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Periodic Solutions for a Delayed Population Model on Time Scales

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Abstract—This paper deals with a delayed single population model on time scales. With the assistance of coincidence degree theory, sufficient conditions for existence of periodic solutions are obtained. Furthermore, the better estimations for bounds of periodic solutions are established.

Keywords—Coincidence degree, continuation theorem, periodic solutions, time scales

I. Introduction

To study the control of a single population of cells, Nazarenko [1] presented the following nonlinear delay differential equation

$$x'(t) = -px(t) + \frac{qx(t)}{r + x^n(t-\tau)}, \qquad (1)$$

where all the coefficients are positive constants, n is a positive integer, $x(t)$ is the size of the population, p is the death rate and the feedback is given by a delayed function.

Taking account of environmental periodic variation, Song considered the nonautonomous differential equation in [2] as follows

$$x'(t) = -p(t)x(t) + \frac{q(t)x(t)}{r + x^n(t-\tau(t))}. \qquad (2)$$

However, the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have non–overlapped generations. Discrete time models can also provide efficient computational models of continuous models for numerical simulations. Thus, the discrete analogy of (2) was considered in [2–3]

$$x(k+1) = x(k)\exp\{-p(k) + \frac{q(k)}{r + x^n(k-\tau(k))}\}. \qquad (3)$$

For system (3), sufficient conditions for existence of periodic solutions were obtained by using different inequality techniques. It is not difficult to find that the methods and main results in [2] and [3] are greatly similar. The calculus on time scales, proposed by Stefan Hilger [4–5], obviously avoid the repetitiveness.

In this paper, we mainly consider the dynamic equation on time scales of the form

$$y^{\Delta}(t) = -p(t) + \frac{q(t)}{r + e^{ny(t-\tau(t))}}, \qquad (4)$$

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where $p(t)$ and $q(t)$ are positive $\omega-$periodic function on time scale $\mathbb{T}$. Set $x(t) = e^{y(t)}$, when the time scale $\mathbb{T}$ is $\mathbb{R}$, system (4) is equivalent to system (2); when $\mathbb{T}$ is $\mathbb{Z}$, system (4) can be reduced to system (3).

The aim of this paper is to establish the periodic solutions of (4) and the approach is based on the continuation theorem in coincidence degree theory, such as [6–7]. Furthermore, we can get the sharp bounds and improve the existence criteria for periodic solutions by the new inequality in [8].

The organization of this paper is as follows. In next section, the basic definitions and theorems are given. In Section 3, we establish our main results for periodic solutions by applying the continuation theorem.

II. Preliminaries

For convenience, we shall first recall some basic definitions and lemmas about time scales which are used in what follows; more details can be found in [5–6]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. Throughout this paper, we assume that the time scale $\mathbb{T}$ is unbounded above and below, such as $\mathbb{R}$, $\mathbb{Z}$ and $\bigcup_{k\in\mathbb{Z}}[2k, 2k+1]$. The following definitions and lemmas about time scales are from [6].

Definition 2.1. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, +\infty)$ are defined, respectively, by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \mu(t) = \sigma(t) - t$. If $\sigma(t) = t$, then t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered).

Definition 2.2. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

In this case, $f^{\Delta}(t)$ is called the delta (or Hilger) derivative of f at t. Moreover, f is said to be delta or Hilger differentiable on $\mathbb{T}$ if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}$. Then we define

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for } r, s \in \mathbb{T}.$$

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T})$.

Lemma 2.4. Every rd-continuous function has an antiderivative.

Lemma 2.5. If $a, b \in \mathbb{T}$, α, $\beta \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T})$,then
(a) $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$;
(b) if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t)\Delta t \geq 0$;
(c) if $|f(t)| \leq g(t)$ on $[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$, then $|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t$.

