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Second Order Admissibilities in Multi-parameter Logistic Regression Model

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***Abstract*—In multi-parameter family of distributions, conditions for a modified maximum likelihood estimator to be second order admissible are given. Applying these results to the multi-parameter logistic regression model, it is shown that the maximum likelihood estimator is always second order inadmissible. Also, conditions for the Berkson estimator to be second order admissible are given.

***Keywords*—Berkson estimator, modified maximum likelihood estimator, Multi-parameter logistic regression model, second order admissibility.

I. Introduction

LOGISTIC regression model is often used as the way of statistical analysis of binary data. In the model, [1] asserted that the minimum logit chi-squared estimator (MLχ^2E for short) is better than the maximum likelihood estimator (MLE) by comparing the exact mean squared errors (MSEs) of the two estimators. The problem which the MLE or the MLχ^2E is better, called the Berkson's bioassay problem, is discussed by many researchers. In this model, there exists a completely sufficient statistic. So the MLχ^2E can be improved by the Rao-Blackwell theorem, which is called Berkson estimator. [2] evaluated the Taylor expansions of the MSEs of these estimators up to the second order and showed that the MSE of the Berkson estimator is asymptotically smaller than that of the MLE. [3] tried to solve the problem in terms of asymptotic admissibility. First, they derived a necessary and sufficient condition for a modified MLE to be second order admissible (SOA) under the quadratic loss function in general setup of one-parameter case. Especially, in the logistic regression model, they showed that the MLE is always second order inadmissible (SOI) and the Berkson estimator is SOA if and only if the number of the doses is greater than or equal to 4. However, in multi-parameter logistic regression model, whether the two estimators are SOA or not is open.

Recently, [4] derived conditions for a modified MLE to be SOA in general setup of two-parameter case. Also, they showed that the MLE is always SOI and the Berkson estimator is SOA if and only if the number of the doses is greater than or equal to 6. The purpose in this article is to extend the results in [4] to the p-parameter case where $p \geq 3$.

This article is organized as follows: In Section 2, we present notations, definition and some theorems for second order admissibility in general setup. In Section 3, we identify whether the MLE and the Berkson estimator are SOA or not in the multi-parameter logistic regression model. Some concluding remarks are given in Section 4. Finally, we give proofs of lemmas.

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II. Preliminaries

In this section, we present necessary condition and sufficient condition for second order admissibility in general setup. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) random vectors according to a probability distribution P_θ, where $\theta = (\theta_1, \theta_2, \ldots, \theta_p)' \in \mathbb{R}^p$ is unknown and $p \geq 3$. Suppose that P_θ has a probability density function $f(x, \theta)$ with respect to some σ-finite measure. Also, we assume the regularity conditions (i) to (v) given in [5]. Then, we consider the estimation problem of θ under the normed quadratic loss function

$$l(\theta, \delta) := (\delta - \theta)' I(\theta)(\delta - \theta) \tag{1}$$

in estimating θ by δ, where $I(\theta)$ is the Fisher information matrix per one observation. Let $\lambda_{\min}(\theta)$ and $\lambda_{\max}(\theta)$ be the smallest and the largest eigenvalues of $I(\theta)$. Henceforth, $\mathrm{tr}\{A\}$ and $|A|$ denote the trace and the determinant of matrix A, and $||\theta|| := \sqrt{\theta'\theta}$. Also, $(d/d\theta)f(\theta) := ((\partial/\partial\theta_1)f(\theta), (\partial/\partial\theta_2)f(\theta), \ldots, (\partial/\partial\theta_p)f(\theta))'$, $(d/d\theta')f(\theta) := ((d/d\theta)f(\theta))'$ for vector valued function $f(\theta)$.

[3] proposed a concept of second order admissibility.

Definition 1: Let $\mathcal{D}$ be a set of first order efficient estimators. An estimator $\delta(\in \mathcal{D})$ of θ is $\mathcal{D}$-second order inadmissible as $n \to \infty$ if there exists an estimator $\delta^*(\in \mathcal{D})$ such that $\lim_{n\to\infty} n^2\{R(\theta, \delta^*) - R(\theta, \delta)\} \leq 0$ for all $\theta \in \mathbb{R}^p$, and $\lim_{n\to\infty} n^2\{R(\theta_0, \delta^*) - R(\theta_0, \delta)\} < 0$ for some $\theta_0 \in \mathbb{R}^p$. An estimator $\delta(\in \mathcal{D})$ is $\mathcal{D}$-second order admissible as $n \to \infty$ if δ is not $\mathcal{D}$-second order inadmissible as $n \to \infty$.

