International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:3, 2010

Two-dimensional Differential Transform Method for Solving Linear and Non-linear Goursat Problem

H. Taghvafard and G. H. Erjaee

Abstract—A method for solving linear and non-linear Goursat problem is given by using the two-dimensional differential transform method. The approximate solution of this problem is calculated in the form of a series with easily computable terms and also the exact solutions can be achieved by the known forms of the series solutions. The method can easily be applied to many linear and non-linear problems and is capable of reducing the size of computational work. Several examples are given to demonstrate the reliability and the performance of the presented method.

Keywords—Quadrature, Spline interpolation, Trapezoidal rule, Numerical integration, Error analysis.

I. Introduction

ONE-dimensional Differential Transform Method (DTM) was first introduced by Zhou [1] for solving linear and non-linear initial value problems in electrical circuit analysis. It has been also used in obtaining series solutions to a wide class of linear and non-linear ordinary differential equations [2-10]. Based on the same methodology, Chen and Ho [11] recently developed the two-dimensional DTM for solving the differential and integral equations. For example, in [12] this method is used for solving a system of differential equations and in [13] for differential-algebraic equations and in [14], [15] is applied to partial differential equations.

The subject of the presented paper is to apply the two–dimensional DTM for solving linear and non-linear Goursat problem which arises in physical phenomena and applied sciences. For this purpose, we consider the standard form of the Goursat problem [16-19]

$$\begin{aligned} &u_{xt} = f(x,t,u,u_x,u_t),\ 0 \le x \le a,\ 0 \le t \le b,\\ &u(x,0) = g(x), \quad u(0,t) = h(t),\\ &g(0) = h(0) = u(0,0). \end{aligned}$$

This equation has been examined by several numerical methods such as Runge-Kutta method, finite difference method, finite elements method, Adomian Decomposition Method (ADM), Variational Iteration Method (VIM) and geometric mean averaging of the functional values of $f(x,t,u,u_x,u_t)$. See for example [16-21] and references therein.

In this paper, we present the applicability and effectiveness of DTM on linear and non-linear Goursat problem. The main advantage of DTM is that it can be applied directly to problems without requiring linearization, discretization or perturbation. Another important advantage is that this method is capable of greatly reducing the size of computational work while accurately providing the series solution with fast convergence rate.

G. H. Erjaee is with the Mathematics Department, Shiraz University, Shiraz, Iran. He is also a research fellow of Mathematics Department of Qatar University, Doha, Qatar.

II. Two-dimensional differential transform

The basic definitions and fundamental operations of the two-dimensional differential transform are defined in [12] as follows:

Consider an analytical function $w(x,t)$ of two variables in domain Ω. Then this function can be represented as a series in $(x_0,t_0) \in \Omega$, using the differential transform

$$W(m,n) = \frac{1}{m!n!}\left[\frac{\partial^{m+n} w(x,t)}{\partial x^m \partial t^n}\right]_{x=x_0,t=t_0}, \qquad (1)$$

by

$$w(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!n!}\left[\frac{\partial^{m+n} w(x,t)}{\partial x^m \partial t^n}\right]_{x=0,t=0} x^m t^n, \quad (2)$$

where $w(x,t)$ is called the inverse transform of $W(m,n)$.

In the following theorem, we summarize fundamental properties of the two-dimensional differential transform (see [12], [14], [22]).

Theorem 1: Let $U(m,n), V(m,n)$ and $W(m,n)$ be the two-dimensional differential transforms of the functions $u(x,t), v(x,t)$ and $w(x,t)$ in $(0,0)$, respectively. Then