Lemma 2.6 ([8]). Let t_1, $t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \to \mathbb{R} \in C_{rd}(\mathbb{T})$ is $\omega-$periodic, then

$$g(t) \leq g(t_1) + \frac{1}{2}\int_k^{k+\omega} |g^\Delta(s)|\Delta s$$

and

$$g(t) \geq g(t_2) - \frac{1}{2}\int_k^{k+\omega} |g^\Delta(s)|\Delta s,$$

the constant factor $\frac{1}{2}$ is the best possible.

For simplicity, we use the following notations throughout this paper. Let $\mathbb{T}$ be ω-periodic, that is $t \in \mathbb{T}$ implies $t+\omega \in \mathbb{T}$,

$$k = \min\{\mathbb{R}^+ \cap \mathbb{T}\}, \quad I_\omega = [k, k+\omega] \cap \mathbb{T},$$
$$\bar{g} = \frac{1}{\omega}\int_{I_\omega} g(s)\Delta s = \frac{1}{\omega}\int_k^{k+\omega} g(s)\Delta s,$$

where $g \in C_{rd}(\mathbb{T})$ is an ω-periodic real function, i.e., $g(t+\omega) = g(t)$ for all $t \in \mathbb{T}$.

Now, we introduce some concepts and a useful result from [9].

Let X, Z be normed vector spaces, $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \ker L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \ker L$, $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$, then it follows that $L|\operatorname{Dom} L \cap \ker P : (I - P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

Next, we state the Mawhin's continuation theorem, which is a main tool in the proof of our theorem.

Lemma 2.7 (Continuation Theorem). Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Suppose

(a) for each $\lambda \in (0, 1)$, every solution u of $Lu = \lambda Nu$ is such that $u \notin \partial\Omega$;
(b) $QNu \neq 0$ for each $u \in \partial\Omega \cap \ker L$ and the Brouwer degree $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$.

Then the operator equation $Lu = Nu$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

III. Main results

Theorem 3.1. If $\bar{q} > r\bar{p}$, then system (4) has at least one $\omega-$periodic solution.

Proof Let $X = Z = \{y \in C(\mathbb{T}, \mathbb{R}) : y(t + \omega) = y(t), \forall t \in \mathbb{T}\}$, $\|y\| = \max_{t \in I_\omega} |y(t)|$, $y \in X($ or $Z)$.

Then X and Z are both Banach spaces when they are endowed with the above norm $\|\cdot\|$.

Let

$$Ny = -p(t) + \frac{q(t)}{r + e^{ny(t-\tau(t))}},$$
$$Ly = y^\Delta(t),$$

and

$$Py = Qy = \frac{1}{\omega}\int_\kappa^{\kappa+\omega} y(t)\Delta t.$$

Obviously, $\ker L = \mathbb{R}$, $\operatorname{Im} L = \{y \in Y : \bar{y} = 0, t \in \mathbb{T}\}$, $\dim \ker L = 1 = \operatorname{codim} \operatorname{Im} L$.

Since $\operatorname{Im} L$ is closed in Z, then L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projections such that $\operatorname{Im} P = \ker L$ and $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$. Furthermore, the generalized inverse (of L) $K_P : \operatorname{Im} L \to \ker P \cap \operatorname{Dom} L$ exists and is given by

$$K_P y = \int_\kappa^t y(s)\Delta s - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^t y(s)\Delta s\Delta t.$$

Thus

$$QNy = \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\left(-p(t) + \frac{q(t)}{r + e^{ny(t-\tau(t))}}\right)\Delta t,$$

and

$$\begin{aligned} K_P(I-Q)Ny \;&=\; \int_\kappa^t y(s)\Delta s - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}\int_\kappa^t y(s)\Delta s\Delta t \\ &\quad - \left(t - \kappa - \frac{1}{\omega}\int_\kappa^{\kappa+\omega}(t-\kappa)\Delta t\right)\bar{y}. \end{aligned}$$

Clearly, QN and $K_P(I - Q)N$ are continuous. Using the Arzela-Ascoli theorem, it is not difficulty to prove that $K_P(I - Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$. In addition, $QN(\bar{\Omega})$ is bounded. Therefore, N is L-compact on $\bar{\Omega}$ with any open bounded set $\Omega \subset X$.