Let $\hat{\theta}_c$ be the modified MLE of θ by function c, that is, $\hat{\theta}_c := \hat{\theta}_{\mathrm{ML}} + c(\hat{\theta}_{\mathrm{ML}})/n$, where $\hat{\theta}_{\mathrm{ML}}$ is the MLE of θ. According to [5], first order efficient estimators can be represented as a modified MLE up to the order $o_p(1/n)$. Therefore, in this paper, we restrict estimators to the class

$$\mathcal{D} := \{\hat{\theta}_c : c \in C^1(\mathbb{R}^p)\},$$

where $C^1(\mathbb{R}^p)$ is the set of all continuously differentiable functions over $\mathbb{R}^p$. Henceforth, second order admissibility means $\mathcal{D}$-second order admissibility as $n \to \infty$ under the normed quadratic loss function (1). Let $b_c(\theta)$ be the asymptotic bias of $\hat{\theta}_c$, that is, $b_c(\theta) := \lim_{n\to\infty} nE_\theta[\hat{\theta}_c - \theta] = b_{\mathrm{ML}}(\theta) + c(\theta)$.

Then, the risk difference between $\hat{\theta}_c$ and $\hat{\theta}_d(\in \mathcal{D})$ is given by

$$\begin{aligned} & R(\theta,\hat{\theta}_d) - R(\theta,\hat{\theta}_c) \\ = & \frac{1}{n^2}\mathrm{tr}\left\{ g(\theta)g'(\theta)I(\theta) + 2b_c(\theta)g'(\theta)I(\theta) + 2\frac{d}{d\theta'}g(\theta) \right\} \\ & + o\left(\frac{1}{n^2}\right) \end{aligned}$$

as $n \to \infty$, where $g(\theta) := d(\theta) - c(\theta)$. Therefore, a necessary and sufficient condition for the modified MLE $\hat{\theta}_c$ to be SOA is that if $g \in C^1(\mathbb{R}^p)$ satisfies

$$\mathrm{tr}\left\{ g(\theta)g'(\theta)I(\theta) + 2b_c(\theta)g'(\theta)I(\theta) + 2\frac{d}{d\theta'}g(\theta) \right\} \le 0 \quad (2)$$

for all $\theta \in \mathbb{R}^p$, then $g(\theta) = 0$ for all $\theta \in \mathbb{R}^p$. Here, we present an assumption, which corresponds to a potential function.

Assumption 1: There exists a continuously differentiable function $\gamma_{c_0} : \mathbb{R}^p \to \mathbb{R}_+$ such that

$$I(\theta)b_{c_0}(\theta) = \frac{d}{d\theta}\log\gamma_{c_0}(\theta)$$

for all $\theta \in \mathbb{R}^p$.

For the modified MLE $\hat{\theta}_{c_0}$ satisfying Assumption 1, (2) is equivalent to

$$\frac{1}{\gamma_{c_0}(\theta)}h'(\theta)I(\theta)h(\theta) \le -2\mathrm{tr}\left\{\frac{d}{d\theta'}h(\theta)\right\}, \quad (3)$$

where $h(\theta) := g(\theta)\gamma_{c_0}(\theta)$.

Theorem 1: Suppose that the modified MLE $\hat{\theta}_{c_0}$ satisfies Assumption 1 and there exists $\xi \in \Xi$ such that

$$H(\theta) := \int_0^\infty \left[\frac{\lambda_{\max}(x)}{\gamma_{c_0}(x)}\right]_{x=\theta+r\omega_\xi} dr < \infty$$

for all $\theta \in \mathbb{R}^p$, where

$$\begin{aligned} \omega_\xi &:= (\omega_{\xi,1}, \omega_{\xi,2}, \ldots, \omega_{\xi,p})', \\ \xi &:= (\xi_1, \xi_2, \ldots, \xi_{p-1})', \\ \omega_{\xi,i} &:= \begin{cases} \cos\xi_1 & (i=1), \\ \cos\xi_i \prod_{j=1}^{i-1}\sin\xi_j & (i=2,\ldots,p-1), \\ \prod_{j=1}^{p-1}\sin\xi_j & (i=p), \end{cases} \\ \Xi &:= \{\xi \in \mathbb{R}^{p-1} : \xi_i \in [0,\pi) \ (i=1,\ldots,p-2), \\ & \qquad \xi_{p-1} \in [0,2\pi)\}. \end{aligned}$$

Furthermore, we assume that the differential of $H(\theta)$ can be obtained by the differential in the integral sign, that is,

$$\frac{\partial}{\partial\theta_i}H(\theta) = \int_0^\infty \frac{\partial}{\partial\theta_i}\left[\frac{\lambda_{\max}(x)}{\gamma_{c_0}(x)}\right]_{x=\theta+r\omega_\xi} dr \quad (4)$$

for $i = 1,\ldots,p$, then $\hat{\theta}_{c_0}$ is SOI.