(a) If $u(x,t) = v(x,t) \pm w(x,t)$, then

$$U(m,n) = V(m,n) \pm W(m,n).$$

(b) If $u(x,t) = av(x,t)$, then

$$U(m,n) = aV(m,n).$$

(c) If $u(x,t) = v(x,t)w(x,t)$, then

$$U(m,n) = \sum_{k=0}^{m}\sum_{l=0}^{n} V(k,n-l)W(m-k,l)$$

(d) If $u(x,t) = \frac{\partial^{r+s} v(x,t)}{\partial x^r \partial t^s}$, then

$$U(m,n) = \frac{(m+r)!}{m!}\frac{(n+s)!}{n!} V(m+r,n+s).$$

(e) If $u(x,t) = e^{av(x,t)}$, then
$U(m,n) =$

$$\begin{cases} e^{aV(0,0)}, m = n = 0,\\ a\sum_{k=0}^{m-1}\sum_{l=0}^{n} \frac{m-k}{m} V(m-k,l)U(k,n-l), m \ge 1,\\ a\sum_{k=0}^{m}\sum_{l=0}^{n-1} \frac{n-l}{n} V(k,n-l)U(m-k,l), n \ge 1. \end{cases}$$

(f) If $u(x,t) = x^k t^h$, then

$$U(m,n) = \delta(m-k, n-h) \tag{3}$$
$$= \begin{cases} 1 & \text{if } m=k, n=h, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

(g) If $u(x,t) = x^k e^{at}$, then

$$U(m,n) = \delta(m-k)\frac{a^n}{n!}.$$

All the results in the parts (f) and (g) have been obtained by setting $x = 0$ and $t = 0$ in (1).

III. Applications and Results

In this section, to see the accuracy of DTM, we apply the method to some linear and non-linear Goursat problems.

A. The linear homogeneous Goursat problem

We first consider the linear homogeneous Goursat problem

$$\begin{aligned} &u_{xt} = L(u), \\ &u(x,0) = g(x), \quad u(0,t) = h(t), \\ &g(0) = h(0) = u(0,0), \end{aligned}$$

where $L(u)$ is a linear function of u.

Example 1: Consider the homogeneous Goursat problem

$$u_{xt} = u, \tag{5}$$
$$u(x,0) = e^x, \quad u(0,t) = e^t, \quad u(0,0) = 1. \tag{6}$$

Taking differential transform of (5), (6) and using Theorem 1, we obtain

$$(m+1)(n+1)U(m+1,n+1) = U(m,n), \tag{7}$$
$$U(m,0) = \frac{1}{m!}, \quad U(0,n) = \frac{1}{n!}, \quad U(0,0) = 1. \tag{8}$$

Substituting (8) into (7) and using the recurrence relation, the results are listed as follows.

$$\begin{aligned} &U(0,0) = 1, \quad U(0,1) = \frac{1}{1!}, \quad U(0,2) = \frac{1}{2!}, \\ &U(0,3) = \frac{1}{3!}, \quad U(1,0) = 1, \quad U(1,1) = 1, \\ &U(1,2) = \frac{1}{2!}, \quad U(1,3) = \frac{1}{3!}, \quad U(2,0) = \frac{1}{2!}, \\ &U(2,1) = \frac{1}{2!}, \quad U(2,2) = \frac{1}{2!2!}, \quad U(2,3) = \frac{1}{2!3!}, \end{aligned}$$

and so on. In general, we have $U(m,n) = \frac{1}{m!n!}$. Substituting all $U(m,n)$ into (2) yiels the solution $u(x,t) = e^{x+t}$. This result is in full agreement with the one obtained in [21] by VIM and in [20] by using ADM [23].

Example 2: Now consider the homogeneous Goursat problem

$$u_{xt} = -2u, \tag{9}$$
$$u(x,0) = e^x, \quad u(0,t) = e^{-2t}, \quad u(0,0) = 1. \tag{10}$$

Applying differential transform of (9), (10) and using Theorem 1, we obtain

$$(m+1)(n+1)U(m+1,n+1) = -2U(m,n), \tag{11}$$
$$U(m,0) = \frac{1}{m!}, \quad U(0,n) = \frac{(-2)^n}{n!}, \quad U(0,0) = 1. \tag{12}$$

