The Effect of Increment in Simulation Samples on a Combined Selection Procedure

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Abstract—Statistical selection procedures are used to select the best simulated system from a finite set of alternatives. In this paper, we present a procedure that can be used to select the best system when the number of alternatives is large. The proposed procedure consists a combination between Ranking and Selection, and Ordinal Optimization procedures. In order to improve the performance of Ordinal Optimization, Optimal Computing Budget Allocation technique is used to determine the best simulation lengths for all simulation systems and to reduce the total computation time. We also argue the effect of increment in simulation samples for the combined procedure. The results of numerical illustration show clearly the effect of increment in simulation samples on the proposed combination of selection procedure.

Keywords—Indifference-Zone, Optimal Computing Budget Allocation, Ordinal Optimization, Ranking and Selection, Subset Selection.

I. Introduction

WE consider optimizing the expected performance of a complex stochastic system that cannot be evaluated exactly, but has to be estimated using simulation. Our goal is to solve the following optimization problem

$$\min_{\theta \in \Theta} J(\theta) \qquad (1)$$

where Θ the feasible solution set and it is finite, huge and has no structure. J is the expected performance measure, L is a deterministic function depends on θ and ξ, so we can write $J(\theta) = E[L(\theta, \xi)]$, θ is a vector representing the system design parameters, and ξ represents all the random effect of the system. If we simulate the system to get estimate of $E[L(\theta, \xi)]$, then the confidence interval of this estimate cannot be improved faster than $1/\sqrt{k}$ where k is the number of samples used to get estimates of $J(\theta)$. This rate maybe good for a problems with small number of alternatives but it is not good enough for the class of complex simulation which we consider in this paper. Thus, one could compromise the objective to get a good enough solution rather than doing extensive simulation.

Ranking and Selection ($R\&S$) procedures, are used to select the best system or a subset that contains the best systems when the number of alternatives is small, see Kim and Nelson [1]. However, for a large scale problems, these procedure will need a huge computational time. In this case, we would relax our objective to finding good systems rather than estimating accurately the performance value for these systems, and this

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is the idea of Ordinal Optimization (OO) procedure, that proposed by Ho et al. [2].

In this paper, we study the effect of the increment in simulation samples, Δ on the performance of the combined selection procedure, that proposed by Almomani and Abdul Rahman [3]. We consider a combined selection procedure that used to selecting a good simulated system with high probability when the number of alternative system is huge. This procedure consists of four stages. In the first stage we use the OO procedure to select randomly a subset that overlaps with the set of the actual best $m\%$ systems with high probability from the feasible solution set Θ. In the second stage, we use Optimal Computing Budget Allocation ($OCBA$) technique to allocate the available computing budget in a way that maximizes the probability of correct selection. This follows by a Subset Selection (SS) procedure to get a smaller subset that contains the best system from the selected subset. In the final stage, we use the Indifference-Zone (IZ) procedure to select the best system among the survivors in the previous stage. This combined procedure is applied on $M/M/1$ queuing system with various values of increment in simulation samples, Δ to study their effect on the performance of the procedure.

Note that, the increment in simulation samples, Δ cannot be too small, to avoid repetition in the increment step in the $OCBA$ algorithm such that will increase the simulation time. On the other hand, if Δ is too large then this will imply a waste in the computation time and will end up with unnecessary high confidence level. Chen et al. [4] and Chen et al. [5] suggested a good choice for the increment in simulation samples, Δ should be between 5 and 10% of the simulated system.

This paper is organized as follows; In the next section, we present a background of OO, $OCBA$, SS, and IZ procedures. In Section 3, we present our combined selection procedure. Section 4, includes the numerical illustration. Finally, in Section 5, we give some concluding remarks.

II. Background

A. Ordinal Optimization (OO)

The OO procedure has emerged as an efficient technique for simulation and optimization. The aim of this procedure is to find good systems, rather than estimating the performance value of these systems accurately. This procedure has been proposed by Ho et al. [2].

Suppose that the Correct Selection (CS) is to select a subset G of g systems from the feasible solution set Θ that contains at least one of the top $m\%$ best systems. Since we

assume that Θ is very huge then the Probability of Correct Selection $P(CS)$ is given by $P(CS) \approx (1 - (1 - \frac{m}{100})^g)$. Now, suppose that the CS is to select a subset G of g systems that contains at least r of the best s systems. Let S be the subset that contains the actual best s systems, then the $P(CS)$ can be obtained using the hyper geometric distribution as, $P(CS) = P(|G \cap S| \geq r) = \sum_{i=r}^{g} \frac{\binom{s}{i}\binom{n-s}{g-i}}{\binom{n}{g}}$. However, since we assume that the number of alternatives is very large then the $P(CS)$ can be approximated by the binomial random variable. Therefore, $P(CS) \approx \sum_{i=r}^{g} \binom{g}{i}(\frac{m}{100})^i(1 - \frac{m}{100})^{g-i}$, where $s/n \times 100\% = m\%$. It is clear that this $P(CS)$ increases when the sample size g increase.

