International Journal of Electrical, Electronic and Communication Sciences
ISSN: 2517-9438
Vol:6, No:2, 2012

Transient Analysis of a Single-Server Queue with Fixed-Size Batch Arrivals

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Abstract—The transient analysis of a queuing system with fixed-size batch Poisson arrivals and a single server with exponential service times is presented. The focus of the paper is on the use of the functions that arise in the analysis of the transient behaviour of the queuing system. These functions are shown to be a generalization of the modified Bessel functions of the first kind, with the batch size B as the generalizing parameter. Results for the case of single-packet arrivals are obtained first. The similarities between the two families of functions are then used to obtain results for the general case of batch arrival queue with a batch size larger than one.

Keywords—batch arrivals, generalized Bessel functions, queue transient analysis, time-varying probabilities.

I. Introduction

The transient analysis of a queuing system with fixed-size batch Poisson arrivals and a single server with exponential service times is presented. One can envision messages arriving at a buffer in fixed size batches of packets and released one packet at time [1]. Transient performance measures for queues have long been recognized as being complementary to the steady-state analysis [2-5], and justifications for transient analysis of traffic in telecommunication systems abound [6]. There often exists a need to understand the initial behaviour of a system. In general queuing systems, arrivals at a service point (e.g. a switch) may occur in batches of different sizes. Depending on the network conditions, the arrivals may be queued for later forwarding. Additionally, there may be cases where the network traffic is diverted suddenly to cope with faults, as in automatic protection switching in which the transport system re-directs traffic when faults and failures occur in subcomponents of the network. In such cases a service point may experience a sudden increase in its load, and this may continue until the original fault has been cleared. After the fault is corrected, traffic reverts to the previous distribution, and this presents another perturbation in the network. There exists now a reduction in the load in parts of the network. The restored element experiences an increased load transient starting from an empty state. This scenario presents the service points with transient conditions that require the kind of analysis attempted here and elsewhere [7].

Numerical inversion of Laplace transforms or generating functions have been used to obtain the transient behaviour of queuing systems [8,9]. Other methods that are equally time-consuming are based on recursive computations. What is presented here is a method that uses a family of functions arising in the analysis of the queuing system. It is shown that these functions are related to the modified Bessel functions of first kind. The relevant properties of the Bessel functions are available in [10] from which a select few are taken for the purposes of this paper. The relationship between the function families is exploited to obtain the results presented. The main key in this exercise is the fact that the empty system probability can be obtained explicitly in closed form for single-packet arrivals.

The analysis proceeds here by first considering single-packet Poisson arrivals, i.e. a batch size of one (B=1). Results for this case can also be found in [11]. In Appendix A the present paper includes additional steps to those of [11] and then proceeds in Appendix B to the general case (B > 1) by using the inherent correspondence between the modified Bessel functions of the first kind and the batch arrival functions. Incidentally this analysis provides a method of implicitly solving the integral equation that is encountered, which elsewhere [7] is solved by a combination of modelling and signal processing. There are other methods [12-14] of solution to such integral equations.

The paper is organized as follows. Section II presents the system model, and sets the main point of reference for the rest of the presentation. Section III presents a discussion of the time-varying probabilities that in the limit of large t eventually yield the steady state results. The section relies on the derivations given Appendix A and Appendix B concerning the relationships between the function families. Section IV presents the expressions for the queuing system statistics needed to assess performance. Section VI gives the results and conclusion. Appendix A gives a detailed derivation of the relevant results for single-packet arrivals (B=1), and Appendix B relies on the results of Appendix B and proceeds to the general case B>1). The appendices carry material that is necessary to support the development of the paper but which, if placed in the main body, would compromise readability.

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II. System Model

Packets arrive at a service point in fixed size batches of B packets according to a Poisson process of mean rate λ arrivals per second. The single server completes the service at the rate of μ packets per second. The probability flow rates are as shown in Fig.1 for a batch size of B packets.

