

# Multi-Stakeholder Road Pricing Game: Solution Concepts

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**Abstract**—A road pricing game is a game where various stakeholders and/or regions with different (and usually conflicting) objectives compete for toll setting in a given transportation network to satisfy their individual objectives. We investigate some classical game theoretical solution concepts for the road pricing game. We establish results for the road pricing game so that stakeholders and/or regions playing such a game will beforehand know what is obtainable. This will save time and argument, and above all, get rid of the feelings of unfairness among the competing actors and road users. Among the classical solution concepts we investigate is Nash equilibrium. In particular, we show that no pure Nash equilibrium exists among the actors, and further illustrate that even “mixed Nash equilibrium” may not be achievable in the road pricing game. The paper also demonstrates the type of coalitions that are not only reachable, but also stable and profitable for the actors involved.

**Keywords**—Road pricing game, Equilibrium problem with equilibrium constraint (EPEC), Nash equilibrium, Game stability.

## I. INTRODUCTION

OVER the past years, vehicle ownership has increased tremendously. It has been realized that the social cost of owning and driving a vehicle does not only include the purchase, fuel, and maintenance fees, but also the cost of man hour loss to congestion and road maintenance, costs of health issues resulting from accidents, exposure to poisonous compounds from exhaust pipes, and high noise level from vehicles. So, to optimize the traffic flow requires a model that optimizes more than one objective which may be in conflict with each other. The model should also consider the user benefit. Optimization of more than one traffic externality is not a novel idea. Road pricing that simultaneously treat time losses, increased fuel consumption, and emission is discussed in [1], [2]. Traffic congestion, air pollution and accident externalities are considered in [3]. Single- and bi-criteria Pareto optimization that deal with users with different values of time and two objectives (time and money) were studied in [4], [5], [6]. Road damage externality is incorporated in the road pricing models of [7]. [8] discusses a road charge design that includes multiple objectives and constraints. In particular, objective functions or constraints considered in their work include social welfare improvement, revenue generation, and distributional equity impact.

All the models mentioned above are based on the idea of multi-objective optimization where one leader decides which

point on the Pareto-front is chosen. They all have one shortcoming; they do not address the issues arising when different stakeholders/autonomous cities with possibly conflicting objectives toll the road. There is need for such models since autonomy of states/cities or regions are increasing becoming popular in the area of infrastructure or road management. In literature, there are few works dealing on these shortcomings: competition among stakeholders<sup>1</sup> with privately owned network with intention of maximizing their toll revenue is studied in [9], [10] - they formulated their problem as equilibrium problem with equilibrium constraints (EPEC). Both toll and capacity competition among private asymmetric roads with congestion in a network with parallel links is studied in [11]. In their paper, [12] analysed the allocative efficiency of private toll roads vis a vis free access and public toll road pricing on a network with two parallel routes joining a common origin and destination. In one of their study regimes, they considered a mixed duopoly with a private road competing with a public toll road. On the other hand, tax competition on a parallel road network when different governments have tolling authority on the different links of the network is studied in [13]. [14] studies the existence and efficiency of oligopoly equilibrium in a congested network with parallel roads, in which operators compete for traffic by simultaneous toll and capacity choices. They establish sufficient conditions for the existence of a pure-strategy oligopoly equilibrium. In contrast to parallel network, [15] studies road pricing in a serial network. They used two links in series where private operators own one link each. The paper investigates the traffic patterns and pricing rules under various regimes of road operation in serial networks. Further discussions on toll competition among operators of serial links can be found in [16] and [17]. [18] also discussed toll and capacity competition among owners of private toll roads on general networks. In their work, they provided a theoretical proof of the existence of the constant  $v/c$  ratio property over general traffic networks. The effects of alternative pricing and investment policies on service level of cross-border transport infrastructure and economic welfare of two neighboring countries are studied in [19]. They showed that the investment rule becomes efficient if infrastructure charge is levied. They established that this result holds for all regimes with charging, regardless their differences in objective functions, financial constraints, and organization of decision units. Using game theory, [20] studied a bilateral monopoly situation on a private highway, involving strategic interactions between a private highway operator and a private transit

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<sup>1</sup> we use stakeholders, leaders and actors interchangeably

operator who uses the same highway for its services.

The studies mentioned in the foregoing literature assume that network or road segments are privately owned or managed by private stakeholders. [21] propose practical pricing schemes that can take into account competition and/or collaboration between different administrative regions of the network. Using numerical examples, they demonstrate that local regional pricing may be beneficial or detrimental to the whole network, depending on the structure and O–D pattern of the network. They showed that cooperation among regions in congestion pricing can improve overall system performance in terms of total social welfare. Again, they only consider competition among separate regions in a network. Furthermore, their results are based on numerical examples. It is on this note that we study the existence of Nash equilibrium among the competing stakeholders on the same network infrastructure. To this, we take into account that private stakeholders (with likely contradicting objectives) that do not own networks influence the implementation of road pricing (or nature of tolls) during policy making. It is also practically feasible that stakeholders of different interests may jointly own the same network infrastructure. This is then a generalised model of [21]. Again, how to incorporate users acceptability of road pricing has not yet been fully discussed in earlier literature; users were modelled to have no say on the imposed tolls. Campaigns on the implementation of road pricing have failed in many cities like Edinburgh (in 2002), Trondheim (in 2005), New York (in 2008), Hong Kong (in 1986), cities in the Netherlands, due to lack of support. This lack of support is due to the fact that the debate on the implementation involves stakeholders with conflicting interests, moreover users are most times never considered on the same level as these stakeholders. In our papers [22], [23], we address these issues and formulate a general model that allows each stakeholder (including users) partake in toll setting. In this paper we further investigate the classical game theoretic solution concepts. In particular, we show that a pure Nash equilibrium may not exist among the actors, and further illustrate that even “mixed Nash equilibrium” may not be achievable in the road pricing game. The paper also demonstrates the type of coalitions that are not only reachable, but also stable and profitable for the actors involved. It is important to mention that road users are modeled on the same level as the stakeholders with one stakeholder representing users’ interest, and as such, making the users *active* in the toll setting game.

This paper is organized as follows: Section II provides the basic traffic models for road pricing. Section III describes the multi-leader road pricing game. Section IV discusses classical solution concepts for the road pricing models. In Section V, we demonstrate our model with two-node network example. Section VI discusses the solutions of Nash equilibrium game, and compares that to the cooperative game solutions. Finally, section VII concludes the paper.

## II. BASIC TRAFFIC MODELS FOR ROAD PRICING.

### A. Notations

Let  $G = (N, A)$  be a network, with  $N$  the set of all nodes and  $A$  the set of (directed) arcs or links in  $G$ . We use the following

notation:

TABLE I  
NOTATIONS

$A$	set of all arcs (links) in $G$
$a$	index for links in $G$
$R$	set of all paths in $G$
$r$	index for paths (routes) in $G$
$W$	set of all OD pairs in $G$
$w$	index for OD pairs in $G$
$f$	path flow vector in $G$
$f_r$	flow on path $r$ in $G$
$v$	vector of link flows in $G$
$v_a$	flow on link $a$ in $G$
$d$	travel demand vector in $G$
$d_w$	demand for the $w^{th}$ OD pair in $G$
$\Gamma$	OD-path incident matrix in $G$
$R_w$	set of all paths connecting $w^{th}$ OD
$\Lambda$	arc-path incident matrix in $G$
$V$	set of feasible flow pattern in $G$
$D(\lambda)$	vector of demand functions in $G$
$D_w(\lambda_w)$	demand function for the OD pair $w$
$B(d)$	inverse demand (or benefit) function
$B_w(d_w)$	inverse demand function for the $w^{th}$ OD pair
$\lambda_w$	least cost to transverse the $w^{th}$ OD pair
$t(v)$	vector of link travel time functions in $G$
$\beta$	monetary value of time per minute (VOT)
$K$	set of all actors in the road pricing game
$C_k(v)$	total network cost function for the $k^{th}$ objective, with $C_k(v) = \sum_{a \in A} C_k^a(v_a)$
$C$	vector of network cost functions in $G$
$Z(v)$	total network cost in $G$ . i.e. $Z(v) = \sum_{k \in K} C_k(v)$