The proof is omitted since it can be shown that $h(\theta) := -\omega_\xi/H(\theta)$ satisfies (3) by the similar way to [4].

Theorem 1 may be complicated since it may be difficult to show the exchangeability of the differential sign and integral one. The next result is easy to handle it.

Corollary 1: Suppose that the modified MLE $\hat{\theta}_{c_0}$ satisfies Assumption 1. If

$$\int_0^\infty \frac{\lambda_{\max}(\theta)}{\gamma_{c_0}(\theta)} d\theta_1 < \infty$$

for all $\theta_2,\ldots,\theta_p$, then $\hat{\theta}_{c_0}$ is SOI.

The proof is omitted since it can be obtained from Theorem 1.

The next lemma is used in Theorem 2. The proof is given in Appendix.

Lemma 1: Suppose that $h \in C^1(\mathbb{R}^p)$. Then,

$$\int_{\mathcal{D}_u} \mathrm{tr}\left\{\frac{d}{d\theta'}h(\theta)\right\} d\theta = u^{p-1}\int_\Xi \omega'_\xi h(u\omega_\xi)J(\xi)d\xi$$

holds for $u > 0$, where Ξ is defined in Theorem 1,

$$\begin{aligned} \mathcal{D}_u &:= \{\theta \in \mathbb{R}^p : \|\theta\| \le u\}, \\ J(\xi) &:= \prod_{i=1}^{p-2}\sin^{p-i-1}\xi_i. \end{aligned}$$

Theorem 2: Suppose that the modified MLE $\hat{\theta}_{c_0}$ satisfies Assumption 1. Put

$$\eta_{c_0}(r) := \int_\Xi \left[\frac{\gamma_{c_0}(\theta)}{\lambda_{\min}(\theta)}\right]_{\theta=r\omega_\xi} J(\xi)d\xi,$$

where ω_ξ is given in Theorem 1. If

$$\int_\varepsilon^\infty \frac{dr}{r^{p-1}\eta_{c_0}(r)} = \infty$$

for some $\varepsilon > 0$, then $\hat{\theta}_{c_0}$ is SOA.

Taking account of Lemma 1, the proof of Theorem 2 can be obtained by the similar way to [4].

III. Admissibilities in logistic regression model

In this section, we consider the second order admissibilities of MLE and the Berkson estimator in p-parameter logistic regression model where $p \ge 3$ is a given number. Suppose that $R_1,\ldots,R_k$ are independently distributed random variables according to the binomial distribution $B(n, P_i(\theta))$ for $i = 1,\ldots,k$ with

$$P_i(\theta) := \frac{1}{1+\exp(-x_i'\theta)},$$

where $\theta := (\theta_1,\theta_2,\ldots,\theta_p)'(\in \mathbb{R}^p)$ is unknown and the doses $x_i = (1, d_{i,2},\ldots,d_{i,p})'$ is known for $i = 1,\ldots,k$. For convenience, we put $d_{i,1} = 1$. Here the condition $|(x_{i_1}, x_{i_2},\ldots,x_{i_p})| \ne 0$ for some $i_1 > \cdots > i_p$ should be assumed. Let $Y_{i1},\ldots,Y_{in}$ be i.i.d. random variables according to the binomial distribution $B(1, P_i(\theta))$ for $i = 1,\ldots,k$. Put $Y_j := (Y_{1j},\ldots,Y_{kj})'$ for $j = 1,\ldots,n$. Then, $Y_1,\ldots,Y_n$ are i.i.d. random vectors and the Fisher information matrix is given by

$$I(\theta) = \sum_{i=1}^k P_i(\theta)(1-P_i(\theta))x_ix_i'.$$

Let $\hat{\theta}_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{B}}$ be the MLE and the Berkson estimator of θ, that is,

$$\hat{\theta}_{\mathrm{B}} = \hat{\theta}_{\mathrm{ML}} + \frac{1}{n}(b_{\mathrm{logit}}(\hat{\theta}_{\mathrm{ML}}) - b_{\mathrm{ML}}(\hat{\theta}_{\mathrm{ML}})),$$

where $b_{\mathrm{logit}}(\theta)$ is the asymptotic bias of the MLχ^2E $\hat{\theta}_{\mathrm{logit}}$ (see (74) in [2]). Then, by the similar arguments to [4], we see that both $\hat{\theta}_{\mathrm{ML}}$ and $\hat{\theta}_{\mathrm{B}}$ satisfy Assumption 1 with

$$\gamma_{\mathrm{ML}}(\theta) = \frac{1}{\sqrt{|I(\theta)|}}, \tag{5}$$

$$\gamma_{\mathrm{B}}(\theta) = \frac{1}{|I(\theta)|}\prod_{i=1}^{k}\left\{P_i(\theta)\exp\left(-\frac{1}{2}x_i'\theta\right)\right\}. \tag{6}$$

To apply Theorems 1 and 2 to the logistic regression model, we prepare some lemmas. The proofs are given in Appendix.