Now, we shall search an appropriate open bounded subset Ω for the application of the continuation theorem, Lemma 2.7. For the operator equation $Lu = \lambda Nu$, where $\lambda \in (0, 1)$, we have

$$y^\Delta(t) = \lambda\left(-p(t) + \frac{q(t)}{r + e^{ny(t-\tau(t))}}\right). \tag{5}$$

Assume that $y(t) \in X$ is a solution of (5) for some $\lambda \in (0, 1)$. Integrating both sides of system (5) over I_ω, we obtain

$$\bar{p}\omega = \int_\kappa^{\kappa+\omega}\frac{q(t)}{r + e^{ny(t-\tau(t))}}\Delta t. \tag{6}$$

Since $y \in X$, there exist $\xi, \eta \in I_\omega$, such that

$$y(\xi) = \min_{t \in I_\omega}\{y(t)\}, \quad y(\eta) = \max_{t \in I_\omega}\{y(t)\}. \tag{7}$$

In view of (6) and (7), we have

$$\bar{p} \leq \frac{\bar{q}}{r + e^{ny(\xi)}}, \qquad \bar{p} \geq \frac{\bar{q}}{r + e^{ny(\eta)}},$$

that is

$$y(\xi) \leq \frac{1}{n}\ln\left(\frac{\bar{q}}{\bar{p}} - r\right),$$
$$y(\eta) \geq \frac{1}{n}\ln\left(\frac{\bar{q}}{\bar{p}} - r\right).$$

Hence,

$$y(t) \leq y(\xi) + \frac{1}{2}\int_{\kappa}^{\kappa+\omega} |y^{\Delta}(t)|\Delta t \leq \ln\left(\frac{\bar{q}}{\bar{p}} - r\right) + \bar{p}\omega := M,$$

$$y(t) \geq y(\eta) - \frac{1}{2}\int_{\kappa}^{\kappa+\omega} |y^{\Delta}(t)|\Delta t \geq \ln\left(\frac{\bar{q}}{\bar{p}} - r\right) - \bar{p}\omega := L.$$

Therefore, we can choose R_1 such that any solution of (5) satisfies

$$\max_{t\in I_\omega} |y(t)| \leq \max\{|M|, |L|\} := R_1.$$

Clearly, R_1 is independent of λ. Let $R = R_1 + R_0$, where R_0 is taken sufficiently large such that $R_0 \geq |\ln\left(\frac{\bar{q}}{\bar{p}} - r\right)|$. Now, we consider the algebraic equations

$$\bar{p} - \frac{\bar{q}}{r + e^{ny}} = 0, \tag{8}$$

every solution y^* of (8) satisfies $\|(y^*)\| < R$. Now, we define $\Omega = \{(y(t) \in X, \|y(t)\| < R\}$. Then it is clear that Ω verifies the requirement (a) of Lemma 2.7. If $y(t) \in \partial\Omega \cap \ker L = \partial\Omega \cap \mathbb{R}$, then $y(t)$ is a constant vector in $\mathbb{R}$ with $\|y(t)\| = |y| = R$, so we have $QNy \neq 0$.

By direct computation, we can obtain $\deg(JQN, \Omega \cap \ker L, 0) = -1 \neq 0$. Now, we have proved that Ω satisfies all conditions of Lemma 2.7. Thus, system (4) has at least one $\omega-$periodic solution in $\mathrm{Dom}\, L \cap \bar{\Omega}$. This completes the proof.

IV. Conclusion

This paper explores the existence of periodic solutions for a single population model with time delay on time scales. Our results indicate that the existence theorem for continuous system (2) can carry over quite easily to its discrete counterpart (3). Hence our results generalize the corresponding results of [2–3]. Since there are many other time scales that not just the set of real numbers $\mathbb{R}$ or the set of integers $\mathbb{Z}$, we can also obtain a much more general result of dynamic system (4) on time scales.

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