

# Improved IDR( $s$ ) method for gaining very accurate solutions

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*Abstract*—The IDR( $s$ ) method based on an extended IDR theorem was proposed by Sonneveld and van Gijzen. The original IDR( $s$ ) method has excellent property compared with the conventional iterative methods in terms of efficiency and small amount of memory. IDR( $s$ ) method, however, has unexpected property that relative residual 2-norm stagnates at the level of less than  $10^{-12}$ . In this paper, an effective strategy for stagnation detection, stagnation avoidance using adaptively information of parameter  $s$  and improvement of convergence rate itself of IDR( $s$ ) method are proposed in order to gain high accuracy of the approximated solution of IDR( $s$ ) method. Through numerical experiments, effectiveness of adaptive tuning IDR( $s$ ) method is verified and demonstrated.

*Keywords*—Krylov subspace methods, IDR( $s$ ), Adaptive tuning, Stagnation of relative residual

## I. INTRODUCTION

**W**E consider to solve a nonsymmetric linear system of equations,

$$A\mathbf{x} = \mathbf{b} \quad (1)$$

where  $A$  is a given nonsymmetric matrix in  $R^{N \times N}$ , and  $\mathbf{x}$  is a solution vector in  $R^N$ , and  $\mathbf{b}$  is a right-hand side vector in  $R^N$ . Krylov subspace methods are effective for solving linear systems of equations [4]. Krylov subspace is defined as follows:

$$K_n(A; \mathbf{r}_0) := \text{span}\{\mathbf{r}_0, A\mathbf{r}_0, \dots, A^{n-1}\mathbf{r}_0\}. \quad (2)$$

Here,  $\mathbf{r}_0 := \mathbf{b} - A\mathbf{x}_0$  is an initial residual vector. The members of Krylov subspace methods, product-type Bi-Conjugate Gradient (BiCG) methods are often used for solving nonsymmetric linear systems of equations. BiCG stabilized (BiCGStab) method [4], and Generalized Product BiCG (GPBiCG) method [6], BiCGSafe method [2] and so on, are some versions of product-type Bi-Conjugate Gradient (BiCG) methods.

In 2008, one of Krylov subspace method, IDR( $s$ ) method is proposed by P. Sonneveld and M. B. van Gijzen [3]. IDR( $s$ ) method is based on the IDR theorem. IDR( $s$ ) method is competitive with or superior to most product-type BiCG methods, and outperforms BiCGStab method when  $s > 1$ .

However, we meet with a phenomenon that relative residual 2-norm of original IDR( $s$ ) method stagnates approximately between  $10^{-12}$  and  $10^{-15}$ . Therefore, we should consider adaptive tuning IDR( $s$ ) method on parameter  $s$  (abbreviated as AT\_IDR( $s$ ) method) for avoidance the stagnation. We will demonstrate effectiveness of AT\_IDR( $s$ ) method for avoidance stagnation of residual by means of some numerical experiments.

This paper is organized as follows. In section 2, we introduce outline of IDR( $s$ ) method. In particular, we describe

IDR theorem and algorithm of IDR( $s$ ) method. In section 3, we present algorithm of AT\_IDR( $s$ ) method in detail. We describe two strategies in order to build AT\_IDR( $s$ ) method and algorithm of AT\_IDR( $s$ ) method, and consider issues on implementation. In section 4, robustness of AT\_IDR( $s$ ) method is demonstrated by numerical experiments. That is, we make clear that AT\_IDR( $s$ ) method can solve systems with high accuracy. Finally, in section 5, we draw concluding remarks.

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## II. IDR( $s$ ) METHOD

### A. IDR theorem

IDR( $s$ ) method is based on the IDR theorem [5][3]. Let  $A$  be any matrix in  $R^{N \times N}$ , and  $\mathbf{v}_0$  be any vector in  $R^N$ , and  $\mathcal{G}_0$  be the complete Krylov space  $K_N(A, \mathbf{v}_0)$ . Let  $S$  denote any space in  $R^N$ , and define the sequence spaces  $\mathcal{G}_j$  ( $j = 1, 2, \dots$ ) as

$$\mathcal{G}_j := (I - \omega_j A)(\mathcal{G}_{j-1} \cap S). \quad (3)$$

Here  $\omega_j$ 's are non-zero scalars. Then, the next two theorems holds.