B. Optimal Computing Budget Allocation ($OCBA$)

The $OCBA$ was proposed to improve the performance of OO by determining the optimal numbers of simulation samples for each system, instead of equally simulating all systems. The goal of this procedure is to allocate the total simulation samples from all systems in a way that maximizes the probability of selecting the best system within a given computing budget. For more details of $OCBA$ see Chen et al. [4], Chen et al. [5], and Chen [6].

Let B be the total sample that available for solving the optimization problem given in (1). Our goal is to allocate these computing simulated samples to maximize the $P(CS)$. In mathematical notation

$$\max_{T_1,\ldots,T_n} P(CS)$$

$$s.t. \quad \sum_{i=1}^{n} T_i = B$$

$$T_i \in \mathbf{N} \qquad i = 1, 2, \ldots, n$$

where $\mathbf{N}$ is the set of non-negative integers, T_i is the number of samples allocated to system i and $\sum_{i=1}^{n} T_i$ denotes the total computational samples and assuming that the simulation times for different systems are roughly the same. To solve this problem Chen et al. [4] proposed the following theorem.

Theorem 1: Given a total number of simulated samples B to be allocated to n competing systems whose performance is depicted by random variables with means $J(\theta_1), J(\theta_2), \ldots, J(\theta_n)$, and finite variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$ respectively, as $B \longrightarrow \infty$, the approximate probability of CS can be asymptotically maximized when

1) $\frac{T_i}{T_j} = \left(\frac{\sigma_i/\delta_{b,i}}{\sigma_j/\delta_{b,j}} \right)^2$; where $i, j \in \{1, 2, \ldots, n\}$ and $i \neq j \neq b$.
2) $T_b = \sigma_b \sqrt{\sum_{i=1, i \neq b}^{n} \frac{T_i^2}{\sigma_i^2}}$

where $\delta_{b,i}$ the estimated difference between the performance of the two systems ($\delta_{b,i} = \overline{J_b} - \overline{J_i}$), and $\overline{J_b} \leq \min_i \overline{J_i}$ for all i. Here $\overline{J_i} = \frac{1}{T_i} \sum_{j=1}^{T_i} \xi_{ij}$, where ξ_{ij} is a sample from ξ_i for $j = 1, \ldots, T_i$.

Proof: See Chen et al. [4]. ∎

C. Subset Selection (SS)

SS procedure screens out the search space and eliminate non-competitive systems and construct a subset that contains the best system with high probability. This procedure is suitable when the number of alternatives is relatively large, and it is used to select a random size subset that contains the actual best system. It is required that $P(\text{CS}) \geq P^*$, where the Correct Selection (CS) is selecting a subset that contains the actual best system, and P^* is a predetermined probability.

The SS procedure dating back to Gupta [7], who presented a single stage procedure for producing a subset containing the best system with a specified probability. Extensions of this work which is relevant to the simulation setting include Sullivan and Wilson [8] who derived a two stage SS procedure that determines a subset of maximum size m that, with a specified probability will contain systems that are all within a pre-specified amount of the optimum.

D. Indifference-Zone (IZ)

The goal of IZ procedure is selecting the best system among n systems when the number of alternatives less than or equal 20. Suppose we have n alternative systems that are normally distributed with unknown means $\mu_1, \mu_2, \ldots, \mu_n$, and suppose that these means are ordered as $\mu_{[1]} \leq \mu_{[2]} \leq \ldots \leq \mu_{[n]}$. We want to select the system that has the best minimum mean $\mu_{[1]}$. The IZ is defined to be the interval $[\mu_{[1]}, \mu_{[1]} + \delta^*]$, where δ^* is a predetermined small positive real number. We are interested in selecting an alternative i^* such that $\mu_{i^*} \in [\mu_{[1]}, \mu_{[1]} + \delta^*]$. Let CS here is selecting an alternative whose mean belongs to the indifference zone. We prefer the CS to take place with high probability, say with a probability not smaller than P^* where $1/n \leq P^* \leq 1$.