Denote by $P_k(t)$ the probability that there are k packets in the system at time t. The probability flow balance equations are given by

$$\frac{dP_k(t)}{dt} + (\lambda+\mu)P_k(t) = \begin{cases} \mu P_1(t) + \mu P_0(t) & k = 0 \\ \mu P_{k+1}(t) & 1 \le k < B \\ \mu P_{k+1}(t) + \lambda P_{k-B}(t) & k \ge B \end{cases} \quad (1)$$

For $P_k(t)$ let P(z,t) be the moment generating function, P*(z,s) the Laplace-Stieltjes transform of P(z,t). Using these in (1) gives

$$\frac{\partial P(z,t)}{\partial t} + \left(\lambda + \mu - \lambda z^{B} - \mu z^{-1}\right)P(z,t) = \mu\left(1 - z^{-1}\right)P_0(t) \quad (2)$$

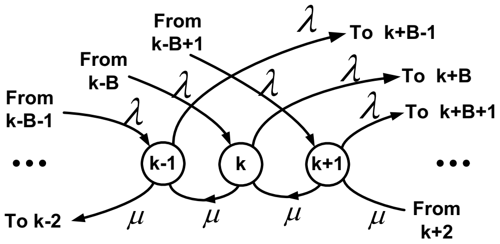

Fig. 1. Probability Flow For an Arrival Batch size B

The solution to (2) is easily seen to be

$$\begin{aligned} P(z,t) &= P(z,0)\exp[-(\lambda+\mu)t]\exp[(\lambda z^{B} + \mu z^{-1})t] \\ &+ \mu(1 - z^{-1})\int_0^t \exp[-(\lambda+\mu)(t-\tau)]\exp[(\lambda z^{B} + \mu z^{-1})(t-\tau)]P_0(\tau)d\tau \end{aligned} \quad (3)$$

with P*(z,s) given by

$$P^*(z,s) = \frac{z^{i+1} + \mu(z-1)P_0^*(s)}{(s+\lambda+\mu)z - \lambda z^{B+1} - \mu} \quad (4)$$

in which $P_k^*(s)$ is the Laplace-Stieltjes transform of $P_k(t)$ and it is assumed that there are i packets in the system initially, giving $P(z,0) = z^i$. The analysis proceeds here by first considering a batch size of one, i.e. single Poisson arrivals (B=1). By using the inherent correspondence between the functions that arise in the analysis and the modified Bessel functions of the first kind, the results for single-packet arrivals are generalized for a batch size B greater than 1. For B=1, the transient expression for empty probability starting with i packets is obtained in Appendix A. The results in Appendix B are used to obtain an equivalent expression for a batch size greater than one (B>1), as shown in Fig.2.

III. PROBABILITIES FOR THE NUMNER OF PACKETS IN THE SYSTEM

The moment generating function (3) contains the exponential function $\exp[(\lambda z^B+\mu z^{-1})t]$ which can be expressed as a two-sided power series

$$\exp\left[\left(\lambda z^{B} + \mu z^{-1}\right)t\right] = \sum_{k=-\infty}^{\infty} z^{k}\left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} V_k^{(B)}(\alpha t) \quad (5)$$

with the following definitions (for $k \ge 0$)

$$V_{-k}^{(B)}(x) = \sum_{l=0}^{\infty} \frac{x^{l(B+1)+k}}{l!(lB+k)!} \quad (6)$$

$$V_{k}^{(B)}(x) = \sum_{l=\sigma_k}^{\infty} \frac{x^{l(B+1)-k}}{l!(lB-k)!} \quad (7)$$

where $\alpha = \left(\mu^{B}\lambda\right)^{1/(B+1)} = \mu(\lambda/\mu)^{1/(B+1)}$ and $\sigma_k = \lceil k/B \rceil$, the smallest integer not less than k/B. Using (5), (6) and (7) in (3) gives

$$\begin{aligned} P_k(t) &= \left(\frac{\lambda}{\mu}\right)^{\frac{k-i}{B+1}} V_{k-i}^{(B)}(\alpha t)\exp[-(\lambda+\mu)t] \\ &+ \left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} \int_0^t \mu P_0(t-\tau)\exp[-(\lambda+\mu)\tau]\left(V_k^{(B)}(\alpha\tau) - \frac{\alpha}{\mu}V_{k+1}^{(B)}(\alpha\tau)\right)d\tau \end{aligned} \quad (8)$$