- (i)  $\mathcal{G}_j \subseteq \mathcal{G}_{j-1}$  for all  $j > 0$ ,
- (ii)  $\mathcal{G}_j = \{\mathbf{0}\}$  for some  $j \leq N$ .

### B. Algorithm of IDR( $s$ ) method

The IDR theorem can be applied by generating residual vectors  $\mathbf{r}_n$  that are forced to be in space  $\mathcal{G}_j$  ( $j \leq N$ ). Then, under assumptions of the IDR theorem, a linear system of equations will be solved after at most  $N$  dimension reduction steps. Algorithm of IDR( $s$ ) method is written as follows:

1. Let  $\mathbf{x}_0$  be an initial guess, and put  $\mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0$
2. For  $n = 0, \dots, s-1$  Do
3.  $\mathbf{v}_n = A\mathbf{r}_n$
4.  $\omega_n = \frac{(\mathbf{v}_n, \mathbf{r}_n)}{(\mathbf{v}_n, \mathbf{v}_n)}$
5.  $\mathbf{q}_n = \omega_n \mathbf{r}_n, \quad \mathbf{e}_n = -\omega_n \mathbf{v}_n,$
6.  $\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{e}_n, \quad \mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{q}_n$
7. End Do
8.  $E_s = (\mathbf{e}_{s-1} \cdots \mathbf{e}_0), \quad Q_s = (\mathbf{q}_{s-1} \cdots \mathbf{q}_0)$
9. Do  $n = s, s+1, \dots$
10. Solve  $\mathbf{c}_n$  from  $P^T E_n \mathbf{c}_n = P^T \mathbf{r}_n$
11.  $\mathbf{v}_n = \mathbf{r}_n - E_n \mathbf{c}_n$
12. If  $\text{mod}(n, s+1) = s$  then
13.  $\mathbf{t}_n = A\mathbf{v}_n$
14.  $\omega_n = \frac{(\mathbf{t}_n, \mathbf{v}_n)}{(\mathbf{t}_n, \mathbf{t}_n)}$