### B. Single Leader Problem Formulation

1) *Stakeholders’ Problem* : We summarize the “tolling problem” for elastic demand where each stakeholder  $k$  would like to solve as if he were the unique leader. We assume that each stakeholder controls a unique objective, and he wishes to minimize his own costs  $C_k(v)$  taking into account the users’ benefit subject to user flow and environmental feasibility conditions. We have also assumed a uni-modal model. A multimodal model is straightforward by adding a superscript on each flow related (dependent) entity, parameter and/or variable to indicate the user class. Using the idea from Beckmann’s formulation [24] the user benefit (UB) is given by

$$UB = \sum_{w \in W} \int_0^{d_w} B_w(\zeta) d\zeta$$

where  $B_w(d_w)$  is the inverse demand or benefit function for the OD pair  $w \in W$ . Assuming that UB is split equally<sup>2</sup> among all stakeholders, the problem of stakeholder  $k$  can then be stated as follows:

<sup>2</sup>In fact UB need not be split equally among the stakeholders, each stakeholder decides if he considers UB or not in his objective. The models do not change.

$$\begin{aligned}
SP_k: \quad \min_{v,d} Z_k &= C_k(v) - \frac{1}{|K|} \sum_{w \in W} \int_0^{d_w} B_w(\zeta) d\zeta \\
s.t. \quad & \\
\left. \begin{aligned} v &= \Lambda f \psi \\ \Gamma f &= d \lambda \\ f &\geq 0 \quad \rho \\ d &\geq 0 \quad \vartheta \\ g(v) &\leq 0 \quad \xi \end{aligned} \right\} & \begin{aligned} &(FeC\_ED) \\ &SideConstraints(SC) \end{aligned}
\end{aligned} \quad (1)$$

Here,  $|K|$  denotes the number of stakeholders. The first set of constraints is the flow feasibility conditions for elastic demand (**FeC\_ED**); the first constraint states that the flow on a link is equal to the sum of all path flows that passes through this link, the second equation states that the sum of flows on all paths originating from origin node  $p$  and ending at destination node  $q$  for an OD pair  $pq$  equals the demand for this OD pair, the third and fourth inequalities simply state that the path flows (and thus the link flows) and all OD demands are non-negative. We have also indicated the corresponding multipliers  $(\psi, \lambda, \xi, \rho, \vartheta)$  in the Karush-Kuhn-Tucker (KKT) conditions (see (1)). The last constraint  $g(v) \leq 0$  (where  $g(v) \in \mathbb{R}^{|A| \times |K|}$ ) contains possible side constraints on the link flow vector  $v$ . These side constraints (which we assume to be convex or linear in  $v$ ) may be standardization constraints such as:

The total emission on certain links should not exceed the stipulated emission standard.

The total noise level on certain links should not exceed the standard allowed dB(A) level.

The number of cars on certain roads should not exceed certain numbers so as to preserve the pavements and reduce accidents, etcetera.

**Assumption 1:** Throughout (and for easiness) we assume that the link cost (travel time) functions are separable, that all functions  $C_k^a(v_a)$  in the objective  $C_k(v)$  are strictly convex and strictly monotonic in  $v_a$  (see (12)), that the inverse demand functions are separable and strictly monotonic, and that the side constraints  $g(v) \leq 0$  are linear.

### C. Multi-objective Model (MO)

In a standard MO model that considers all stakeholders, one has to solve [25], [26] a program such as:

$$\begin{aligned}
\min_{v,d} Z &= (SP_t, SP_e, SP_n, SP_s, SP_i, \dots) \\
s.t. \quad & FeC\_ED \& SC
\end{aligned} \quad (2)$$

Where the indices,  $(t, e, n, s, i, \dots)$  refer to different objectives (say travel time, emission, noise, safety, ...). More precisely, one has to find a point on the Pareto front of this program. In what follows we will consider the Pareto point given as the minimizer of the (special) MO program (system monetary costs  $Z = \sum_{k \in K} SP_k$ ):

$$\begin{aligned}
MO: \quad \min_{v,d} Z &:= \sum_{k \in K} C_k(v) - \sum_{w \in W} \int_0^{d_w} B_w(\zeta) d\zeta \\
s.t. \quad & FeC\_ED \& SC
\end{aligned} \quad (3)$$

Note that by choosing different weight factors for the objectives in the MO, we can model preferences for some externalities.

1) (Road) User Problem - UP : Without loss of generality, we assume that the only determinant of user's route choice behaviour is the travel costs and benefits of a trip. Under Assumption 1, the well-known Beckmann's formulation of Wardrop's user equilibrium (UE) [24] describes the users' behaviour mathematically by the convex program:

$$\begin{aligned}
UP: \quad \min_{v,d} \quad & \sum_{a \in A} \int_0^{v_a} \beta t_a(u) du - \sum_{w \in W} \int_0^{d_w} B_w(\zeta) d\zeta \\
s.t. \quad & FeC\_ED
\end{aligned}$$

### D. First and Second-best Pricing

To solve the toll pricing problem in presence of one leader, first and second best pricing techniques are mostly used. The first-best pricing idea is based on a comparison between the KKT-conditions for MO and the KKT-conditions for UP. In general the first best prices are not unique. We summarize the result in the following corollary (see [27] for a proof - the straight forward extension to multiple objectives can be found in [28]).

#### Corollary 1.

Suppose  $(\bar{v}, \bar{d})$  is a solution for the MO, then any social toll vector  $\theta$  (with toll  $\theta_a$  on link  $a$ ) satisfying the following set of linear conditions is a toll such that  $(\bar{v}, \bar{d})$  is also the elastic user equilibrium with respect to costs  $\beta t(v) + \theta$ :

$$\begin{aligned}
& \sum_{a \in A} (\beta t_a(\bar{v}_a) + \theta_a) \delta_{ar} \geq B(\bar{d}_w) \quad \forall r \in R_w, \forall w \in W \\
& \sum_{a \in A} (\beta t_a(\bar{v}_a) + \theta_a) \bar{v}_a = \sum_{w \in W} B(\bar{d}_w) \bar{d}_w
\end{aligned} \quad (4)$$

or in short

$$\begin{aligned}
\Lambda^T (\beta t(\bar{v}) + \theta) &\geq \Gamma^T B(\bar{d}) \\
(\beta t(\bar{v}) + \theta)^T \bar{v} &= B(\bar{d})^T \bar{d}
\end{aligned}$$

We will refer to (4) as equilibrium constraint for elastic demand (**EqC\_ED**). For fixed demand, the matrix form of (4) becomes

$$\begin{aligned}
\Lambda^T (\beta t(\bar{v}) + \theta) &\geq \Gamma^T \lambda \\
(\beta t(\bar{v}) + \theta)^T \bar{v} &= \bar{d}^T \lambda
\end{aligned} \quad (5)$$

where  $\lambda$  is a free vector (of multipliers, see (1)) with components  $\lambda_w$  representing the minimum route travel cost for a given OD pair. In case we use (5), we will refer to it as equilibrium constraint for fixed demand (**EqC\_FD**). One of the possible tolls is given by the "first pricing" toll (see [27] for proof on single objective, [28] extended the proof to multi-objective):

$$\theta_{sc} = \sum_{k \in K} |K| \nabla C_k(\bar{v}) - \beta t(\bar{v}) + |K| \nabla g(\bar{v}) \xi \quad (6)$$