Since this expression appears to be cumbersome, it seems reasonable to introduce here the functions $h_k(t)$ and $q_k(t)$ defined by

$$h_k(t) = V_k^{(B)}(\alpha\tau) - \frac{\alpha}{\mu}V_{k+1}^{(B)}(\alpha\tau) \quad (9)$$

$$q_k(t) = \left(\frac{\lambda}{\mu}\right)^{\frac{-k}{B+1}} V_{-k}^{(B)}(\alpha\tau) \quad (10)$$

to cast (8) more compactly as

$$\begin{aligned} P_k(t) &= q_{i-k}(t)\exp[-(\lambda+\mu)t] \\ &+ \left(\frac{\alpha}{\mu}\right)^{k} \int_0^t \mu P_0(t-\tau)\exp[-(\lambda+\mu)\tau]h_k(\tau)d\tau \end{aligned} \quad (11)$$

By setting k=0 the above equation yields the following integral equation for $P_0(t)$.

$$\begin{aligned} P_0(t) &= q_i(t)\exp[-(\lambda+\mu)t] \\ &+ \int_0^t \mu P_0(t-\tau)\exp[-(\lambda+\mu)\tau]h_0(\tau)d\tau \end{aligned} \quad (12)$$

There are many methods of solution for such an integral equation. It is possible to use transforms (e.g. Laplace-Stieltjes transforms) to solve the integral equation. The required transforms would be expressed as power series in inverse powers of the Laplace variable s. Numerical techniques may be employed to invert the resulting transforms. Several authors have presented methods of solution that express the unknown function ($P_0(t)$ in this case) in terms of the derivatives of the known function [12,13]. Other methods exist for convolution kernels that have special forms, such as Bessel functions[14]. An indirect solution of the integral equation (12) is via modelling and signal processing is presented in [7].

Single-Packet Arrivals (B = 1)

$$P_0(t) = \exp[-(\lambda+\mu)t] \sum_{m=i+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^{m/2} \frac{m}{t\sqrt{\mu\lambda}} I_m\left(2t\sqrt{\mu\lambda}\right)$$

$\Downarrow$

Multiple-Packet Arrivals (B >1)

$$P_0(t) = \exp[-(\lambda+\mu)t] \sum_{m=i+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^{\frac{m}{B+1}} \frac{m}{\alpha t} V_{-m}^{(B)}(\alpha t)$$

Fig. 2. $P_0(t)$ Expression for B>1 inferred from relations between batch arrival functions and the modified Bessel functions.

The method used in this paper exploits the natural similarities between the functions $V_k^{(B)}(\cdot)$ and $I_k(\cdot)$, the modified Bessel functions of the first kind. Appendix A derives the expression for $P_0(t)$ for the single-packet arrivals (B=1) in readiness for generalizations given in Appendix B, which presents many of the relevant relations needed in the conclusion of this paper. Fig.2 gives the empty probability as inferred from the discussions in Appendix A and Appendix B.

It has been found necessary to use $V_{-m}^{(B)}(\alpha\, t)$ because it avoids the parameter σ_k and the results obtained are in agreement with those found using other methods.

$$P_0(t) = \exp[-(\lambda+\mu)t] \sum_{m=i+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^{\frac{m}{B+1}} \frac{m}{\alpha t} V_{-m}^{(B)}(\alpha\, t) \tag{13}$$

Equation (B9) can be re-written as

$$\frac{m}{\alpha\, t} V_{-m}^{(B)}(\alpha\, t) = V_{-(m-1)}^{(B)}(\alpha\, t) - B\, V_{-(m+B)}^{(B)}(\alpha\, t) \tag{14}$$

which enables (10) to be expressed as

$$P_0(t) = \exp[-(\lambda+\mu)]t \sum_{m=i+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^{\frac{m}{B+1}} \left(V_{-(m-1)}^{(B)}(\alpha\, t) - B\, V_{-(m+B)}^{(B)}(\alpha\, t)\right) \tag{15}$$

The known expression for $P_0(t)$ can be substituted in (8) to obtain the rest of the probabilities $P_k(t)$.