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15.      $\mathbf{e}_n = -E_n \mathbf{c}_n - \omega_n \mathbf{t}_n$ 
16.      $\mathbf{q}_n = -Q_n \mathbf{c}_n + \omega_n \mathbf{v}_n$ 
17.     Else
18.      $\mathbf{q}_n = -Q_n \mathbf{c}_n + \omega_n \mathbf{v}_n, \mathbf{e}_n = -A \mathbf{q}_n$ 
19.     End If
20.      $\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{e}_n, \mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{q}_n$ 
21.     if  $\|\mathbf{r}_{n+1}\|_2 / \|\mathbf{r}_0\|_2 \leq \epsilon$  then stop
22.      $E_n = (\mathbf{e}_{n-1} \cdots \mathbf{e}_{n-s}), Q_n = (\mathbf{q}_{n-1} \cdots \mathbf{q}_{n-s})$ 
23.     End Do

```

### C. How to make matrix $P$

We discuss how to make matrix  $P$ . The matrix  $P$  is defined as  $P = (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_s)$ . The every entries of  $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_s$  are random numbers between 0.0 and 1.0. Then, matrix  $P$  is orthonormalized by modified Gram-Schmidt method as

$$(\mathbf{p}_i, \mathbf{p}_j) = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

We note that it is necessary to make matrix only once before the iteration process of IDR( $s$ ) method.

### III. AT\_IDR( $s$ ) METHOD

In this section, we discuss how to build AT\_IDR( $s$ ) method which improves convergence property by tuning parameter  $s$  of IDR( $s$ ) method.

#### A. To build AT\_IDR( $s$ ) method

For the larger parameter  $s$ , computation time of IDR( $s$ ) method per one iteration becomes longer. Therefore we adopt two significant strategies as follows:

- 1) **The first strategy:** Detect stagnation of residual.
- 2) **The second strategy:** Reset parameter  $s$  as the original value.

We describe the first strategy. First, we compute variation rate of residual norm  $\sigma_n := \frac{\|\mathbf{r}_n\|_2 - \|\mathbf{r}_{n-1}\|_2}{\|\mathbf{r}_{n-1}\|_2}$ . Second, we regard it as occurrence of stagnation, when the index "sentinel" of  $\sigma_n < \delta$  in consecutive times. We remark that one must give parameters  $\delta$  and *sentinel* before iteration process of AT\_IDR( $s$ ) method as *sentinel* = 5,  $\delta = 0.1$ . The second strategy is clear in trivial, so we omit description of the second strategy.

#### B. Diagram of adaptive tuning AT\_IDR( $s$ ) method on parameter $s$

In this section, we discuss adaptive technique of AT\_IDR( $s$ ) method using a diagram. Fig. 1 shows the diagram of adaptive tuning AT\_IDR( $s$ ) method on parameter  $s$ . At first, we compute the variation rate of residual 2-norm  $\sigma_n$ . Next, we check whether  $\sigma_n < \delta$  or not. If  $\sigma_n < \delta$ , we increase *count* by one, and regard it as occurrence of stagnation of residual when *count* = *sentinel*. On the other hand, if  $\sigma_n \geq \delta$ , we reset *count* as 0, and regard it as avoidance of stagnation of residual. When stagnation of residual occurs, we increase parameter  $s$  by one, and reset *count* as 0. When stagnation of residual is avoided, we reset parameter  $s$  as the original value.

#### C. Algorithm of AT\_IDR( $s$ ) method

We present algorithm of AT\_IDR( $s$ ) method as follows: At lines number 9 and between 23 and 31, we detect the stagnation of the residual norm, and tune parameter  $s$ .

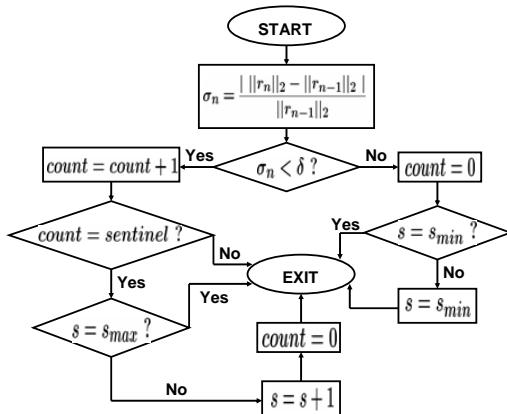
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1.  Let  $\mathbf{x}_0$  be an initial guess, and put  $\mathbf{r}_0 = \mathbf{b} - A \mathbf{x}_0$ 
2.  For  $n = 0, \dots, s - 1$  Do
3.      $\mathbf{v}_n = A \mathbf{r}_n$ 
4.      $\omega_n = \frac{(\mathbf{v}_n, \mathbf{r}_n)}{(\mathbf{v}_n, \mathbf{v}_n)}$ 
5.      $\mathbf{q}_n = \omega_n \mathbf{r}_n, \mathbf{e}_n = -\omega_n \mathbf{v}_n,$ 
6.      $\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{e}_n, \mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{q}_n$ 
7.  End Do
8.   $E_s = (\mathbf{e}_{s-1} \cdots \mathbf{e}_0), Q_s = (\mathbf{q}_{s-1} \cdots \mathbf{q}_0)$ 
9.   $s_{min} = s$ 
10. Do  $n = s, s + 1, \dots$ 
11.  Solve  $\mathbf{c}_n$  from  $P^T E_n \mathbf{c}_n = P^T \mathbf{r}_n$ 
12.   $\mathbf{v}_n = \mathbf{r}_n - E_n \mathbf{c}_n$ 
13.  If  $\text{mod}(n, s + 1) = s$  then
14.      $\mathbf{t}_n = A \mathbf{v}_n$ 
15.      $\omega_n = \frac{(\mathbf{t}_n, \mathbf{v}_n)}{(\mathbf{t}_n, \mathbf{t}_n)}$ 
16.      $\mathbf{e}_n = -E_n \mathbf{c}_n - \omega_n \mathbf{t}_n$ 
17.      $\mathbf{q}_n = -Q_n \mathbf{c}_n + \omega_n \mathbf{v}_n$ 
18.  Else
19.      $\mathbf{q}_n = -Q_n \mathbf{c}_n + \omega_n \mathbf{v}_n, \mathbf{e}_n = -A \mathbf{q}_n$ 
20.  End If
21.   $\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{e}_n, \mathbf{x}_{n+1} = \mathbf{x}_n + \mathbf{q}_n$ 
22.  If  $\|\mathbf{r}_{n+1}\|_2 / \|\mathbf{r}_0\|_2 \leq \epsilon$  then stop
23.   $\sigma_n = \frac{\|\mathbf{r}_{n+1}\|_2 - \|\mathbf{r}_n\|_2}{\|\mathbf{r}_n\|_2}$ 
24.  If  $\sigma_n < \delta$  then
25.      $count = count + 1$ 
26.     If  $count = \text{sentinel}$  and  $s < s_{max}$  then
27.         $count = 0, s = s + 1$ 
28.     End If
29.  Else
30.      $count = 0, s = s_{min}$ 
31.  End If
32.   $E_n = (\mathbf{e}_{n-1} \cdots \mathbf{e}_{n-s}), Q_n = (\mathbf{q}_{n-1} \cdots \mathbf{q}_{n-s})$ 
33.  End Do