If there are extra conditions on the toll vector  $\theta$  (e.g., some links  $a \in Y$  are non-tollable ( $\theta_a = 0$ )) there might be no feasible

first-best pricing toll. In this case one has to find a second-best pricing vector, and instead of solving a standard program MO together with (4), one has to solve the following bi-level program also called a mathematical program with equilibrium constraints (MPEC):

$$\begin{aligned} \min_{d,v,\theta} Z &= \sum_k C_k(v) - \sum_{w \in W} \int_0^{d_w} B_w(\zeta) d\zeta \\ \text{s.t.} \quad & \Lambda^T (\beta t(v) + \theta) \geq \Gamma^T B(d) \\ & (\beta t(v) + \theta)^T v = B(d)^T d \\ & \theta_a = 0 \quad \forall a \in Y \\ & g(v) \leq 0 \\ & FeC\_ED \end{aligned} \quad (7)$$

### III. MULTI-LEADER MODEL IN ROAD PRICING

In the foregoing models, we discussed a one leader road pricing problem using the MO program. Such models have their shortcomings; when one decision maker (*dm*), (e.g the government) controls the traffic flow of a transportation system through road pricing, then it is likely that some other stakeholders affected by activities of transportation may not be happy with the decisions made by this *dm*. This is because when the *dm* models the MO road pricing problem, all traffic externalities are simultaneously considered with or without preference for any externality (see MO (3)). When preference is given, say, to congestion, then the effect of the preferred externality subdues the effect of other externalities, and this may translate to huge costs for some stakeholders. For example, lower travel time (say high speeds) may translate to more accidents (costs for insurance companies). Even without preference to any externality, it is intuitive that stakeholders still will prefer to partake in toll setting to safeguard their interests. The main problem of a classical approach from multi-objective optimization is the following: supposing that each stakeholder can influence the toll setting, why should a (independent) player accept a situation which he can improve by changing the tolls?

In such a situation the classical concept of Nash equilibrium in game theory gives an appropriate alternative model. Such models are used in economics in situations where independent players may influence the market with their strategies in order to optimize their specific objective.

The question we like to address from game theoretical/economic point of view is; what happens when each stakeholder optimizes his objective by tolling the network, given that other stakeholders are doing the same? Formally, we introduce the mathematical and economic theory behind.

#### A. Mathematical and Economic Theory

The Mathematical Program with Equilibrium Conditions (MPEC) (7) described in the previous section is a Stackelberg game where a leader (*dm*) moves first followed by sequential move of other players (road users). If we assume that various stakeholders are allowed to set toll (or at least

influence the tolls) on the network, then, users are influenced not only by just one leader as in Stackelberg game, but by more than one decision maker. In a multi-leader-multi-follower game/problem, the leaders take decisions (search for toll vectors  $\theta^k, k \in K$ , that optimize their respective objectives) at the upper level which influence the followers (users) at the lower level. The followers then react accordingly (user/Wardrop equilibrium), which in turn may cause the leaders to update their individual decisions leading to lower level players reactions again. These updates continue until a stable situation is reached. A stable state is reached if no stakeholder can improve his objective by unilaterally changing his toll. Note however, that given the stable state decision tolls of leaders, the lower level stable situation is given by the (unique) Wardrop's equilibrium (see (10)). So the bi-level game can be seen as a single (upper) level game with additional equilibrium conditions (for the lower level).

In the above non-cooperative scenario, each actor continuously solves a *program with equilibrium conditions* which is influenced by other actors' *program with equilibrium conditions*, and this translates to an *equilibrium problem subject to equilibrium condition*. [29]. Since a stable state upper level tolls will lead to a (unique) Wardrop's equilibrium in the lower level (due to Assumption 1), our aim therefore is to find a Nash toll vector for the leaders (see Fig. 1).

**Remark 1:** The theory described above does not necessarily mean that stakeholders have different toll collecting machines/booths on the links. Our model describes the Nash toll vector that can be agreed upon during policy making or debate.

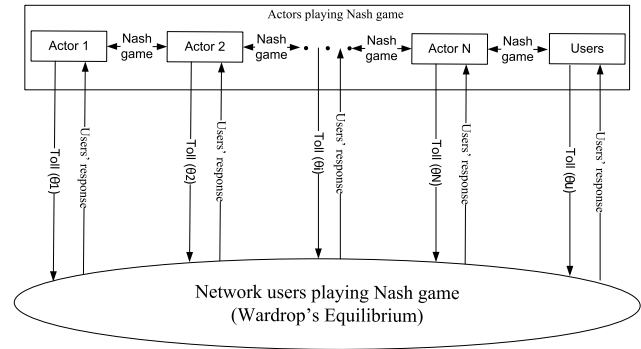


Fig. 1. Multi-leader-multi-follower Nash Game

#### B. Mathematical Models for the Bi-level Nash Equilibrium Game (EPEC)

We now mathematically introduce the toll pricing game and the concept of Nash equilibrium (NE) [30], [31] as outlined in subsection III A.

Assume that Assumption 1 holds. This in particular ensures that (for given costs) the Wardrop equilibrium (WE)  $(v, d)$  is unique. Let  $\theta^k$  be the link toll vector of player  $k \in K$ . We use  $\theta^{-k}$  to denote all toll vectors in  $K \setminus k$ . In the Nash game, for given  $\theta^{-k}$ , the  $k^{th}$  stakeholder tries to find a solution toll  $\bar{\theta}^k$

for the following problem:

$$\Psi_k(\bar{\theta}^k, \bar{\theta}^{-k}) = \min_{\theta^k} \Psi_k(\theta^k, \bar{\theta}^{-k})$$

where for given  $\theta^k$  (and  $\bar{\theta}^{-k}$ )

$$\Psi_k(\theta^k, \bar{\theta}^{-k}) := \min_{v^k, d^k} Z_k = C_k(v^k) - \frac{1}{|K|} \sum_{w \in W} \int_0^{d_w^k} B_w(\zeta) d\zeta$$

s.t

$$\begin{aligned} \Lambda^T \left( \beta t(v^k) + \theta^k + \sum_{j \in K \setminus k} \bar{\theta}^j \right) &\geq \Gamma^T B(d^k) \\ \left( \beta t(v^k) + \theta^k + \sum_{j \in K \setminus k} \bar{\theta}^j \right)^T v^k &= B(d^k)^T (d^k) \\ \text{and} \\ v^k &= \Lambda f^k \\ \Gamma f^k &= d^k \\ f^k &\geq 0 \\ (\theta^k &\geq 0) \end{aligned} \quad (8)$$

The concept of a Nash equilibrium is to look for a situation where for fixed strategies  $\bar{\theta}^{-k}$  of the opponent players, the best that player  $k$  can do is to choose his own toll to be  $\bar{\theta}^k$ . A NE is thus a whole set of toll vectors  $\bar{\theta} = (\bar{\theta}^k, k \in K)$  such that for each player  $k$  the following holds:

$$\Psi_k(\bar{\theta}^k, \bar{\theta}^{-k}) \leq \Psi_k(\theta^k, \bar{\theta}^{-k}) \quad (9)$$

for all feasible tolls  $\theta^k$  and  $\forall k \in K$

See that in the optimization problem above, each leader  $k$  can only change his own link toll vector  $\theta^k$ . The strategies  $\bar{\theta}^j, j \neq k$  of the other leaders are fixed in  $k$ 's problem. The first system of constraints are the equilibrium constraints and the second system are the feasibility conditions.

**Remark 2:** Note that users represented in the upper level as an autonomous stakeholder seek users' interests, for example, a lower link and/or network tolls.

#### IV. SOLUTION CONCEPTS

##### A. Existence of Nash Equilibrium

In this subsection we analyse the existence of Nash equilibrium in our tolling game. We show below that this simple standard Nash equilibrium concept (of (8) & (9)) is not always applicable to the tolling problem. The main reason lies in the special structure of the problems  $\Psi_k(\bar{\theta}^k, \bar{\theta}^{-k})$  in (8) leading to the following fact:

**Fact:** Due to Assumption 1, for given vectors  $\bar{\theta}^k, k \in K$  the corresponding solution  $(\bar{v}, \bar{d})$  of the system (8) (i.e., the elastic demand user equilibrium with respect to the costs  $[\beta t(v) + \sum_{j \in K} \bar{\theta}^j]$ ) is uniquely given. Therefore it holds:

**Assertion:** If  $\bar{\theta}$  is a Nash equilibrium, then all corresponding solution vectors

$$(\bar{v}^k, \bar{d}^k) = (\bar{v}, \bar{d}), k \in K \text{ of } \Psi_k \quad (10)$$

are identical.