Lemma 2: The determinant of the Fisher information matrix is represented as

$$|I(\theta)| = \sum_{i_1>\cdots>i_p} P_{i_1}(\theta)(1-P_{i_1}(\theta))\cdots P_{i_p}(\theta)(1-P_{i_p}(\theta)) \times |(x_{i_1}, x_{i_2}, \ldots, x_{i_p})|^2.$$

Lemma 3: For all $\theta_1 > 0$, $\theta_2, \ldots, \theta_p \in \mathbb{R}$, the followings hold:

(i) $\mathrm{tr}\{I(\theta)\} \le C_1(\theta_{\{1\}})e^{-\theta_1}$,

(ii) $|I(\theta)| \le C_2(\theta_{\{1\}})e^{-p\theta_1}$,

(iii) $\displaystyle\prod_{i=1}^{k}\left\{P_i(\theta)\exp\left(-\frac{1}{2}x_i'\theta\right)\right\} \ge C_3(\theta_{\{1\}})\exp\left(-\frac{k}{2}\theta_1\right)$

for some $C_1(\theta_{\{1\}})$, $C_2(\theta_{\{1\}})$ and $C_3(\theta_{\{1\}})$, where $\theta_{\{1\}} := (\theta_2, \ldots, \theta_p)'$.

Lemma 4: For $l_i \ne l_j$, put

$$\Xi_{l_1,\ldots,l_p} := \Big\{\xi \in \Xi : |x_{l_1}'\omega_\xi| \le |x_{l_2}'\omega_\xi| \le \cdots \le |x_{l_p}'\omega_\xi| \le |x_i'\omega_\xi| \ (i \ne l_1, \ldots, l_p)\Big\}.$$

For all $\xi \in \Xi_{l_1,l_2,\ldots,l_p}$ and $r > 0$, the followings hold:

(i) $\mathrm{tr}\{I(r\omega_\xi)\} \le C_4\exp(-r|x_{l_1}'\omega_\xi|)$,

(ii) $\displaystyle |I(r\omega_\xi)| \ge C_5\exp\left(-r\sum_{m=1}^{p}|x_{l_m}'\omega_\xi|\right)$,

(iii) $\displaystyle\prod_{i=1}^{k}\left\{P_i(r\omega_\xi)\exp\left(-\frac{r}{2}x_i'\omega_\xi\right)\right\} \le \exp\left(-\frac{r}{2}\sum_{i=1}^{k}|x_{l_m}'\omega_\xi|\right)$,

where C_4 and C_5 are constants.

Theorem 3: The MLE $\hat{\theta}_{\mathrm{ML}}$ of θ is always SOI.

Proof: From (5), Lemma 3 and the fact $\lambda_{\max}(\theta) \le \mathrm{tr}\{I(\theta)\}$, we have

$$\frac{\lambda_{\max}(\theta)}{\gamma_{\mathrm{ML}}(\theta)} \le C_1(\theta_{\{1\}})\sqrt{C_2(\theta_{\{1\}})}\ \exp\left\{-\frac{1}{2}(p+2)\theta_1\right\}$$

for all $\theta_1 > 0$, $\theta_2, \cdots, \theta_p \in \mathbb{R}$. Therefore, we see that

$$\begin{aligned}&\int_0^\infty \frac{\lambda_{\max}(\theta)}{\gamma_{\mathrm{ML}}(\theta)}d\theta_1 \\ \le\ & \int_0^\infty C_1(\theta_{\{1\}})\sqrt{C_2(\theta_{\{1\}})}\ \exp\left\{-\frac{1}{2}(p+2)\theta_1\right\}d\theta_1 \\ <\ & \infty,\end{aligned}$$

which implies the second order inadmissibility of $\hat{\theta}_{\mathrm{ML}}$ by Corollary 1. ■

Theorem 4: The Berkson estimator $\hat{\theta}_{\mathrm{B}}$ is SOI if $p \le k \le 2p+1$.