The IZ procedure consists of two stages. In the first stage, all systems are sampled using t_0 simulation runs to get an initial estimate of the expected performance measure and their variances. Next, depending on the information obtained in the first stage, how many more samples are needed in the second stage for each system in order to guarantee that $P(\text{CS}) \geq P^*$ is computed. Rinott [9] has presented a procedure that is applicable when the data are normally distributed and all systems are simulated independently of each others. This procedure consists of two stages for the case when variances are completely unknown. On the other hand, Tamhane and Bechhofer [10] has presented a simple procedure that is valid when variances may not be equal.

To achieve the CS with high probability, $R\&S$ procedures need a huge computational time, so it is not practical when n is large. Therefore the combined procedures were proposed to reduce the competent system. Nelson et al. [11] proposed a two-stage subset selection procedure. The first stage is to reduce the number of competitive systems. These systems are carried out to the second stage that involved with the IZ procedure using the information gathered from the first stage. Alrefaei and Almomani [12] proposed two sequential algorithms for selecting a subset of k systems that is contained in the set of the top s systems. Another comprehensive review

of $R\&S$ procedures can be found in Bechhofer et al. [13], Goldsman and Nelson [14], and Kim and Nelson [15].

III. THE COMBINED SELECTION PROCEDURE

Our combined procedure consists four procedures, OO, $OCBA$, SS, and IZ. Initially, using OO procedure, a subset G is randomly selected from a feasible solution set that overlaps with the set that contains the actual best $m\%$ systems with probability $(1-\alpha_1)$. Then using the $OCBA$ procedure to allocate the available computing budget. This is follows with SS procedure to get a smaller subset I that contains the best system among the previous selected subset with probability $(1 - \alpha_2)$. Finally, we apply IZ procedure to select the best system from set I with probability $(1 - \alpha_3)$.

Algorithm:-

Setup: Specify the values of g and k, where $|G| = g$ and $|G'| = k$. Also consider that the number of initial simulation samples $t_0 \geq 2$, the indifference zone δ^*, and $t = t_{(1-\alpha_2/2)\frac{1}{g-1},t_0-1}$ from the t-distribution. Let $T_1^l = T_2^l = \ldots = T_g^l = t_0$, and determine the total computing budget B. Note that, G is the selected subset from Θ, that satisfies $P(G$ contains at least one of the best $m\%$ systems$) \geq 1 - \alpha_1$, and G' is the selected subset from G, where $g \geq k$. Also note that, l represents the iteration number.
Select a subset G of size g randomly from Θ and for each system i in G, where $i = 1, \ldots, g$ take a random samples of t_0 observations y_{ij} $(j = 1, \ldots, t_0)$.

Initialization: Calculate the sample mean $\bar{y}_i^{(1)}$ and variances and s_i^2, where $\bar{y}_i^{(1)} = \frac{\sum_{j=1}^{T_i^l} y_{ij}}{T_i^l}$ and $s_i^2 = \frac{\sum_{j=1}^{T_i^l} (y_{ij} - \bar{y}_i^{(1)})^2}{T_i^l - 1}$, for all $i = 1, \ldots, g$.
Order the systems in G according to their sample averages; $\bar{y}_{[1]}^{(1)} \leq \bar{y}_{[2]}^{(1)} \leq \ldots \leq \bar{y}_{[g]}^{(1)}$. Then select the best k systems from the set G, and represent this subset as G'.

Stopping Rule: If $\sum_{i=1}^{g} T_i^l \geq B$, then stop. Otherwise, Randomly select a subset G'' of the $g - k$ alternatives from $\Theta - G'$, let $(G = G' \bigcup G'')$.

Simulation Budget Allocation: Increase the computing budget by Δ and compute the new budget allocation, $T_1^{l+1}, T_2^{l+1}, \ldots, T_g^{l+1}$, by using Theorem 1.
Perform additional $\max\{0, T_i^{l+1} - T_i^l\}$ simulations for each system i, $i = 1, \ldots, g$, let $l \longleftarrow l + 1$. Go to **Initialization**.

Screening: Set $I = \{i : 1 \leq i \leq k$ and $\bar{y}_i^{(1)} \geq \bar{y}_j^{(1)} - [W_{ij} - \delta^*]^-, \forall i \neq j\}$, where $W_{ij} = t \left(\frac{s_i^2}{T_i} + \frac{s_j^2}{T_j} \right)^{1/2}$ for all $i \neq j$, and $[x]^- = x$ if $x < 0$ and $[x]^- = 0$ otherwise.
If I contains a single index, then this system is the best system. Otherwise, for all $i \in I$, compute the second sample size $N_i = \max\{T_i, \lceil (\frac{hs_i}{\delta^*})^2 \rceil\}$, where $h = h(1 - \alpha_3/2, t_0, |I|)$ be the Rinott [9] constant and can be obtained from tables of Wilcox [16].