*Proof:* Given that  $\bar{\theta}^k$  solves system (8) for all actors  $k \in K$ , then it means that at Nash equilibrium among the actors, the link toll vector  $\bar{\theta}$  is given by  $\bar{\theta} = \sum_{k \in K} \bar{\theta}^k$ , where  $\bar{\theta}_a = \sum_{k \in K} \bar{\theta}_a^k, \forall a \in A$ . Due to Assumption 1, this toll vector  $\bar{\theta}$  yields a unique flow pattern  $(\bar{v}, \bar{d})$ . Of course the users do not differentiate the tolls (per actor  $k$ ), what they experience is the total toll vector  $\bar{\theta}$ , and as such, the vector  $\bar{\theta}$  (together with the travel time costs) determines the unique user/Wardrop equilibrium flow  $(\bar{v}, \bar{d})$  for the system. ■

1) *Unrestricted Toll Values:* From the relation (10) we can directly deduce the following results.

##### Corollary 2.

- Suppose the leaders can toll all links with no restrictions (no constraint  $\theta^k \geq 0$  in (8)), then, for the tolling game with elastic demand, there does not exist a Nash equilibrium in general. Moreover, in this game the players do not have any incentive to cooperate.
- When the demand is fixed, even under the extra conditions  $\theta^k \geq 0$  in (8), there does not exist a Nash equilibrium in general.

*Proof:* We will even show that in (the general) case where not all players have the same solution  $(v^k, d^k)$  in their own program  $SP_k$  (see (1)) there will never be a Nash equilibrium of the form of (9).

(a) Assume  $\bar{\theta}$  is a Nash equilibrium with  $(\bar{v}, \bar{d}, \bar{\theta}^k)$  the solution of player  $k$ . Recall that (by (10)) all user flows  $(\bar{v}, \bar{d})$  are the same at Nash. By assumption, at least one of the players, say player  $\ell$ , has a different ideal (or optimal) link flow  $(\bar{v}, \bar{d})$  in  $SP_k$  (since players are assumed to have conflicting objectives) and by our discussion in Section 2, player  $\ell$  can achieve this flow in  $\Psi_\ell(\bar{\theta}^\ell, \bar{\theta}^{-\ell})$  by choosing e.g., the first best pricing toll

$$\bar{\theta}^\ell = |K| \nabla C_\ell(\bar{v}) - \beta t(\bar{v}) - \sum_{k \in K \setminus \ell} \bar{\theta}^k \quad (11)$$

where  $|K|$  is the number of players. Note that this toll  $\bar{\theta}^\ell$  may be negative. Since at any stage of the game, any player  $k$  can always achieve his ideal flow in  $SP_k$ , it is clear that no equilibrium can be reached and that players do not have any reason to cooperate if they can always achieve  $SP_k$  on their own.

(b) The same clearly holds in the case of fixed demand. However, in this case we can always achieve a first best pricing toll in (5) satisfying  $\bar{\theta}^\ell \geq 0$ : To see this, note that for fixed demand, any leader  $\ell \in K$  has the following valid toll vectors as part of a whole polyhedron (see proof below) that achieve the ideal flow vector for leader  $\ell$

$$\bar{\theta}^\ell = [\alpha(\nabla C_\ell(\bar{v})) - \beta t(\bar{v})] - \sum_{k \in K \setminus \ell} \bar{\theta}^k; \text{ where } \alpha > 0 \quad (12)$$

By making  $\alpha$  large enough (in view of strict monotonicity) we can assure  $\bar{\theta}^\ell \geq 0$ . ■

*Proof of (12):* Suppose  $\tilde{v}$  is an ideal flow vector that solves (1) (omitting the UB - fixed demand) for player  $\ell$ , now let  $\theta^\ell$  be the corresponding toll vector satisfying (5), this means that  $\tilde{v}$  is solution of the LP

$$\min_v (\beta t(\tilde{v}) + \theta^\ell)^T v \quad s.t. \quad v \in V$$

where  $\beta t(v)$  is a vector of link travel time functions. Obviously  $\tilde{v}$  also solves the following LP

$$\min_v \alpha (\beta t(\tilde{v}) + \theta^\ell)^T v \quad s.t. \quad v \in V \quad \text{where } \alpha > 0$$

but,

$$\begin{aligned} \alpha (\beta t(\tilde{v}) + \theta^\ell)^T v &= \left( (\beta t(\tilde{v}) + \theta^\ell) + (\alpha - 1) (\beta t(\tilde{v}) + \theta^\ell) \right)^T v \\ &= (\beta t(\tilde{v}) + [\theta^\ell + (\alpha - 1) (\beta t(\tilde{v}) + \theta^\ell)])^T v \end{aligned}$$

this means that with  $\theta^\ell$ , any vector

$$\tilde{\theta}^\ell = \theta^\ell + (\alpha - 1) (\beta t(\tilde{v}) + \theta^\ell) = \alpha (\beta t(\tilde{v}) + \theta^\ell) - \beta t(\tilde{v})$$

is a valid toll vector as well. Recall that for fixed demand and for one objective, the marginal social cost (MSC) toll or the so called first best toll given by (see (6) - elastic demand equivalent):

$$\theta_{sc}^\ell = \theta^\ell = \nabla C_\ell(\tilde{v}) - \beta t(\tilde{v})$$

is one toll vector that achieves the ideal flow vector  $\tilde{v}$ , therefore

$$\begin{aligned} \tilde{\theta}^\ell &= \alpha (\beta t(\tilde{v}) + \theta^\ell) - \beta t(\tilde{v}) = \alpha (\beta t(\tilde{v}) + (\nabla C_\ell(\tilde{v}) - \beta t(\tilde{v}))) \\ &\quad - \beta t(\tilde{v}) \\ &= \alpha (\nabla C_\ell(\tilde{v})) - \beta t(\tilde{v}) \end{aligned}$$

In the presence of other actors' toll  $\sum_{k \in K \setminus \ell} \tilde{\theta}^k$ ,  $\tilde{\theta}^\ell$  now becomes

$$\tilde{\theta}^\ell = \alpha (\nabla C_\ell(\tilde{v})) - \beta t(\tilde{v}) - \sum_{k \in K \setminus \ell} \tilde{\theta}^k; \quad \text{where } \alpha > 0 \quad \blacksquare$$

We emphasize that extra restrictions on the tolls  $\theta^k$  may play in favor of the existence of a Nash equilibrium.

Generally, what can we say on the existence of NE? A well-known theorem in game theory [32] states that the game has a Nash equilibrium if the following conditions are met:

- The strategy sets for each player are compact and convex, and each player's utility function  $\Psi_k(\theta^k, \tilde{\theta}^{-k})$  is continuous and quasi-convex in his strategy  $\theta^k$ . In general, we do not expect such properties on  $\Psi_k(\theta^k, \tilde{\theta}^{-k})$ , in fact, we show in [23] that the utility functions may not be convex in the strategy sets.

## V. TWO-NODE NETWORK EXAMPLE

### A. Pure Nash Equilibrium

We use a simple example to illustrate how changes in cost functions (on the same network) effect the existence of Nash equilibrium.