Proof: By the similar argument to the proof of Theorem 3, we have

$$\frac{\lambda_{\max}(\theta)}{\gamma_{\mathrm{B}}(\theta)} \le \frac{C_1(\theta_{\{1\}})C_2(\theta_{\{1\}})}{C_3(\theta_{\{1\}})}\exp\left\{-\frac{1}{2}(2p+2-k)\theta_1\right\}$$

for all $\theta_1 > 0$, $\theta_2, \ldots, \theta_p \in \mathbb{R}$. Therefore, we see that $\hat{\theta}_{\mathrm{B}}$ is SOI if $p \le k \le 2p+1$. ■

Theorem 5: The Berkson estimator $\hat{\theta}_{\mathrm{B}}$ is SOA if

$$2(p-1)|x_{l_1}'\omega_\xi| - 4\sum_{m=1}^{p}|x_{l_m}'\omega_\xi| + \sum_{m=1}^{k}|x_{l_m}'\omega_\xi| > 0 \quad (\forall\xi \in \Xi_{l_1,l_2,\ldots,l_p}) \tag{7}$$

holds for all $l_1, l_2, \ldots, l_p (l_i \ne l_j)$, where $\Xi_{l_1,l_2,\ldots,l_p}$ is defined in Lemma 4. In particular, if $k \ge 4p-3$, the Berkson estimator $\hat{\theta}_{\mathrm{B}}$ is SOA.

Proof: Note that $\eta_{\mathrm{B}}(r)$ is written by

$$\eta_{\mathrm{B}}(r) = \sum_{l_i \ne l_j}\int_{\Xi_{l_1,l_2,\ldots,l_p}} \frac{\gamma_{\mathrm{B}}(r\omega_\xi)}{\lambda_{\min}(r\omega_\xi)}J(\xi)d\xi.$$

It is easy to show that

$$\frac{1}{\lambda_{\min}(\theta)} \le \frac{\mathrm{tr}^{p-1}\{I(\theta)\}}{|I(\theta)|}.$$

So, from (6) and Lemma 4, we have

$$\begin{aligned}&\frac{\gamma_{\mathrm{B}}(r\omega_\xi)}{\lambda_{\min}(r\omega_\xi)} \\ \le\ & \frac{\mathrm{tr}^{p-1}\{I(r\omega_\xi)\}}{|I(r\omega_\xi)|^2}\prod_{i=1}^{k}\left\{P_i(r\omega_\xi)\exp\left(-\frac{r}{2}x_i'\omega_\xi\right)\right\} \\ \le\ & C_6\exp\left\{-\frac{r}{2}\left(2(p-1)|x_{l_1}'\omega_\xi| - 4\sum_{m=1}^{p}|x_{l_m}'\omega_\xi| + \sum_{m=1}^{k}|x_{l_m}'\omega_\xi|\right)\right\}\end{aligned}$$

for all $\xi \in \Xi_{l_1,\ldots,l_p}$, $r > 0$, where C_6 is a constant. If (7) holds, we have $\lim_{r\to\infty} r^{p-1}\eta_{\mathrm{B}}(r) = 0$. So, we get the first part from Theorem 2. Next, we show the second part. From the assumption $p \ge 3$, we can show that

$$\begin{aligned}&2(p-1)|x_{l_1}'\omega_\xi| - 4\sum_{m=1}^{p}|x_{l_m}'\omega_\xi| + \sum_{m=1}^{k}|x_{l_m}'\omega_\xi| \\ =\ & (2p-5)|x_{l_1}'\omega_\xi| - 3\sum_{m=2}^{p}|x_{l_m}'\omega_\xi| + \sum_{m=p+1}^{k}|x_{l_m}'\omega_\xi| \\ \ge\ & (2p-5)|x_{l_1}'\omega_\xi| \\ >\ & 0\end{aligned}$$

if $k \ge 4p-3$. This completes the proof. ■

IV. Concluding remarks

In one-parameter logistic regression model, [3] showed that the MLE is always SOI and the Berkson estimator is SOA if only if the number of the doses is greater than or equal to 4 under the quadratic loss function. The loss function in [3] is different from the one in [4] and this article, however the admissibility result under the quadratic loss function coincides with the one under the normed quadratic loss function. Table 1 summarizes the admissibility or inadmissibility of the two estimators for $p = 1, \dots, 4$. The symbol — means that the case can not be considered since the number of the doses k is larger than the dimension of the parameter space p. The part of ? means that the case is still open. There are some reasons why it is open. One is that the lower bound of $\lambda_{\min}(\theta)$ is not sharp enough to show the complete inadmissibility of $\hat{\theta}_{\mathrm{B}}$. Another is that we applied Corollary 1 not but Theorem 1 to show the inadmissibility. It seems that showing the validity of (4) is not so easy. The authors conjecture that there exist both cases.