Take additional $N_i - T_i$ random samples of y_{ij} for each system $i \in I$, and compute the overall sample means for $i \in I$ as $\bar{y}_i^{(2)} = \frac{\sum_{j=1}^{N_i} y_{ij}}{N_i}$
Select system $i \in I$ with the smallest $\bar{y}_i^{(2)}$ as the best.

Remarks:-

- Increment in simulation samples, Δ, is a positive integer representing the additional number of simulation samples that is used in the **Simulation Budget Allocation** step. Note that, if Δ is too small, then we need to repeat this step many times.

- Nelson et al. [11] have shown that with probability at least $1 - (\alpha_2 + \alpha_3)$ our combined selection procedure selects the best system in the subset G. Therefore, if G contains at least one of the top $m\%$ systems, then our procedure selects a good system with probability $1 - (\alpha_2 + \alpha_3)$. On the other hand, from the OO procedure we can show that the selected set G, contains at least one of the best $m\%$ systems with probability $(1 - \alpha_1) = 1 - \left(1 - \frac{m}{100}\right)^g$. Therefore, $P($the selected system in the combined selection procedure is in the top $m\%$ systems$) \geq \left(1 - \left(1 - \frac{m}{100}\right)^g\right)(1 - (\alpha_2 + \alpha_3)) \geq 1 - \left(\left(1 - \frac{m}{100}\right)^g + \alpha_2 + \alpha_3\right)$.

IV. NUMERICAL EXAMPLE

In our example, we consider the queuing systems when the inter arrival times and the service times are exponentially distributed and the system has one server. This queuing system is known as **M/M/1** queuing system. Our goal is selecting one of the best $m\%$ systems that has the minimum average waiting time per customer from n $M/M/1$ queuing systems.

As a measure of selection quality for our combined selection procedure, we use the Probability of Correct Selection $(P(CS))$, and the Expected Opportunity Cost $(E(OC))$ of a potentially incorrect selection, where the Opportunity Cost (OC) is the difference of unknown mean between the selected best system and the actual best system. More details of $E(OC)$ in He et al. [17], and Chick and Wu [18]. In this example we take the $E(OC)$ for our procedure as the absolute value of the difference of unknown mean between the selected best system and the actual best system. We estimate the $P(CS)$ for our combined selection procedure by counting the number of times we successfully find the best system that belongs to the actual $m\%$ best subset out of 100 independent replication.

To study the effect of the increment in simulation samples, Δ, we consider five different values; $\Delta = 10, 20, 50, 80, 100$. We apply our procedure in two experiments of $M/M/1$ queuing systems under some assumptions. In the first experiment, we assume the arrival rate λ is fixed and the service rate μ belong to the interval $[a, b]$. In particular, take $\lambda = 1$ and $\mu \in [4, 5]$. Suppose we have 1000 of $M/M/1$ queuing systems, and we discretize the problem by assuming that $\Theta = \{4.001, 4.002, \ldots 5.000\}$. Therefore, the best queuing system would be the 1000th queuing system with $\mu_{1000} = 5.0$. Let $n = 1000$, $g = 50$, $\alpha_2 = \alpha_3 = 0.005$, $\delta^* = 0.05$, $k = 10$, $B = 500$ and $t_0 = 10$. Suppose we want to select one of

the best (5%) systems, then our target is the systems from 951 to 1000. The correct selection is defined as selecting the system that belongs to $\{\mu_{951}, \mu_{952}, \ldots, \mu_{1000}\}$. We can calculate the analytical Probability of the Correct Selection as $P(CS) \geq 1 - \left(\left(1 - \frac{5}{100}\right)^{50} + 0.005 + 0.005\right) \geq 0.91$.

Table I contains the results of this experiment, with 100 replications for selecting one of the best (5%) systems. From the table, $\overline{\sum_{i=1}^{g} T_i}$ and $\overline{\sum_{i \in I} N_i}$ for each are referring to the average number of the total sample size in **Stopping Rule** and **Screening** respectively in our algorithm, and $\overline{E(OC)}$ denotes the average number of Expected Opportunity Cost.

In the second experiment, we applying the same parameters from the first experiment above, but we change the initial sample size t_0 from 10 to 50, and we assume that the total budget $B = 2500$. The results are shown in Table II for 100 replications.