In this example we consider a network of two links;  $a$  and  $b$ , and two actors; actor  $I$  and actor  $II$ . Actors are respectively interested in minimizing two different types of "traffic" costs

$C^I$  and  $C^{II}$  for the network. We use  $\chi^I$  and  $\chi^{II}$  to describe the link cost (negative utility) functions for the "traffic" costs  $C^I$  and  $C^{II}$  respectively.

$$\chi^I = \begin{cases} \chi_a^I = \begin{cases} 2v_a & \text{if } \theta_a^I = 0 \\ 2v_a + OC & \text{otherwise} \end{cases} & \text{for link } a \\ \chi_b^I = \begin{cases} 2.5v_b & \text{if } \theta_b^I = 0 \\ 2.5v_b + OC & \text{otherwise} \end{cases} & \text{for link } b \end{cases}$$

$$\chi^{II} = \begin{cases} \chi_a^{II} = \begin{cases} 2v_a + 2 & \text{if } \theta_a^{II} = 0 \\ 2v_a + 2 + OC & \text{otherwise} \end{cases} & \text{for link } a \\ \chi_b^{II} = \begin{cases} 3v_b & \text{if } \theta_b^{II} = 0 \\ 3v_b + OC & \text{otherwise} \end{cases} & \text{for link } b \end{cases}$$

$$C^I = e^T \chi^I \text{ and } C^{II} = v^T \chi^{II}$$

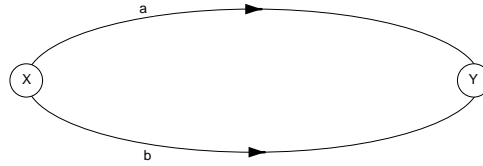


Fig. 2 Two-Node Network

where  $v_a + v_b = 2$ ,  $\theta_q^i$  is the toll of player  $i$  ( $I$  or  $II$ ) on link  $q$  ( $a$  or  $b$ ),  $e = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ,  $v = \begin{pmatrix} v_a \\ v_b \end{pmatrix}$ , and  $OC$  is the operational cost for a toll booth. We set  $OC = 0.55$  per toll booth. We take  $\chi^{II}$  (when  $\theta_a^{II} = \theta_b^{II} = 0$ ) to represent the travel time function of the links, in other words, actor  $II$  cares for travel time cost of the system ( $C^{II} = v^T \chi^{II}$ ), and of course the operational cost if he wants to set a toll (booth) on a link. Actors can only choose strategies from the following set of discrete toll:  $\theta_q^i \in \{0, 1, 2, 3, 4, 5, 6\}$ . For each move by an actor, the resulting flow is in User Equilibrium (UE).

*Result of the tolling game*

$$v = \begin{pmatrix} v_a \\ v_b \end{pmatrix}, \quad SC = (Total) System Cost = C^I + C^{II}$$

The first table represents the user equilibrium (UE) on toll free network. The UE flow w.r.t  $\chi^{II}$  on link  $a$  and  $b$  are respectively  $0.8$  and  $1.2$ . In the result table, the tolls as well as  $OC$  on the links are represented in a vector form  $(x, y)$ , the first entry is for player  $I$ , and the second entry for player  $II$ .  $SC$  is the social or system cost which represents the total cost experienced in the system. The arrow that points downwards indicates *moves (process of tolling the network)* by actors. An actor can add or remove tolls depending on which strategy optimizes his objective. In the second table, that is, *the first move by actor I*, he (actor  $I$ ) sets his maximum possible link toll (i.e. 6) on link  $b$  so as to shift traffic to link  $a$  (see  $C^I$ ).

TABLE II  
RESULT OF THE TWO-NODE NETWORK EXAMPLE

Link	Tolls	v	OC	C <sup>I</sup>	C <sup>II</sup>	Path cost	SC
a	(0,0)	0.8	(0,0)	1.6	2.9	3.6	4.5
b	(0,0)	1.2	(0,0)	3	4.3	3.6	7.3
				<b>4.6</b>	<b>7.2</b>		<b>11.8</b>
I							
a	(0,0)	2	(0,0)	4	12	6	16
b	(6,0)	0	(0.55,0)	0.55	0	6	6.55
				<b>4.55</b>	<b>12</b>		<b>16.55</b>
II							
a	(0,5)	1	(0,0.55)	2	4	9	6
b	(6,0)	1	(0.55,0)	3.05	3.55	9	6.6
				<b>5.05</b>	<b>7.55</b>		<b>12.6</b>
I							
a	(0,5)	0	(0,0.55)	0	0.55	7	0.55
b	(0,0)	2	(0,0)	5	12	6	17
				<b>5</b>	<b>12.55</b>		<b>17.55</b>
II							

Nash equilibrium does not exist

Observe the cost of 0.55 under OC for player I. Under the column C<sup>I</sup>, comes the total link costs for players I and II, the boldfaced numbers are the total network cost for the players, the total system cost (SC) is also in bold; for instance, in the first move of player II (third table),  $C_a^{II} = v_a \chi_a^{II} = v_a(2v_a + 2) = 1(2 * 1 + 2) = 4$ , and  $C_b^{II} = v_b \chi_b^{II} = v_b(3v_b + OC) = 1(3 * 1 + 0.55) = 3.55$ , and  $C^{II} = C_a^{II} + C_b^{II} = 4 + 3.55 = 7.55$ . Notice that operational cost OC = 0 unless a player has set a toll (booth) on a link (in that case, OC = 0.55)

It turns out that this game has no Nash equilibrium (NE) and the actors will always have incentive to perpetually change their strategies irrespective of what other player does.

#### Remarks about this example game

- 1) The system optimal result for the example above is given by:

$$v_a = 0.95, v_b = 1.05, C^I = 4.47, C^{II} = 7.01, SC = 11.48.$$

- 2) In general, the Nash equilibrium solution boundary of the game is given as follows:

for OC:  $0.5 < OC < 0.6$  pure NE does not exist,  
otherwise pure NE exists

showing that the existence of NE can depend on the cost of operating the toll booths.

- 3) If we ignore the OC in the model, then NE always exist among the two actors. In this case, all actors use up to their maximum toll with  $\theta_a = (6,0)$ , and  $\theta_b = (0,6)$ . Without boundary restriction on the link tolls, the two leaders will infinitely keep on increasing the link tolls (actor I on link a and II on link b). This is true since the demand is fixed. In fixed demand models, it is assumed that users will always travel no matter the cost of travel. On the other hand, with elastic demand, infinite tolls by the leaders will, of course, result in no travel scenario which is of no benefit to the actors.

- 4) By mere interchanging the link cost functions of actor I, that is,

$$\chi^I = \begin{cases} \chi_a^I = \begin{cases} 2.5v_a & \text{if } \theta_a^I = 0 \\ 2.5v_a + OC & \text{otherwise} \end{cases} & \text{for link a} \\ \chi_b^I = \begin{cases} 2v_b & \text{if } \theta_b^I = 0 \\ 2v_b + OC & \text{otherwise} \end{cases} & \text{for link b} \end{cases}$$

pure Nash equilibrium exists for any value of OC.

- 5) The cost function of the type described in this example (which includes the cost of operating the toll booths) has more practical intuition than that described in 3. This means that in practice, Nash equilibrium may not exist for such road pricing game.