TABLE I
Admissibilities of MLE and Berkson estimator for $p = 1, \dots, 4$

<table>
<tr><th colspan="2"></th><th>$p=1$
[3]</th><th>$p=2$
[4]</th><th>$p=3$
[Th.3,4,5]</th><th>$p=4$
[Th.3,4,5]</th></tr>
<tr><td colspan="2">$\hat{\theta}_{\mathrm{ML}}$</td><td>SOI</td><td>SOI</td><td>SOI</td><td>SOI</td></tr>
<tr><td rowspan="10">$\hat{\theta}_{\mathrm{B}}$</td><td>$k=1$</td><td rowspan="3">SOI</td><td>—</td><td rowspan="2">—</td><td rowspan="3">—</td></tr>
<tr><td>$k=2$</td><td rowspan="4">SOI</td></tr>
<tr><td>$k=3$</td><td rowspan="4">SOI</td></tr>
<tr><td>$k=4$</td><td rowspan="7">SOA</td><td rowspan="5">SOI</td></tr>
<tr><td>$k=5$</td></tr>
<tr><td>$k=6,7$</td><td rowspan="5">SOA</td></tr>
<tr><td>$k=8$</td><td>?</td></tr>
<tr><td>$k=9$</td><td rowspan="3">SOA</td></tr>
<tr><td>$k=10,11,12$</td><td>?</td></tr>
<tr><td>$k \geq 13$</td><td>SOA</td></tr>
</table>

Appendix A

In this section, we give the proofs of lemmas presented in the previous Sections.

Proof of Lemma 1: Changing the variables as

$$\begin{cases} \theta_1 := r\cos\xi_1, \\ \theta_i := r\cos\xi_i \prod_{j=1}^{i-1}\sin\xi_j \quad (i = 2, \dots, p-1), \\ \theta_p := r\prod_{j=1}^{p-1}\sin\xi_j, \end{cases}$$

where $r > 0$ and $\xi \in \Xi$, we see that

$$r = \|\theta\|, \quad \xi_i = \mathrm{Tan}^{-1}\frac{\sqrt{\sum_{j=i+1}^{p}\theta_j^2}}{\theta_i} \quad (i = 1, \dots, p-1),$$

and the Jacobian is $r^{p-1}J(\xi)$. Put $H(r,\xi) := h(r\omega_\xi)$. Then, by the chain rule, we have

$$\begin{aligned} &\frac{\partial}{\partial\theta_l}h_l(\theta) \\ &= \begin{cases} \dfrac{\partial}{\partial r}H_1(r,\xi)\cos\xi_1 - \dfrac{\partial}{\partial\xi_1}H_1(r,\xi)\dfrac{\sin\xi_1}{r} & (l=1), \\ \dfrac{\partial}{\partial r}H_l(r,\xi)\cos\xi_l\displaystyle\prod_{i=1}^{l-1}\sin\xi_i \\ \quad +\dfrac{\cos\xi_l}{r}\displaystyle\sum_{i=1}^{l-1}\dfrac{\partial}{\partial\xi_i}H_l(r,\xi)\cos\xi_i\dfrac{\prod_{j=i+1}^{l-1}\sin\xi_j}{\prod_{j=1}^{i-1}\sin\xi_j} \\ \quad -\dfrac{1}{r}\dfrac{\partial}{\partial\xi_l}H_l(r,\xi)\dfrac{\sin\xi_l}{\prod_{j=1}^{l-1}\sin\xi_j} & (l=2,\dots,p-1), \\ \dfrac{\partial}{\partial r}H_p(r,\xi)\displaystyle\prod_{i=1}^{p-1}\sin\xi_i \\ \quad +\dfrac{1}{r}\displaystyle\sum_{i=1}^{p-1}\dfrac{\partial}{\partial\xi_i}H_p(r,\xi)\cos\xi_i\dfrac{\prod_{j=i+1}^{p-1}\sin\xi_j}{\prod_{j=1}^{i-1}\sin\xi_j} & (l=p). \end{cases} \end{aligned}$$