Substituting (12) into (11) and using the recurrence relation, we have

$$\begin{aligned} &U(0,0) = 1, \quad U(0,1) = \frac{-2}{1!}, \quad U(0,2) = \frac{(-2)^2}{2!}, \\ &U(0,3) = \frac{(-2)^3}{3!}, U(1,0) = 1, \quad U(1,1) = \frac{-2}{1!}, \\ &U(1,2) = \frac{(-2)^2}{2!}, \quad U(1,3) = \frac{(-2)^3}{3!}, \quad U(2,0) = \frac{1}{2!}, \\ &U(2,1) = \frac{-2}{2!}, \quad U(2,2) = \frac{(-2)^2}{2!2!}, \quad U(2,3) = \frac{(-2)^3}{2!3!}, \end{aligned}$$

and so on. In general, we have $U(m,n) = \frac{(-2)^n}{m!n!}$. Substituting all $U(m,n)$ into (2), the solution is $u(x,t) = e^{x-2t}$. This result is again in full agreement with the one obtained in [21] by VIM and in [20] by using ADM.

B. The linear inhomogeneous Goursat problem

We now consider the linear inhomogeneous Goursat problem

$$\begin{aligned} &u_{xt} = L(u) + w(x,t), \\ &u(x,0) = g(x), \quad u(0,t) = h(t), \\ &g(0) = h(0) = u(0,0), \end{aligned}$$

where $L(u)$ is a linear function of u.

Example 3: We first consider the linear inhomogeneous Goursat problem

$$u_{xt} = u - t, \tag{13}$$
$$u(x,0) = e^x, \quad u(0,t) = t + e^t, \quad u(0,0) = 1. \tag{14}$$

Taking differential transform of (13), (14) and using Theorem 1, we obtain

$$(m+1)(n+1)U(m+1,n+1) = U(m,n) - \delta(m, n-1), \tag{15}$$
$$U(m,0) = \frac{1}{m!}, \quad U(0,n) = \frac{1}{n!} + \delta(n-1), \quad U(0,0) = 1. \tag{16}$$

Substituting (16) into (15) and using the recurrence relation, we have

$$\begin{aligned} &U(0,0) = 1, \quad U(0,1) = \frac{1}{1!} + 1, \quad U(0,2) = \frac{1}{2!}, \\ &U(0,3) = \frac{1}{3!}, \quad U(1,0) = 1, \quad U(1,1) = 1, \\ &U(1,2) = \frac{1}{2!}, \quad U(1,3) = \frac{1}{3!}, \quad U(2,0) = \frac{1}{2!}, \\ &U(2,1) = \frac{1}{2!}, \quad U(2,2) = \frac{1}{2!2!}, \quad U(2,3) = \frac{1}{2!3!}, \end{aligned}$$

and so on. In general, we have

$$U(m,n) = \begin{cases} 2 & \text{if} \quad m=0, n=1, \\ \frac{1}{m!n!} & \text{otherwise.} \end{cases}$$

Substituting all $U(m,n)$ into (2) yields the solution $u(x,t) = t + e^{x+t}$. This result is in full agreement with the one obtained in [20] by using ADM and in [21] by using VIM.

Example 4: Consider the linear inhomogeneous Goursat problem

$$u_{xt} = u + 4xt - x^2t^2, \tag{17}$$
$$u(x,0) = e^x, \quad u(0,t) = e^t, \quad u(0,0) = 1. \tag{18}$$

Taking differential transform of (17), (18) and using Theorem 1, we obtain

$$\begin{aligned} &(m+1)(n+1)U(m+1,n+1) = \\ &U(m,n) + 4\delta(m-1,n-1) - \delta(m-2,n-2), \quad (19) \\ &U(m,0) = \frac{1}{m!}, \quad U(0,n) = \frac{1}{n!} + \delta(n-1), \quad (20) \\ &U(0,0) = 1. \quad (21) \end{aligned}$$