#### B. Mixed Nash Equilibrium

The matrix representation of the two-player cost minimization game above with best response strategies (0,0) or (0,6) on links (a,b) for player I, and (0,0) or (5,0) on links (a,b) for player II (see Table II or the matrix in the section "interpretation" below) is given by:

$$\begin{matrix} & q & 1-q \\ \begin{matrix} p \\ 1-p \end{matrix} & \begin{pmatrix} 4.55, 12.00 \\ 4.60, 7.20 \end{pmatrix} & \begin{pmatrix} 5.05, 7.55 \\ 5.00, 12.55 \end{pmatrix} \end{matrix}$$

Observe of course that the game has no pure NE. In the mixed strategy game, player I has the strategy  $(p, 1-p)$  of playing (Top, Bottom) and player II, the strategy  $(q, 1-q)$  of playing (Left, Right), where  $p$  and  $q$  are probabilities. The best reply functions for both players are given below:

$$\beta_I(q) = \begin{cases} \{(1,0)\} & \text{if } \frac{1}{2} < q \leq 1 \\ \{(p, 1-p) | 0 \leq p \leq 1\} & \text{if } q = \frac{1}{2} \\ \{(0,1)\} & \text{if } 0 \leq q < \frac{1}{2} \end{cases}$$

$$\beta_{II}(p) = \begin{cases} \{(1,0)\} & \text{if } 0 \leq p < \frac{5.35}{9.8} \\ \{(q, 1-q) | 0 \leq p \leq 1\} & \text{if } p = \frac{5.35}{9.8} \\ \{(0,1)\} & \text{if } \frac{5.35}{9.8} < p \leq 1 \end{cases}$$

Graphically:

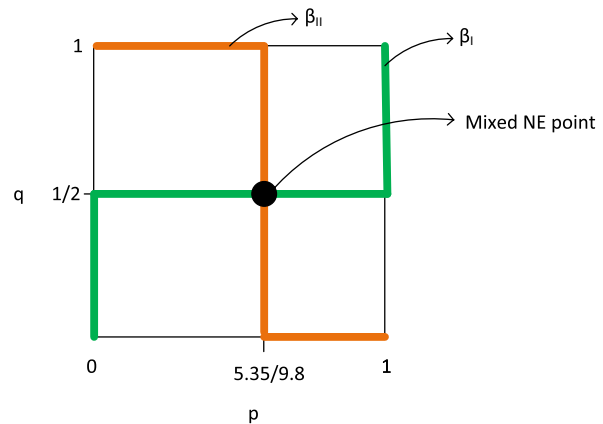


Fig. 3 Graphical representation of the mixed NE

The Mixed Nash equilibrium point  $(p, q)$  corresponds to  $(\frac{5.35}{9.8}, \frac{1}{2})$ . So, the strategies of player *I* and player *II* that will lead to mixed Nash equilibrium are  $(\frac{5.35}{9.8}, \frac{4.45}{9.8})$  and  $(\frac{1}{2}, \frac{1}{2})$  respectively. The expected cost is  $C^I = 4.80$ ,  $C^{II} = 9.82$ ,  $SC = 14.62$ .

#### Interpretation

A (mixed) Nash equilibrium exists among the players if we can find a (mixed) strategy tolls among the actors such that the equilibrium cost point  $C^I = 4.80$ ,  $C^{II} = 9.82$  is reached. In other words, actors now choose link tolls from the line  $[0, 6]$  instead of discrete set  $\{0, 1, 2, 3, 4, 5, 6\}$ .

The following tolls matrix with the associated probabilities for players *I* and *II* (the right matrix is the cost matrix) translates to the mixed Nash equilibrium cost  $C^I = 4.80$ ,  $C^{II} = 9.82$ ,  $SC = 14.62$  which we deduced above:

$$\begin{array}{cc} \hat{q} & 1-\hat{q} \\ \hat{p} & \begin{pmatrix} (0,6), (0,0) & (0,6), (5,0) \\ (0,0), (0,0) & (0,0), (5,0) \end{pmatrix}; \begin{pmatrix} \frac{5.35}{9.8} & \frac{4.45}{9.8} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \\ 1-\hat{p} & \begin{pmatrix} (0,6), (0,0) & (0,6), (5,0) \\ (0,0), (0,0) & (0,0), (5,0) \end{pmatrix}; \begin{pmatrix} 4.55, 12.00 & 5.05, 7.55 \\ 4.60, 7.20 & 5.00, 12.55 \end{pmatrix} \end{array}$$

Each pair of entry represents toll actions of the players *I* and *II* respectively. The first entry of each action is a players toll action on link *a* and the second entry, his action on link *b*. For example, the Top-Right entry of the toll matrix has the entry  $(0, 6), (5, 0)$  and this means that player *I* has no toll on link *a* and tolled 6 on link *b*, while player *II* tolled 5 on link *a* and nothing on link *b*. It turns out that there is no choice probability  $(\hat{p}, \hat{q})$  for the actors in the toll matrix game such that the game coincides with the mixed NE of the cost matrix game. This is because, there is no flow vector  $(v_a, v_b)$  such that  $C^I = 4.80(-0.55)$ , and  $C^{II} = 9.82(-0.55)$  hold at the same time. Even with finite number of players, and finite number of turns, we cannot find a mixed toll vector such that Nash equilibrium exists among the actors.

Recall that Nash [30] proved that his famous and most used theorem that every game with finite number of players in which each player can choose from finitely many pure strategies has at least one (Nash) equilibrium. "Unfortunately", our game has only two players and two possible actions per player, yet we cannot find a pair of strategy  $(\hat{p}, \hat{q})$  such that the resulting cost is in equilibrium. This lies on the fact that the expected costs  $C^I = 4.80$ ,  $C^{II} = 9.82$  is not linear in the strategy  $(\hat{p}, \hat{q})$ , and thus, the cost functions for the players under mixed strategy  $(\hat{p}, \hat{q})$  may not be continuous or quasi-convex. In fact, the strategy  $(\hat{p}, \hat{q})$  has no direct implication on the cost (utility) matrix since

- 1) the strategies for the players which are the tolls are generally not convex on the cost functions (see [23])
- 2) the tolls in the toll matrix are not uniquely determined, and this means that the strategy  $(\hat{p}, \hat{q})$  is not uniquely given, and
- 3) the non-unique tolls lead to a unique Wardrop's equilibrium which determines the cost matrix, and
- 4) we acknowledge that the traffic flow is not linear.

## VI. NASH EQUILIBRIUM AND COOPERATIVE GAME

### A. Stationary Points of Cooperative and Non-cooperative Game

#### Nash Equilibrium/Non-cooperative Game Problem

Let tolls be bounded and suppose that Nash equilibrium exists, then it will be interesting to know how far the Nash flow vector deviates from the "optimal flow" vector resulting from grand coalition game or MO solution of system (14). In other words, we are interested in knowing how far the competition among the actors can worsen the optimal system cost. Let us consider the fixed demand case.

From (8) and (10) we know that each stakeholder  $k \in K$  solves the following problem:

$$\min_{v, \theta^k} Z_k = C_k(v)$$

s.t

$$\begin{aligned} \Lambda^T \left( \beta t(v) + \theta^k + \sum_{j \in K \setminus k} \tilde{\theta}^j \right) &\geq \Gamma^T \lambda \quad [\vartheta^k] \\ \left( \beta t(v) + \theta^k + \sum_{j \in K \setminus k} \tilde{\theta}^j \right)^T v &= \tilde{d}^T \lambda \quad [\eta^k] \\ \text{and} \\ v &= \Lambda f \quad [\psi] \\ \Gamma f &= \tilde{d} \quad [\zeta] \\ f &\geq 0 \quad [\rho] \\ (\theta^k &\geq 0) \quad [\sigma^k] \end{aligned} \quad (13)$$

Recall from (10) that the link flows (and thus the minimum path cost  $\lambda$ ) are no longer actor dependent. Due to non-uniqueness of path flows, it is possible to have different path flows for different actor, but then, without loss of generality, we take one path flow pattern  $f$  for all actors and omit the superscript  $k$  on  $f$ . The Greek letters  $(\vartheta^k, \eta^k, \psi, \zeta, \rho, \sigma^k)$  are KKT multipliers associated with the constraints. The sign  $\sim$  indicates fixed parameters in the above optimization problem. System (13) involving all players is called an *equilibrium problem subject to equilibrium condition*, see also [33] for analysis of such game.

#### Assumption 2.

- 1) We make the reasonable/practical assumption that with positive flows, the total network cost is never zero, i.e.  $(\tilde{d})^T \lambda > 0$
- 2) We assume that the *linear independence constraint qualification (LICQ)* holds for (13) & (14).