```

#### D. Issues on implementation

In the above algorithm of AT\_IDR( $s$ ) method, there are issues on implementation. That is, they are to build matrices  $P, Q_n$  and  $E_n$ . Number of columns of matrices  $P, Q_n$  and  $E_n$  is  $s$ . Therefore we have to change number of columns of matrices  $P, Q_n$  and  $E_n$ , if parameter  $s$  increases. However it is difficult to implement matrices with dynamic allocation. We show how to build matrices  $P, Q_n$  and  $E_n$ . First, we set  $s_{max}$  as upper limit of parameter  $s$ . Second, matrices  $P, Q_n$  and  $E_n$  are built as an  $N \times s_{max}$  matrix of constant size. Third, from  $s + 1$  to  $s_{max}$  columns of matrices  $P, Q_n$  and  $E_n$  are ignored when  $s < s_{max}$  for simplicity.

Fig. 1. Diagram of adaptive tuning AT\_IDR(s) method on parameter  $s$ .

#### IV. NUMERICAL EXPERIMENTS

In this section we discuss numerical experiments of comparing performance of AT\_IDR(s) method with IDR(s) method. All computations are carried out in double precision floating-point arithmetic on a PC with a POWER5 processor (1.9GHz). Intel Fortran Compiler90 ver 7.1 and compile option -O3 -qtune=power5 -qarch=pwr5 -qhot was used. In all cases the iteration was started with the initial guess solution  $x_0 = 0$ . The maximum iterations was fixed as 10000. We set parameters of AT\_IDR(s) method as  $sentinel = 5$ ,  $\delta = 0.1$ . Four test matrices are from University of Florida Sparse Matrix Collection[1]. Description of test matrices is shown in Table I. In this Table, "nnz" means number of nonzero entries, and "ave. nnz" means number of nonzero entries per single row.

TABLE I  
SPECIFICATIONS OF TEST MATRICES.

group/matrix	dimension	nnz	ave. nnz
Watson/ChemMaster1	40,401	201,201	4.98
Watson/Baumann	112,211	748,331	6.67
FEMLAB/Sme3Da	12,504	874,887	69.97
FEMLAB/Sme3Db	29,067	2,081,063	71.59
FEMLAB/Sme3Dc	42,930	3,148,656	73.34
Quaglino/ViscoPlastic1	4,326	61,166	14.14
Quaglino/ViscoPlastic2	32,769	381,326	11.64

##### A. Numerical Results

Tables 2–8 for matrices Sme3Dc, ChemMaster1, ViscoPlastic2, Baumann, Sme3Da, Sme3Db and Viscoplatic1 show iterations and CPU time in seconds of IDR(s) and AT\_IDR(s) methods when the stopping criterion, i.e.,  $\frac{\|r_n\|_2}{\|r_0\|_2}$  is less than  $10^{-12}$ ,  $10^{-13}$  and  $10^{-14}$ , respectively. In Tables, "max" means that iterative methods did not converge until maximum iterations. "break" means also that all computations were halted because of huge numerical errors during iteration process. "itr:"

means also number of iterations. Some observations are gained from Tables 2–8.

- AT\_IDR(s) method performs well compared with the original IDR(s) method.
- IDR(s) method does not converge often when  $\epsilon$  for convergence criterion is set as  $10^{-13}$ ,  $10^{-14}$  and  $s$  is more than 4.
- On the other hand, AT\_IDR(s) method converges for almost cases.