IV. QUEUING SYSTEM STATISTICS

The statistics such as the average number in the system and the variance of this number can be obtained by taking appropriate derivatives of (4) with respect to z, setting z=1, and inverting the indicated Laplace-Stieltjes transforms.

$$\left.\frac{\partial P^*(z,s)}{\partial z}\right|_{z=1} = \frac{i}{s} + \frac{\mu}{s} P_0^*(s) - (1-\rho)\frac{\mu}{s^2} \tag{16}$$

$$\left.\frac{\partial^2 P^*(z,s)}{\partial z^2}\right|_{z=1} = \frac{i^2 - i}{s} - 2\frac{\mu}{s} P_0^*(s) - 2(1-\rho)\frac{\mu}{s^2} P_0^*(s) + \big((B+1)\rho - 2(1-\rho)(i-1)\big)\frac{\mu}{s^2} + 2(1-\rho)^2 \frac{\mu^2}{s^3} \tag{17}$$

with $\rho = B\lambda/\mu$ as the offered load. These equations can be inverted to obtain the corresponding expression in the time domain.

$$\left.\frac{\partial P(z,t)}{\partial z}\right|_{z=1} = i + \int_0^t \mu P_0(\tau)d\tau - (1-\rho)(\mu\, t) \tag{18}$$

$$\left.\frac{\partial^2 P^*(z,s)}{\partial z^2}\right|_{z=1} = i^2 - i - 2\int_0^t \mu P_0(\tau)d\tau - 2(1-\rho)\int_0^t \mu P_0(\tau)(t-\tau)d\tau + \big((B+1)\rho - 2(1-\rho)(i-1)\big)\mu t + (1-\rho)^2(\mu t)^2 \tag{19}$$

From these, the mean number $\overline{N}(t)$ of packets in the system and its variance $\sigma_N^2(t)$ are obtained as functions of time.

$$\overline{N}(t) = \left.\frac{\partial P(z,t)}{\partial z}\right|_{z=1} = i + \int_0^t \mu P_0(\tau)d\tau - (1-\rho)(\mu\, t) \tag{20}$$

$$\sigma_N^2(t) = \left.\frac{\partial^2 P(z,t)}{\partial z^2}\right|_{z=1} - \left(\overline{N}(t)\right)^2 + \overline{N}(t) \tag{21}$$

The statistics obtained here are useful during the transient stage and also after steady state has been achieved. They address the buffer requirements, and give an indication of the possible waiting time before exiting the system. The more "customers" there in the system the longer will be the waiting time.

V. RESULTS AND CONCLUSION

Fig.3 gives the empty probability for the system starting from two conditions. The curves beginning at the top left of the figure are for the empty initial state (i=0), while those beginning from the bottom are obtained when the system starts from a non-empty state (i=5). Clearly the two sets of curves merge as they approach and eventually reach their steady state levels. Furthermore, the results agree with those obtained by other methods [7].

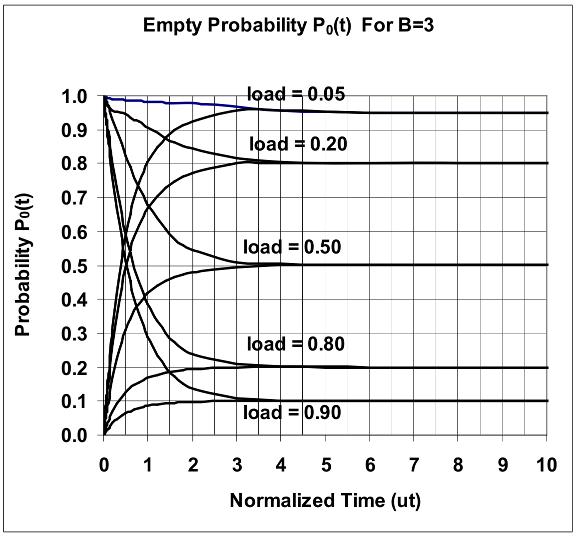

Fig. 3 The Empty Probability. One Set of curves begins from 1 (System Initially Empty, i=0), and the other set begins from zero (System Initially non-Empty, i=5),