Suppose for every leader  $k$ ,  $L_k$  is the Lagrangian and that  $(\bar{v}, \bar{\theta}^k)$  solves (13) at (Nash) equilibrium, then with assumption 2, there exist multipliers  $(\bar{\vartheta}^k, \bar{\eta}^k, \bar{\psi}, \bar{\zeta}, \bar{\rho}, \bar{\sigma}^k)$  such that the following KKT conditions hold  $\forall k \in K$ :



**KKT 13**

$$\begin{aligned}
L_k &= C_k(v) + \left[ \Gamma^T \lambda - \Lambda^T \left( \beta t(v) + \theta^k + \sum_{j \in K \setminus k} \bar{\theta}^j \right) \right]^T \bar{\vartheta}^k \\
&\quad + \left[ \left( \beta t(v) + \theta^k + \sum_{j \in K \setminus k} \bar{\theta}^j \right)^T v - (\bar{d})^T (\lambda) \right] \bar{\eta}^k \\
&\quad + (\Lambda f - v)^T \bar{\psi} + (\bar{d} - \Gamma f)^T \bar{\zeta} - f^T \bar{\rho} - (\theta^k)^T \bar{\sigma}^k \\
\nabla_v L_k &= \nabla C_k(v) - \beta \left( \Lambda^T \nabla t(v) \right)^T \bar{\vartheta}^k \\
&\quad + \left( \beta t(v) + \theta^k + \sum_{j \in K \setminus k} \bar{\theta}^j + \beta v^T \nabla t(v) \right) \bar{\eta}^k - \bar{\psi} = 0 \\
\nabla_f L_k &= \Lambda^T \bar{\psi} - \Gamma^T \bar{\zeta} - \bar{\rho} = 0 \\
\nabla_{\theta^k} L_k &= -\Lambda \bar{\vartheta}^k + \bar{v} \bar{\eta}^k - \bar{\sigma}^k = 0 \\
f^T \bar{\rho} &= 0 \\
(\bar{\theta}^k)^T \bar{\sigma}^k &= 0 \\
\bar{\vartheta}^k, \bar{\rho}, \bar{\sigma}^k &\geq 0, \left[ \Gamma^T \lambda - \Lambda^T \left( \beta t(v) + \bar{\theta}^k + \sum_{j \in K \setminus k} \bar{\theta}^j \right) \right]^T \bar{\vartheta}^k = 0
\end{aligned}$$

**Grand Coalition or Cooperative Game Problem**

The grand coalition (GC) game with a toll vector  $\theta = \sum_{k \in K} \theta^k$  (assuming that GC assigns  $\theta^k$  to each actor  $k \in K$ ) is formulated as follows:

$$\begin{aligned}
\min_{v, \theta^k} Z &= \sum_{k \in K} C_k(v) \\
s.t. \quad & \\
\Lambda^T \left( \beta t(v) + \sum_{k \in K} \theta^k \right) &\geq \Gamma^T \lambda \quad [\vartheta] \\
\left( \beta t(v) + \sum_{k \in K} \theta^k \right)^T v &= (\bar{d})^T \lambda \quad [\eta] \\
\text{and} \quad & \\
v &= \Lambda f \quad [\psi] \\
\Gamma f &= \bar{d} \quad [\zeta] \\
f &\geq 0 \quad [\rho] \\
(\theta^k &\geq 0) \quad [\sigma^k] \quad \forall k \in K
\end{aligned}$$

**Remark 3**

- The grand coalition game in system (14) minimizes the entire system cost, and thus, resulting in "Pareto" optimal system flow  $\bar{v}$ .
- Since systems (13) & (14) have a non-linear constraint respectively, the efficient way to solve the systems so that we reach the global optimum is to:
  - 1) Solve the convex system for an optimal flow  $\bar{v}$  by omitting the first two left constraints and the last (the toll) constraint in systems (13) & (14).
  - 2) then, fixing the optimal flow  $\bar{v}$ , we search for a feasible toll vector  $\bar{\theta}$  that satisfies the omitted constraints in (1). Observe that now all systems are linear. This

is the same as solving the linear system (5) together with non-negativity of the tolls.

- The solution steps above apply to both first and second best pricing schemes.

Now, suppose  $L$  is the Lagrangian and that  $\bar{v}$  and  $\bar{\theta}^k \forall k \in K$  solves the grand coalition game (14), then with assumption 2, there exist multipliers  $(\bar{\vartheta}, \bar{\eta}, \bar{\psi}, \bar{\zeta}, \bar{\rho}, \bar{\sigma}^k)$  such that the following KKT conditions hold:

**KKT 14**

$$\begin{aligned}
L &= \sum_{k \in K} C_k(v) + \left[ \Gamma^T \lambda - \Lambda^T \left( \beta t(v) + \sum_{k \in K} \theta^k \right) \right]^T \bar{\vartheta} \\
&\quad + \left[ \left( \beta t(v) + \sum_{k \in K} \theta^k \right)^T v - (\bar{d})^T (\lambda) \right] \bar{\eta} \\
&\quad + (\Lambda f - v)^T \bar{\psi} + (\bar{d} - \Gamma f)^T \bar{\zeta} - f^T \bar{\rho} - (\theta^k)^T \bar{\sigma}^k \\
\nabla_v L &= \sum_{k \in K} \nabla C_k(v) - \beta \left( \Lambda^T \nabla t(v) \right)^T \bar{\vartheta} \\
&\quad + \left( \beta t(v) + \sum_{k \in K} \theta^k + \beta v^T \nabla t(v) \right) \bar{\eta} - \bar{\psi} = 0 \\
\nabla_f L &= \Lambda^T \bar{\psi} - \Gamma^T \bar{\zeta} - \bar{\rho} = 0 \\
\nabla_{\theta^k} L &= -\Lambda \bar{\vartheta} + \bar{v} \bar{\eta} - \bar{\sigma}^k = 0 \\
f^T \bar{\rho} &= 0, \quad (\bar{\theta}^k)^T \bar{\sigma}^k = 0 \quad \forall k \in K \\
\bar{\vartheta}, \bar{\rho}, \bar{\sigma}^k &\geq 0 \quad \forall k \in K, \left[ \Gamma^T \lambda - \Lambda^T \left( \beta t(v) + \sum_{k \in K} \bar{\theta}^k \right) \right]^T \bar{\vartheta} = 0
\end{aligned}$$

Let tolls be bounded and suppose that Nash equilibrium exists, then, (theoretically) the (stationary point) solution of the Nash game converges to a stationary point of the cooperative game. We thus state the following corollary:

**Corollary 3.** With assumption 2, there exist multipliers  $(\bar{\vartheta}^k, \bar{\eta}^k, \bar{\psi}, \bar{\zeta}, \bar{\rho}, \bar{\sigma}^k)$  such that KKT 13 holds for all  $k$  at (Nash) equilibrium, furthermore, the corresponding (stationary) vector  $(\bar{v}, \bar{\theta})$  that solves the Nash game (9) or (13) is also a stationary (possibly a local or global solution) for the grand coalition (GC) game (14), where  $\bar{\theta} \in \mathbb{R}^{|K|}$ .