In order to make out how robust AT\_IDR(s) is for analysis with high accuracy, we made stopping criterion more degree by degree. When the stopping criterion is  $\frac{\|r_n\|_2}{\|r_0\|_2} \leq 10^{-15}$  which is almost as same as the so-called machine epsilon of  $2.2 \times 10^{-16}$ , IDR(s) method doesn't converged for all cases. On the other hand, AT\_IDR(s) method converged for all cases. We can understand effectiveness of AT\_IDR(s) method for analysis with high accuracy.

Figs.2–3 display relative residual history of IDR(s) and AT\_IDR(s) methods for matrices ChemMaster1 and ViscoPlastic2. In these Figures, we show relative residual history of IDR(s) method in red solid line and AT\_IDR(s) method in green dashed line, and variation of parameter  $s$  of AT\_IDR(s) method in blue plot. From Figs.2–3, the following observations can be made as below.

- Results for matrix ChemMaster1 as shown in Fig.2:
  - Relative residual norm of IDR(s) method stagnates for all parameter  $s$ .
  - Relative residual norm of AT\_IDR(1) method converges without stagnation.
  - Relative residual norm of AT\_IDR(4) and AT\_IDR(8) methods stagnate after 150 iterations, and converge after parameter  $s$  is tuned.
  - If parameter  $s$  is tuned at around 150 iterations, iterations of AT\_IDR(4) and AT\_IDR(8) may be lower.
- Results for matrix ViscoPlastic2 as shown in Fig.3:
  - Relative residual 2-norm of IDR(s) method diverges at  $s$  is equal to 1, and stagnates at  $s$  is equal to 4 or 8.
  - Relative residual 2-norm of AT\_IDR(1) method stagnates after 3000 iterations, and converges at 3939 iterations.
  - Relative residual 2-norm of AT\_IDR(4) and AT\_IDR(8) methods converges without stagnation.
  - Parameter  $s$  of AT\_IDR(s) is tuned too much, and CPU times of AT\_IDR(s) per one iteration is longer than that of IDR(s) method.
  - When we set parameter  $\delta$  for smaller value or  $sentinel$  for larger value, parameter  $s$  is tuned more moderately, and AT\_IDR(s) method converges faster.

From the above many observations, we can see that AT\_IDR(s) method is more robust than IDR(s) method. However, we should find out optimum parameters  $\delta$  and  $sentinel$  to improve convergence property of AT\_IDR(s) method.

TABLE II  
CONVERGENCE OF IDR(s) AND AT\_IDR METHODS FOR MATRIX SME3DC.

method	s	$\epsilon = 10^{-12}$		$\epsilon = 10^{-13}$		$\epsilon = 10^{-14}$	
		itr.	time	itr.	time	itr.	time
IDR(s)	1	4048	292.5	max	-	max	-
	2	2096	153.5	2325	168.3	max	-
	4	1216	91.5	max	-	max	-
	8	1011	78.4	max	-	max	-
AT_IDR	1	6683	388.2	6690	386.9	7072	409.7
	2	3305	194.6	3318	194.8	3600	210.5
	4	1200	73.9	1239	76.3	1277	78.4
	8	986	62.4	995	63.2	1022	64.9

TABLE III  
CONVERGENCE OF IDR(s) AND AT\_IDR METHODS FOR MATRIX CHEMMASTER1.

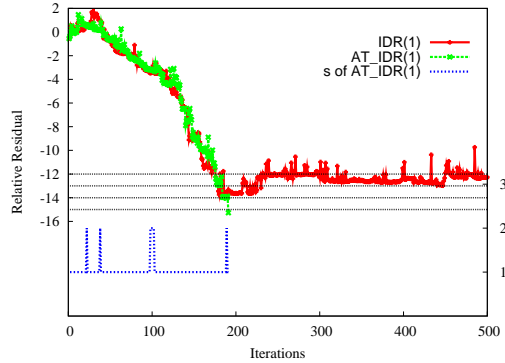
method	s	$\epsilon = 10^{-12}$		$\epsilon = 10^{-13}$		$\epsilon = 10^{-14}$	
		itr.	time	itr.	time	itr.	time
IDR(s)	1	176	0.81	179	0.82	max	-
	2	162	0.78	max	-	max	-
	4	295	1.46	max	-	max	-
	8	max	-	max	-	max	-
AT_IDR	1	176	0.85	184	0.90	188	0.89
	2	249	1.23	262	1.30	270	1.33
	4	151	0.85	330	1.82	341	1.86
	8	147	1.00	289	1.90	297	1.94

TABLE IV  
CONVERGENCE OF IDR(s) AND AT\_IDR METHODS FOR MATRIX VISCOPLASTIC2.

