infty$. Therefore, the slave system synchronize the master system in the sense of AFPLS (4). This completes the proof. □

III. Numerical simulation

In order to demonstrate the validity of proposed ideas, some numerical simulations are presented. Fourth-order Runge-Kutta method with sampling time $0.0001[\mathrm{sec}]$ is used to solve the system of differential equations (3).
The system parameters are used by $a = 10, b = 28, c = 1, d = 8/3$ in numerical simulation. The initial conditions for master and slave system are given by $x(0) = (0, 1, 1)^T$ and $y(0) = (-2.5, -1.5, -1.65)^T$, respectively. The scaling functions for functional synchronization are taken for a choice as

$$\begin{aligned} \alpha_1(t) &= 1.1 + \sin 2t \\ \alpha_2(t) &= 1.1 + \cos\frac{\pi}{2}t \\ \alpha_3(t) &= e^{-0.07t} + 4. \end{aligned} \quad (12)$$

Fig 2. displays the scaling functions. Also, $\tau = 1$ and control gain $k_i = 1(i = 1, 2, 3)$ are chosen.

Case 1. First, we consider the problem of FPLS without uncertain parameters. From data given above, the control input is applied at $30[\mathrm{sec}]$ to show the effect of synchronization.
Fig 3. shows that error signals of FPLS go to zero asymptotically. It means FPLS occurs between lag state of x and current state of y.

Case 2. Second, we revisit the Case 1 to show AFPLS with uncertain four parameters a, b, c, d. The initial guess of parameters are chosen as $a_1(0) = 0, b_1(0) = 0, c_1(0) = 0, d_1(0) = 0$. Fig 4. displays error signals for this case, in which $\|e_i(t)\|(i = 1, 2, 3)$ go to zero just as we intended. Finally, Fig 5. displays parameters go to the values of master parameters: $\lim_{t\to\infty} a_1(t) = 10, \lim_{t\to\infty} b_1(t) = 28, \lim_{t\to\infty} c_1(t) = 1, \lim_{t\to\infty} d_1(t) = 8/3$.

IV. Conclusion

In this paper, we have investigated the synchronization problem of two Lorenz systems with four unknown parameters. An adaptive control scheme is presented for functional lag synchronization. Numerical simulations show that our novel idea is effective for AFPLS of Lorenz systems. The final remark is that the proposed method is applicable to any chaotic systems.

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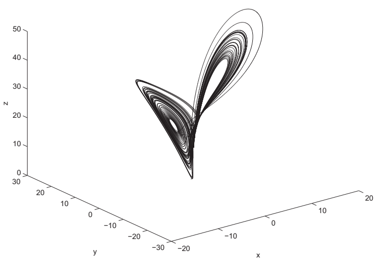

Fig. 1. Chaotic motion of Lorenz system

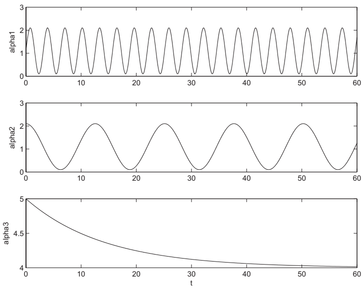

Fig. 2. Scaling functions

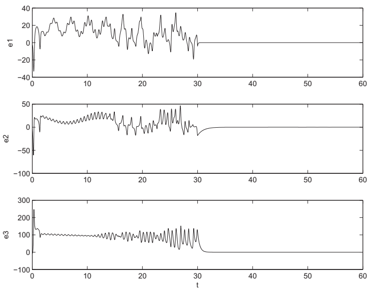

Fig. 3. Error signals: Case 1

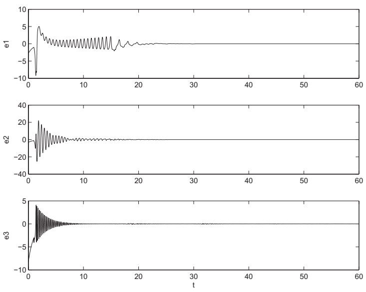

Fig. 4. Error signals: Case 2

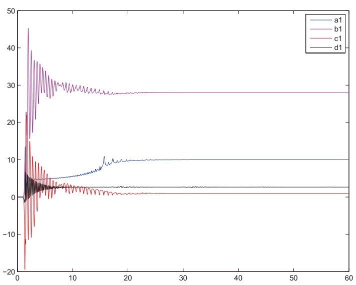

Fig. 5. Estimated values for unknown parameters