*Proof:* Since  $(\bar{\vartheta}^k, \bar{\eta}^k, \bar{\psi}, \bar{\zeta}, \bar{\rho}, \bar{\sigma}^k)$  exist, then, there exist  $(\bar{\vartheta}, \bar{\eta}, \bar{\psi}, \bar{\zeta}, \bar{\rho}, \bar{\sigma}^k) = \sum_{k \in K} (\bar{\vartheta}^k, \bar{\eta}^k, \bar{\psi}, \bar{\zeta}, \bar{\rho}, \bar{\sigma}^k)$  such that the corresponding vector  $(\bar{v}, \bar{\theta})$  of system (13) solves KKT 14. For

instance, see from KKT 14 that

$$\begin{aligned}
 \nabla_v L &= \sum_{k \in K} \nabla C_k(\check{v}) - \beta \left( \Lambda^T \nabla l(\check{v}) \right)^T \check{\theta} \\
 &+ \left( \beta l(\check{v}) + \sum_{k \in K} \theta^k + \beta \check{v}^T \nabla l(\check{v}) \right) \check{\eta} - \check{\psi} \\
 &= \sum_{k \in K} \nabla C_k(\check{v}) - \beta \left( \Lambda^T \nabla l(\check{v}) \right)^T \sum_{k \in K} \check{\theta}^k \\
 &+ \left( \beta l(\check{v}) + \sum_{k \in K} \theta^k + \beta \check{v}^T \nabla l(\check{v}) \right) \sum_{k \in K} \check{\eta}^k - \sum_{k \in K} \check{\psi} \\
 &= \sum_{k \in K} \left( \nabla C_k(\check{v}) - \beta \left( \Lambda^T \nabla l(\check{v}) \right)^T \check{\theta}^k \right. \\
 &\quad \left. + \left( \beta l(\check{v}) + \sum_{k \in K} \check{\theta}^k + \beta \check{v}^T \nabla l(\check{v}) \right) \check{\eta}^k - \check{\psi} \right) = 0 \\
 &\text{(see KKT 13)}
 \end{aligned}$$

#### Remark 4

- Observe that other KKT conditions of KKT 13 and KKT 14 are the same.
- Corollary 3 is comparable to Proposition 5.5 in [29]. We do not assume a completely separable system though.
- The corollary can be extended to the Nash game between any form of coalitions that the stakeholders deem profitable.

#### B. Stability of solutions

We now discuss in detail cooperative game solutions when the demand is fixed. Consider a cooperative game with a characteristic function  $u: 2^K \rightarrow \mathbb{R}$ ,  $S \rightarrow u(S)$ ,  $S \subseteq K$ , relative to a partition  $p = \{S_1, S_2, S_3, \dots, S_r\}$  of  $K$  such that  $\bigcap_i S_i = \emptyset$  and  $\bigcup_i S_i = K$ . We treat each  $S_i$  as a single player. Each coalition  $S_i \in p$  competes with all other coalitions  $S_j \in p$ ,  $i \neq j$ . In terms of objectives, the game is a Nash equilibrium game between coalitions  $S_i \in p$  with (collective) objective of each coalition  $S_i$  given by (see also system (13)):

$$\min_{v, \theta} \sum_{k \in S_i} C_k(v)$$

s.t

feasibility conditions

EqC-FD

Now, suppose this game has a unique Nash equilibrium  $(\theta(p), v(p))$ . We define

$$u(S_i, p) = - \sum_{k \in S_i} C_k(v(p)) \quad (15)$$

as the corresponding outcome and utility  $u_k(v(p))$  for each  $k \in K$ . In addition, we define the utility  $u(S)$  of a subset  $S \subseteq K$  as the "worst case utility" for  $S$  as

$$u(S) = \min_{p, S \subseteq p} u(S, p)$$

In particular,  $u(K) = u(K, \{K\})$

**Definition 1. Core:** we define a core as follows

$$\text{core} : \left\{ x \in \mathbb{R}^{|K|} \mid x(K) = u(K), x(S) \geq u(S), \forall S \subseteq K \right\}$$

where  $x(S) = \sum_{k \in S} x_k$ .

**Definition 2. A partition**  $p = \{S_1, S_2, S_3, \dots, S_r\}$  is stable if there exist some allocations  $x \in \mathbb{R}^{|K|}$  with  $x(S_i) = u(S_i, p)$ , such that

$$x(S) \geq u(S), \forall S \subseteq K$$

**Remark 5:** By definition, if  $\text{core} \neq \{0\}$  then the grand coalition  $p = \{K\}$  is stable ( $x \in \text{core}$  yields the allocation in definition 2).

**Lemma 1.** If a partition  $p = \{S_1, S_2, S_3, \dots, S_r\}$  of  $K$  is stable, then the core  $\neq \{0\}$

*Proof:* Assume  $p = \{S_1, S_2, S_3, \dots, S_r\}$  is stable, then there exist some allocations  $x \in \mathbb{R}^{|K|}$  with  $x(S_i) = u(S_i, p)$ , such that

$$x(S) \geq u(S), \forall S \subseteq K$$

But then, we have

$$x(K) = \sum_{k \in K} x_k = \sum_i x(S_i) = \sum_i u(S_i, p) \leq u(K) \leq x(K)$$

so equality holds and  $x \in \text{core}$ . The first inequality follows from (15) (and the fact that  $u(K)$  is the maximizer of  $-\sum_{k \in K} C_k(v(p))$ ), and the second inequality follows from the stability condition on  $p$ .

In particular,

$$\sum_i u(S_i, p) = u(K)$$

**Corollary 4.** A necessary condition for a partition  $p = \{S_1, S_2, S_3, \dots, S_r\}$  of  $K$  to be stable is that

$$\sum_i x(S_i) = u(K)$$

*Proof:* Proof follows from the proof of Lemma 1. ■

**Corollary 5.** The resulting Nash equilibrium flow vector  $\bar{v}$  for any partition set  $p$  is a stationary (possibly local or global) point  $\check{v}$  of the grand coalition program (14).

*Proof:* The proof follows from corollary 3. ■

**Remark 6:** In general, we do not expect the game to have a core.

We illustrate this statement with a simple example. Such example can be constructed from traffic models. Suppose we have three players (I, II, III) and five possible partitions  $\{p_1, p_2, p_3, p_4, p_5\}$  of  $K$ , where  $K = \{I, II, III\}$ . The table

below gives the outcome for each of the possible coalitions in each partition set:

Partition sets \ Coalition	$u_k(v(p))$			$\sum_{k \in K} u_k$
	I	II	III	
$p_1 (\{I\}, \{II\}, \{III\})$	6	8	7	21
$p_2 (\{I, III\}, \{II\})$	10	7	10	27
$p_3 (\{I, II\}, \{III\})$	12	11	6	29
$p_4 (\{II, III\}, \{I\})$	5	12	9	26
Grand Coalition $p_5 (\{I, II, III\})$	14	5	12	31
Worst Case $u(\{k\}) = \min_k (v(p))$	5	5	6	16
Gain w.r.t. $p_5$	9	0	6	15
Egalit. Gain Sharing (EGS) = $\frac{15}{3}$	5	5	5	15
Final outcome = $u(\{k\}) + EGS$	10	10	11	31

We have assumed Egalitarian sharing rule of the total gain w.r.t. grand coalition  $p_5$  among the players. Observe that players I and II will be better off playing coalition  $\{I, II\}$  in partition set  $p_3$  instead of GC, but then, player III will switch to coalition  $\{II, III\}$  in partition set  $p_4$  promising player II a positive utility of 13 units. Such move by players III and II will trigger player I to form the coalition  $\{I, III\}$  in partition  $p_2$  by assuring player III a positive utility of 10 units and so on. This change of strategies may continue indefinitely in this example. Observe that though the grand coalition presents the maximum total utility for the game, it is not stable, and by remark 5, the core is empty.

**Remark 7:** The results in corollary 3, 4, 5 and Lemma 1 are still valid for elastic demand model.

## VII. CONCLUSION

We study the classical game theoretical solution concepts ranging from Nash solutions and cooperative solutions to core of the road pricing game. We showed that in general, the road pricing game has no Nash equilibrium (both in pure and mixed strategies). With bound restrictions on tolls, the game may possess Nash equilibrium. We then show that a stationary Nash equilibrium point coincides with that of the grand coalition game. We further proved that if side payments are allowed within coalitions in the cooperative game, then a partition is stable if the core is non empty, and the total utility of any stable partition is the same as that of the grand coalition game.

Since the models used in this paper centered on classical optimization formulations, the number of variables can grow uncontrollably big when the network is large. This calls for an efficient optimization heuristic which can transform the analytical models into heuristic algorithms capable of handling large networks. Extension of the road pricing game to include dynamic traffic model will be the next step of research.

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