Using this relation and integration by parts, we have

$$\begin{aligned} &\int_{\mathcal{D}_u}\frac{\partial}{\partial\theta_1}h_1(\theta)d\theta \\ &= \int_\Xi\int_0^u\left\{\frac{\partial}{\partial r}H_1(r,\xi)\cos\xi_1 - \frac{\partial}{\partial\xi_1}H_1(r,\xi)\frac{\sin\xi_1}{r}\right\} \\ &\quad \times r^{p-1}J(\xi)drd\xi \\ &= \int_\Xi\cos\xi_1 J(\xi)\left\{\int_0^u r^{p-1}\frac{\partial}{\partial r}H_1(r,\xi)dr\right\}d\xi \\ &\quad -\int_0^u\int_{\Xi_{\{1\}}} r^{p-2}\prod_{i=2}^{p-2}\sin^{p-i-1}\xi_i \\ &\qquad\qquad \times\int_0^\pi\sin^{p-1}\xi_1\frac{\partial}{\partial\xi_1}H_1(r,\xi)d\xi_1 d\xi_{\{1\}}dr \\ &= u^{p-1}\int_\Xi J(\xi)\cos\xi_1 H_1(u,\xi)d\xi, \end{aligned}$$

where $d\xi_{\{1\}} := d\xi_2\cdots d\xi_{p-1}$ and $\Xi_{\{1\}} := \{(\xi_2,\dots\xi_{p-1}) : \xi \in \Xi\}$. Similarly, we have

$$\begin{aligned} \int_{\mathcal{D}_u}\frac{\partial}{\partial\theta_l}h_l(\theta)d\theta &= u^{p-1}\int_\Xi J(\xi)H_l(u,\xi)\cos\xi_l\prod_{j=1}^{l-1}\sin\xi_j d\xi \\ &\qquad\qquad (l=2,\dots,p-1), \\ \int_{\mathcal{D}_u}\frac{\partial}{\partial\theta_p}h_p(\theta)d\theta &= u^{p-1}\int_\Xi J(\xi)H_p(u,\xi)\prod_{j=1}^{p-1}\sin\xi_j d\xi. \end{aligned}$$

Therefore, it follows that

$$\int_{\mathcal{D}_u}\mathrm{tr}\left\{\frac{d}{d\theta}h(\theta)\right\}d\theta = u^{p-1}\int_\Xi\omega'_\xi h(u\omega_\xi)J(\xi)d\xi.$$

This completes the proof. ■

Lemma 1 may be proved by the divergence theorem, which is easier than the proof of Lemma 1.

Proof of Lemma 2: Let S_p be the symmetric group of degree p and let $\mathrm{sgn}(\sigma)$ be the sign of $\sigma \in S_p$. Since the

(α, β)-th element of $I(\theta)$ is

$$I_{\alpha\beta}(\theta) = \sum_{i=1}^{k} P_i(\theta)(1 - P_i(\theta))d_{i,\alpha}d_{i,\beta},$$

we have

$$\begin{aligned}
&|I(\theta)| \\
&= \sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) \sum_{i_1=1}^{k} P_{i_1}(\theta)(1 - P_{i_1}(\theta))d_{i_1,1}d_{i_1,\sigma(1)} \\
&\qquad \cdots \sum_{i_p=1}^{k} P_{i_p}(\theta)(1 - P_{i_p}(\theta))d_{i_p,p}d_{i_p,\sigma(p)} \\
&= \sum_{i_1=1}^{k} \cdots \sum_{i_p=1}^{k} P_{i_1}(\theta)(1 - P_{i_1}(\theta)) \cdots P_{i_p}(\theta)(1 - P_{i_p}(\theta)) \\
&\qquad \times d_{i_1,1} \cdots d_{i_p,p} \sum_{\sigma\in S_p} \mathrm{sgn}(\sigma) d_{i_1,\sigma(1)} \cdots d_{i_p,\sigma(p)} \\
&= \sum_{i_1=1}^{k} \cdots \sum_{i_p=1}^{k} P_{i_1}(\theta)(1 - P_{i_1}(\theta)) \cdots P_{i_p}(\theta)(1 - P_{i_p}(\theta)) \\
&\qquad \times d_{i_1,1} \cdots d_{i_p,p}|(x_{i_1}, \ldots, x_{i_p})| \\
&= \sum_{\sigma\in S_p} \sum_{i_{\sigma(1)}>\cdots>i_{\sigma(p)}} P_{i_1}(\theta)(1 - P_{i_1}(\theta)) \\
&\qquad \cdots P_{i_p}(\theta)(1 - P_{i_p}(\theta))d_{i_1,1} \cdots d_{i_p,p}|(x_{i_1}, \ldots, x_{i_p})|.
\end{aligned}$$