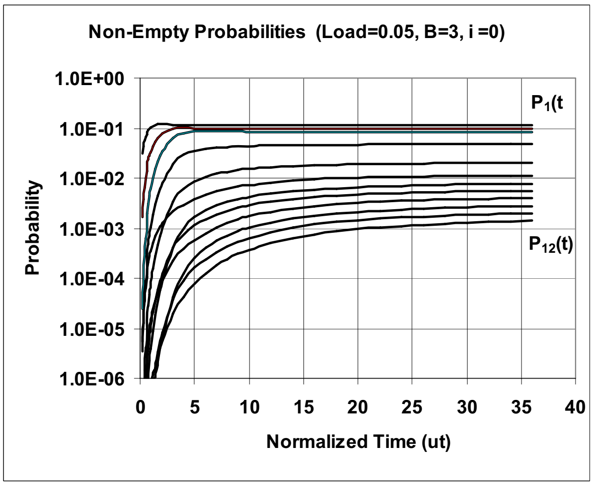

Fig. 4 Non-Empty Probabilities. $P_1(t)$ is on top and $P_{12}(t)$ is at the bottom. The values of $P_k(t)$ are found to decrease as the index is decreased.

International Journal of Electrical, Electronic and Communication Sciences
ISSN: 2517-9438
Vol:6, No:2, 2012

Fig.4 shows the non-empty probabilities, $P_k(t)$ for k=1,2,3, …, as a function of time (normalized time μt). The trace at the top is for $P_1(t)$ and the one at the bottom is for P_{12}. Other traces are omitted for obvious reasons. The probabilities are found to decrease as the index increases. The results obtained are in support of those obtained earlier.

Fig.5 shows the mean number of packets in system as a function of time starting from the empty state (i=0). Evidently these traces approach the correct steady state values, although they take some time to stabilize. Fig.6 shows the same statistic for a system starting from a no-empty queue (i=5). The same steady state values are reached in this figure as well.

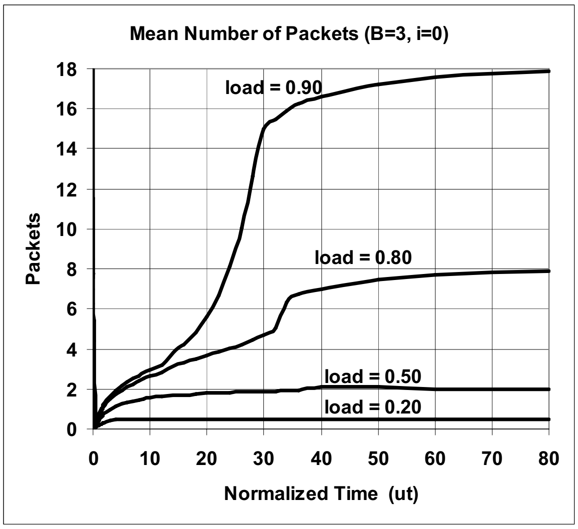

Fig. 5 Mean number of Packets in the system as a function of normalized time (μt) starting from an empty queue (i=0)