From Table I and Table II, we note the changing in $\overline{\sum_{i=1}^{g} T_i}$ as we change the value of increment in simulation samples, Δ. Clearly the value of $\overline{\sum_{i=1}^{g} T_i}$ increases when Δ increase, and this is expected since T_i are calculated in our algorithm after we increase the computing budget by Δ. However, the amount of the increase in $\overline{\sum_{i=1}^{g} T_i}$ is relatively small when we increase the value of Δ. Furthermore, we can see that the values of $\overline{\sum_{i=1}^{g} T_i}$ in Table I is less than the value of $\overline{\sum_{i=1}^{g} T_i}$ in Table II. This is due to the changing of the initial sample size t_0 from 10 to 50. Also note that, the value of $\overline{\sum_{i \in I} N_i}$ approximately the same in all cases of Δ in both experiments.

Moreover, from the Table I and Table II we can see that the $P(CS)$ for our combined selection procedure is very closed to the analytical $P(CS)$. The highest $P(CS)$ occurs in Table I when $\Delta = 50$ with the $P(CS) = 90\%$ comparing with the other values of Δ. Also, the highest value of $P(CS)$ occurs in Table II when $\Delta = 50$ with the $P(CS) = 85\%$ comparing with the other values of Δ.

We also notice that the $\overline{E(OC)}$ for our procedure in Table I are high and are not so close with the analytical $\overline{E(OC)}$. The reason for this is that with a small t_0 will end up with a poor estimate for the initial mean and variance. It means that we get a bad estimator for the means for the actual best system and the best systems that are choose by our procedure, but we note here the $\overline{E(OC)}$ is not affected by Δ. Whereas in Table II we note that the value of $\overline{E(OC)}$ for our procedure is very small and closed to the analytical $\overline{E(OC)}$, which also indicate that the $\overline{E(OC)}$ is not affected by Δ.

V. CONCLUSION

This paper discuss the effect of increment in simulation samples, Δ on the performance of a combined selection procedure that is used to selecting a good simulated system when the number of alternatives is large. The procedure consists four stages. Initially, using OO procedure, a subset G is randomly selected form a feasible solution set that overlap with the set that contains the actual best $m\%$ systems with high probability. Then $OCBA$ procedure is used to allocate the available computing budget. This is follows with SS procedure to get a smaller subset I with high probability, that contains the best system among the previous selected subset,

where $|I| \leq 20$. Finally, IZ procedure is applied to select the best system from that set I. We apply this procedure in two experiments of $M/M/1$ queuing system under some parameters setting, with five different values for Δ, to study the effect of Δ on our combined selection procedure. From the numerical results we note that the increase in $\overline{\sum_{i=1}^{g} T_i}$ with the increment simulation in samples, Δ. Also, we note that, there is no effect from Δ on the $\overline{\sum_{i \in I} N_i}$ and $P(CS)$. Moreover, there is also no substantial effect from the Δ on the $\overline{E(OC)}$. However, we should be careful when we choose the value of increment in simulation samples, Δ, since we need to increase the computing budget by Δ if Δ too small and also, by choosing a large value of Δ will lead to a waste in the computation time.

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TABLE I
THE NUMERICAL ILLUSTRATION FOR $n = 1000, g = 50, m\% = 5\%, k = 10, t_0 = 10, B = 500$

Δ	10	20	50	80	100
$\overline{\sum_{i=1}^{g} T_i}$	1423	1879	3338	4815	5856
$\overline{\sum_{i \in I} N_i}$	3732	3919	4266	4694	4686
Procedure $P(CS)$	79%	84%	90%	83%	88%
Analytical $P(CS)$	91%	91%	91%	91%	91%
Procedure $\overline{E(OC)}$	0.014393922	0.012408734	0.013044338	0.013937230	0.015563045
Analytical $\overline{E(OC)}$	0.009587749	0.004870680	0.006655096	0.008899322	0.008017380

TABLE II
THE NUMERICAL ILLUSTRATION FOR $n = 1000, g = 50, m\% = 5\%, k = 10, t_0 = 50, B = 2500$

Δ	10	20	50	80	100
$\overline{\sum_{i=1}^{g} T_i}$	5188	5599	7050	8428	9380
$\overline{\sum_{i \in I} N_i}$	2660	2552	2568	2776	2603
Procedure $P(CS)$	83%	76%	85%	73%	81%
Analytical $P(CS)$	91%	91%	91%	91%	91%
Procedure $\overline{E(OC)}$	0.006947018	0.006561811	0.006334258	0.005562929	0.007340014
Analytical $\overline{E(OC)}$	0.009958084	0.009390408	0.009271699	0.009432285	0.009389010