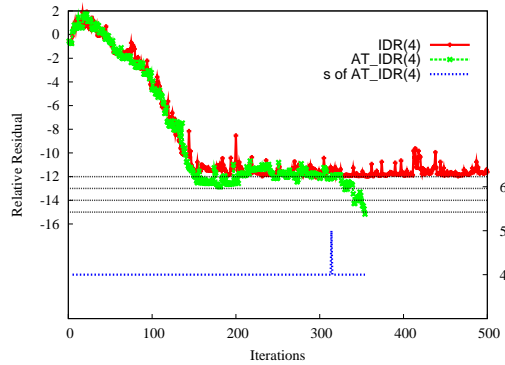
method	s	$\epsilon = 10^{-12}$		$\epsilon = 10^{-13}$		$\epsilon = 10^{-14}$	
		itr.	time	itr.	time	itr.	time
IDR(s)	1	max	-	max	-	max	-
	2	max	-	max	-	max	-
	4	1393	6.79	1520	7.46	max	-
	8	1288	7.24	1451	8.23	max	-
AT_IDR	1	1851	8.34	2650	11.90	3058	13.65
	2	1607	7.60	1834	8.69	1887	8.86
	4	1337	6.90	1490	7.72	1611	8.32
	8	1289	7.93	1460	9.09	1552	9.68

TABLE V  
CONVERGENCE OF IDR(s) AND AT\_IDR METHODS FOR MATRIX BAUMANN.

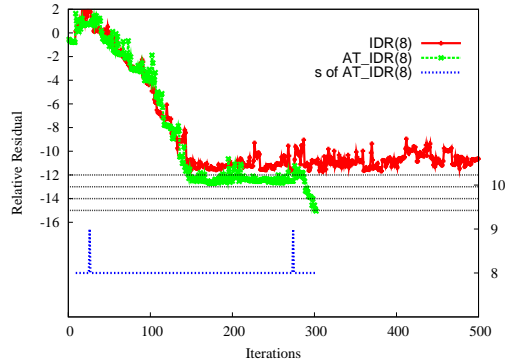
method	s	$\epsilon = 10^{-12}$		$\epsilon = 10^{-13}$		$\epsilon = 10^{-14}$	
		itr.	time	itr.	time	itr.	time
IDR(s)	1	break	-	break	-	break	-
	2	max	-	max	-	max	-
	4	669	11.80	max	-	max	-
	8	max	-	max	-	max	-
AT_IDR	1	2621	38.69	2695	39.64	2760	40.87
	2	1304	20.87	1418	22.72	1437	22.90
	4	569	10.61	581	10.90	613	11.48
	8	500	12.31	520	12.83	541	13.38