Changing the variables as $i_{\sigma(1)} = i_1, \ldots, i_{\sigma(p)} = i_p$, we get

$$\begin{aligned}
&|I(\theta)| \\
&= \sum_{\sigma\in S_p} \sum_{i_1>i_2>\cdots>i_p} P_{i_1}(\theta)(1 - P_{i_1}(\theta)) \cdots P_{i_p}(\theta)(1 - P_{i_p}(\theta)) \\
&\quad \times d_{i_{\sigma^{-1}(1)},1} \cdots d_{i_{\sigma^{-1}(p)},p}|(x_{i_{\sigma^{-1}(1)}}, \ldots, x_{i_{\sigma^{-1}(p)}})| \\
&= \sum_{i_1>i_2>\cdots>i_p} P_{i_1}(\theta)(1 - P_{i_1}(\theta)) \cdots P_{i_p}(\theta)(1 - P_{i_p}(\theta)) \\
&\quad \times \sum_{\sigma\in S_p} d_{i_{\sigma(1)},1} \cdots d_{i_{\sigma(p)},p}|(x_{i_{\sigma(1)}}, \ldots, x_{i_{\sigma(p)}})|\mathrm{sgn}(\sigma) \\
&= \sum_{i_1>i_2>\cdots>i_p} P_{i_1}(\theta)(1 - P_{i_1}(\theta)) \cdots P_{i_p}(\theta)(1 - P_{i_p}(\theta)) \\
&\quad \times |(x_{i_1}, x_{i_2}, \ldots, x_{i_p})|^2.
\end{aligned}$$

■

Proof of Lemma 3: It is easily obtained that

$$\begin{aligned}
P_i(\theta)(1 - P_i(\theta)) &\leq \exp(-|x_i'\theta|) \\
&\leq \exp\left(-\theta_1 + \sum_{m=2}^{p} |d_{i,m}\theta_m|\right)
\end{aligned}$$

for all $\theta_1 > 0$, $\theta_2, \ldots, \theta_p \in \mathbb{R}$. According to

$$\mathrm{tr}\{I(\theta)\} = \sum_{i=1}^{k}\sum_{j=1}^{p} d_{i,j}^2 P_i(\theta)(1 - P_i(\theta))$$

and Lemma 2, we have (i) and (ii). Furthermore, it can be shown that

$$\begin{aligned}
&P_i(\theta)\exp\left(-\frac{1}{2}x_i'\theta\right) \\
&\quad = \frac{1}{\exp(x_i'\theta/2) + \exp(-x_i'\theta/2)} \\
&\quad \geq \frac{1}{2}\exp\left(-\frac{1}{2}|x_i'\theta|\right) \\
&\quad \geq \frac{1}{2}\exp\left\{-\frac{1}{2}\left(\theta_1 + \sum_{m=2}^{p}|d_{i,m}\theta_m|\right)\right\}.
\end{aligned}$$

Hence, we get (iii). ■

Proof of Lemma 4: Since

$$\frac{1}{4}\exp(-r|x_i'\omega_\xi|) \leq P_i(r\omega_\xi)(1 - P_i(r\omega_\xi)) \leq \exp(-r|x_i'\omega_\xi|)$$

for all $\xi \in \Xi_{l_1,l_2,\ldots,l_p}$ and $r > 0$, we get

$$\begin{aligned}
\mathrm{tr}\{I(r\omega_\xi)\} &\leq \sum_{i=1}^{k}\sum_{j=1}^{p} d_{i,j}^2 \exp(-r|x_i'\omega_\xi|) \\
&\leq \exp(-r|x_{l_1}'\omega_\xi|)\sum_{i=1}^{k}\sum_{j=1}^{p} d_{i,j}^2.
\end{aligned}$$

By using the result of Lemma 2, we get

$$\begin{aligned}
&|I(r\omega_\xi)| \\
&\geq \frac{1}{4^p}\sum_{i_1>\cdots>i_p} |(x_{i_1}, x_{i_2}, \ldots, x_{i_p})|^2 \exp\left(-r\sum_{m=1}^{p}|x_{i_m}'\omega_\xi|\right) \\
&\geq \frac{1}{4^p}|(x_{l_1}, x_{l_2}, \ldots, x_{l_p})|^2 \exp\left(-r\sum_{m=1}^{p}|x_{l_m}'\omega_\xi|\right).
\end{aligned}$$

Furthermore, the inequality

$$\begin{aligned}
&P_i(r\omega_\xi)\exp\left(-\frac{r}{2}x_i'\omega_\xi\right) \\
&\quad = \frac{1}{\exp(rx_i'\omega_\xi/2) + \exp(-rx_i'\omega_\xi/2)} \\
&\quad \leq \exp\left(-\frac{r}{2}|x_i'\omega_\xi|\right) \leq \exp\left(-\frac{r}{2}|x_{l_i}'\omega_\xi|\right)
\end{aligned}$$

implies (iii). ■

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