Substituting (20) and (21) into (19) and using the recurrence relation, we have

$$\begin{aligned} &U(0,0) = 1, \quad U(0,1) = \frac{1}{1!}, \quad U(0,2) = \frac{1}{2!}, \\ &U(0,3) = \frac{1}{3!}, \quad U(1,0) = 1, \quad U(1,1) = 1, \\ &U(1,2) = \frac{1}{2!}, \quad U(1,3) = \frac{1}{3!}, \quad U(2,0) = \frac{1}{2!}, \\ &U(2,1) = \frac{1}{2!}, \quad U(2,2) = \frac{5}{2!2!}, \quad U(2,3) = \frac{1}{2!3!}, \end{aligned}$$

and so on. In general, we have

$$U(m,n) = \begin{cases} \frac{5}{4} & \text{if} \quad m=2, n=2, \\ \frac{1}{m!n!} & \text{otherwise.} \end{cases}$$

Substituting all $U(m,n)$ into (2), the solution is $u(x,t) = x^2t^2 + e^{x+t}$. This result is in full agreement with the one obtained in [21] by VIM and in [20] by using ADM.

C. The non-linear Goursat problem

Here, we apply DTM to non-linear Goursat problem.

Example 5: We first consider the non-linear Goursat problem

$$u_{xt} = -u^3 + x^3 + 3x^2t + 3xt^2 + t^3 \tag{22}$$
$$u(x,0) = x, \quad u(0,t) = t, \quad u(0,0) = 0. \tag{23}$$

Taking differential transform of (22), (23) and using Theorem 1, we obtain

$$\begin{aligned} &(m+1)(n+1)U(m+1,n+1) \\ &= -\sum_{r=0}^{m}\sum_{l=0}^{m-r}\sum_{s=0}^{n}\sum_{p=0}^{n-s} U(r,n-s-p)U(l,s) \\ &U(m-r-l,p) + \delta(m-3,n) + 3\delta(m-2,n-1) \\ &+3\delta(m-1,n-2) + \delta(m,n-3), \end{aligned}$$

$$U(m,0) = \delta(m-1), \quad U(0,n) = \delta(n-1), \quad U(0,0) = 0. \tag{24}$$

Substituting (24) into (24) and using the recurrence relation, we obtain

$$U(m,n) = \begin{cases} 1 & \text{if} \quad m=1, n=0, \\ 1 & \text{if} \quad m=0, n=1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the solution is $u(x,t) = x + t$ which is in full agreement with the one obtained in [21] by VIM.

Example 6: We finally consider the non-linear Goursat problem

$$u_{xt} = e^{-2u} \tag{25}$$
$$u(x,0) = 0, \quad u(0,t) = 0, \quad u(0,0) = 0. \tag{26}$$

Applying differential transform of (25), (26) and using Theorem 1, we obtain

$$(m+1)(n+1)U(m+1,n+1) = V(m,n), \tag{27}$$
$$U(m,0) = 0, \quad U(0,n) = 0, \quad U(0,0) = 0, \tag{28}$$

where

$$V(m,n) =$$

$$\begin{cases} e^{-2U(0,0)}, \quad m=0 \text{ and } n=0, \\ -2\sum_{k=0}^{m-1}\sum_{l=0}^{n} \frac{m-k}{m} U(m-k,l)V(k,n-l), m \geq 1, \\ -2\sum_{k=0}^{m}\sum_{l=0}^{n-1} \frac{n-l}{n} U(k,n-l)V(m-k,l), n \geq 1 \end{cases} \tag{29}$$

Substituting (28) and (29) into (27) and using the recurrence relation, we obtain

$$\begin{aligned} &V(m,0) = \delta(m), && U(m,1) = \delta(m-1), \\ &V(m,1) = -2\delta(m-1), && U(m,2) = -\frac{1}{2}\delta(m-2), \\ &V(m,2) = 3\delta(m-2), && U(m,3) = \frac{1}{3}\delta(m-3), \end{aligned}$$

and so on. In general, one gets

$$\begin{aligned} V(m,n) &= (-1)^n(n+1)\delta(m-n), \\ U(m,n) &= \frac{(-1)^{n+1}}{n}\delta(m-n), \end{aligned} \tag{30}$$

and thus

$$U(m,n) = \begin{cases} \frac{(-1)^{n+1}}{m}, & \text{if} \quad m=n, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the solution is $u(x,t) = \ln(1+xt)$ which is the exact solution.