(a) s = 1

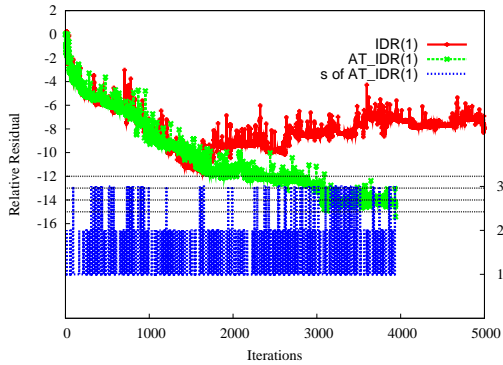


(b) s = 4

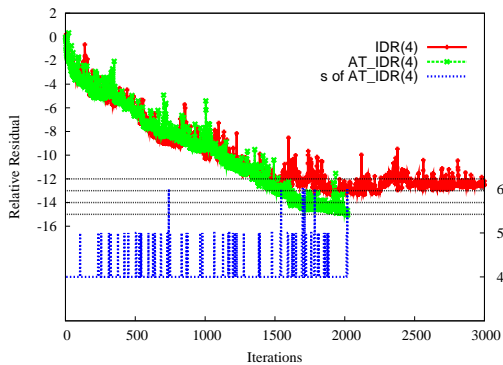


(c) s = 8

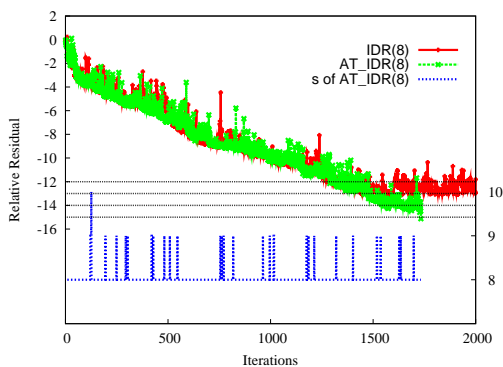
Fig. 2. Relative residual history of IDR(s) and AT\_IDR(s) methods, and variation of parameter s of AT\_IDR(s) method for matrix ChemMaster1.



(a)  $s = 1$



(b)  $s = 4$



(c)  $s = 8$

Fig. 3. Relative residual history of IDR( $s$ ) and AT\_IDR( $s$ ) methods, and variation of parameter  $s$  of AT\_IDR( $s$ ) method for matrix ViscoPlastic2.

TABLE VI  
CONVERGENCE OF IDR( $s$ ) AND AT\_IDR METHODS FOR MATRIX SME3DA.

method	$s$	$\epsilon = 10^{-12}$		$\epsilon = 10^{-13}$		$\epsilon = 10^{-14}$	
		itr.	time	itr.	time	itr.	time
IDR( $s$ )	1	1799	15.01	1976	16.19	2133	17.61
	2	891	7.77	max	-	max	-
	4	657	6.14	max	-	max	-
	8	max	-	max	-	max	-
AT_IDR	1	4127	31.37	4299	31.94	4299	31.94
	2	1020	8.37	1045	8.53	1087	8.74
	4	648	5.68	664	5.93	781	6.73
	8	523	5.03	529	5.02	542	5.18

TABLE VII  
CONVERGENCE OF IDR( $s$ ) AND AT\_IDR METHODS FOR MATRIX SME3DB.

method	$s$	$\epsilon = 10^{-12}$		$\epsilon = 10^{-13}$		$\epsilon = 10^{-14}$	
		itr.	time	itr.	time	itr.	time
IDR( $s$ )	1	2250	85.21	2887	109.09	2934	110.01
	2	1293	50.11	max	-	max	-
	4	max	-	max	-	max	-
	8	max	-	max	-	max	-
AT_IDR	1	5429	191.56	5468	192.52	5499	193.44
	2	1304	47.79	1361	49.88	1393	51.76
	4	886	33.94	894	34.22	916	35.60
	8	684	27.86	703	28.63	721	29.05

V. CONCLUSION

We proposed AT\_IDR( $s$ ) method for purpose of resolving stagnation of residual by tuning parameter  $s$  of IDR( $s$ ) method adaptively. We can conclude that AT\_IDR( $s$ ) method converges when IDR( $s$ ) method doesn't converge because of stagnation of relative residual norm. AT\_IDR( $s$ ) method is more robust than IDR( $s$ ) method.

As future work, we have two goals. The first goal is to find out optimum parameters  $\delta$  and *sentinel* to improve convergence rate of AT\_IDR( $s$ ) method. The second goal is to devise more effective adaptive tuning technique for IDR( $s$ ) method.

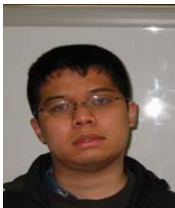
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TABLE VIII  
CONVERGENCE OF IDR( $s$ ) AND AT\_IDR METHODS FOR MATRIX VISCOPLASTIC1.

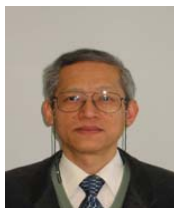
method	$s$	$\epsilon = 10^{-12}$		$\epsilon = 10^{-13}$		$\epsilon = 10^{-14}$	
		itr.	time	itr.	time	itr.	time
IDR( $s$ )	1	1226	0.66	1388	0.76	max	-
	2	1098	0.65	1192	0.67	max	-
	4	985	0.64	1059	0.64	1152	0.71
	8	936	0.67	974	0.74	1268	0.89
AT_IDR	1	1367	0.78	1491	0.85	1697	1.01
	2	1120	0.66	1219	0.72	1366	0.81
	4	967	0.63	1062	0.69	1130	0.72
	8	932	0.69	987	0.75	1078	0.81

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