IV. Conclusions

We successfully applied the two–dimensional DTM to find the approximate series solutions of linear and non-linear Goursat problem. The presented method reduces the computational difficulties existing in the other traditional methods and all the calculations can be done by simple manipulations. Several examples of linear and non-linear Goursat problem were tested by applying DTM and the results have revealed remarkable performances.

ACKNOWLEDGMENT

This work has been partially supported by the Qatar National Priorities Research Program under the grant number NPRP 08-056-1-014.

REFERENCES

[1] Zhou JK. Differential transform and its applications for electrical circuits. Wuhan. China: Huazhong University Press; 1986.
[2] Chen CL, Lin SH, Chen, CK. Application of Taylor transformation to nonlinear predictive control problem. Appl Math Model 1996;20:699-710.
[3] Chiou JS, Tzeng JR. Application of the Taylor transform to non-linear vibration problems. ASME J Vib Acoust 1996;118:83-87.
[4] Jang MJ, Chen CL. Analysis of the response of a strongly non-linear damped system using a differential transformation technique. Appl Math Comput 1997;88:137-151.
[5] Chen CL, Liu YC. Differential transformation technique for steady nonlinear heat conduction problems. Appl Math Comput 1998;95:155-164.
[6] Yu LT, Chen CK. The solution of the Blasius equation by the differential transformation method. Math Comput Model 1998; 28:101-111.
[7] Yu LT, Chen CK. Application of Taylor transformation to optimize rectangular fins with variable thermal parameters. Appl Math Model 1998;22:11-21.
[8] Kuo BL. Application of the differential transformation method to the solutions of Falkner-Skan wedge flow. Acta Mech 2003;164:161-174.
[9] Kuo BL. Thermal boundary–layer problems in a semi–infinite plate by the differential transformation method. Appl Math Comput 2004;150:303-320.
[10] Chen CK, Chen SS. Application of the differential transformation method to a nonlinear conservative system. Appl Math Comput 2004;154:431-441.
[11] Chen CK, Ho SH. Solving partial differential equations by two-dimensional differential transform. Appl Math Comput 1999;106:171-179.
[12] Ayaz F. Solutions of the system of differential equations by differential transform method. Appl Math Comput 2004; 147:547-567.
[13] Ayaz F. Application of differential transform method to differential-algebraic equations. Appl Math Comput 2004;152:649-657.
[14] Chen CK. Solving partial differential equations by two-dimensional differential transform. Appl Math Comput 1999;106:171-179.
[15] Jang MJ, Chen CK, Liu YC. Two-dimensional differential transform for partial differential equations. Appl Math Comput 2001;121:261-270.
[16] Wazwaz AM. On the numerical solution of the Goursat problem. Appl Math Comput 1993;59:89-95.
[17] Wazwaz AM. The decomposition method for approximate solution to the Goursat problem. Appl Math Comput 1995;69:299-311.
[18] Day JT. A Runge-Kutta method for the numerical solution of the Goursat problem in hyperbolic partial dfferential equations. Comput J 1966;9:81-83.
[19] Evans DJ, Sanugi BB. Numerical solution of the Goursat problem by a nonlinear trapezoidal formula. Appl Math Lett 1988;1:221-223.
[20] Wazwaz AM. Partial Differential Equations: Methods and Applications, Balkema Publishers, The Netherlands, 2002.
[21] Wazwaz AM. The variational iteration method for a reliable treatment of the linear and the nonlinear Goursat problem. Appl Math Comput 2007;193:455-462.
[22] Chang SH, Chang IL. A new algorithm for calculating two–dimensional differential transform of nonlinear functions. Appl Math Comput 2009;215:2486-2494.
[23] Adomian G. Solving Frontier Problems of Physics: The Decomposition Method, Kluwer, Boston, 1994.