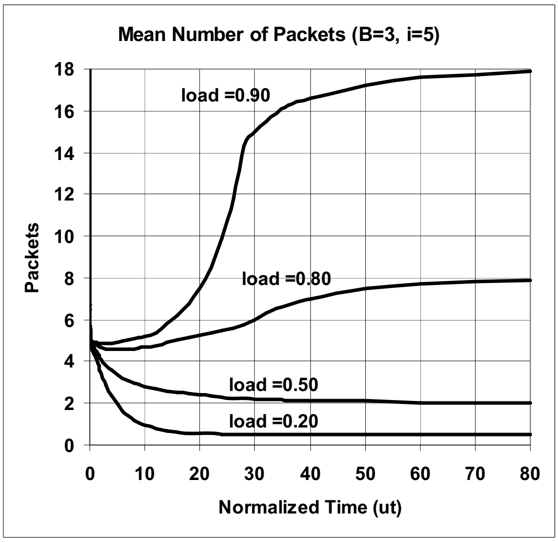

Fig. 6 Mean number of packets in the system as a function of normalized time (μt) starting from a non-empty queue (i=5)

The paper has addressed the problem of obtaining the transient solution to a queuing system in which packets arrive according to a Poisson process in fixed size batches, with exponential service time. The analysis has exposed an underlying family of functions that bear resemblance with the modified Bessel functions of the first kind. After establishing this resemblance, the paper has proceeded to exploit the isomorphism with the case of single-packet arrivals to obtain results for the general case (B>1). the analysis replaces the occurrences $I_k(.)$ replaced by $V_k^{(B)}(.)$ of the appropriate argument to obtain the required expressions. These functions are in some sense a generalization of the modified Bessel functions of the first kind, and are shown in Appendix B to share similarities.

For the case of single-packet arrivals (B=1), an explicit closed form solution is found. Coupling this with the relations between the two families of functions exposes inherent similarities. This is immediately exploited in the analysis by replacing the occurrences of $I_k(.)$ by $V_k^{(B)}(.)$ with the appropriate argument, and using the relevant relations to obtain the required expressions. The conclusions drawn here as before indicate that steady state results are not sufficient to capture the queue behaviour following a perturbation. Transient analysis is required, and this paper has provided a means to accomplish this task for the fixed size batch arrivals. The conclusions drawn here are in agreement with what is already known.

APPENDIX A. EMPTY PROBABILITY FOR A BATCH SIZE OF ONE (B=1)

The moment generating function P(z,t) has Laplace-Stieltjes transform P*(z,s) given by

$$P^*(z,s) = \frac{z^{i+1} + \mu(z-1)P_0^*(s)}{(s+\lambda+\mu)z - \lambda z^2 - \mu} \qquad \text{(A1)}$$

For this case of single arrivals (B=1) exact solutions can be found, since the denominator in (A1) is quadratic in z, can be easily factored to yield the zeros z_1 and z_2 as

$$z_1 = \frac{(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu) - 4\mu\lambda}}{2\lambda} = \frac{2\mu}{(s+\lambda+\mu) + \sqrt{(s+\lambda+\mu) - 4\mu\lambda}}$$

$$z_2 = \frac{(s+\lambda+\mu) + \sqrt{(s+\lambda+\mu) - 4\mu\lambda}}{2\lambda} = \frac{2\mu}{(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu) - 4\mu\lambda}} \qquad \text{(A2)}$$

It can be shown [11] that $|z_1| < 1$ and $|z_2| \geq 1$. The function $P^*(z,s)$ must be analytic within the unit disc $\{z : |z| \leq 1\}$. Therefore by Rouché's Theorem, there we be a pole-zero cancellation to remove the pole in the unit disc. This gives

$$P_0^*(s) = \frac{z_1^{i+1}}{\mu(1 - z_1)} \qquad \text{(A3)}$$

Before obtaining the expression for $P_0(t)$, it is noted that the exponential function in (3) for B=1 can expressed as a two-sided series in z according to

$$\exp\left[\left(\lambda z + \mu z^{-1}\right)t\right] = \sum_{k=-\infty}^{\infty} z^k \left(\frac{\lambda}{\mu}\right)^{k/2} I_k\left(2t\sqrt{\mu\lambda}\right) \qquad \text{(A4)}$$

where $I_k(.)$ is the modified Bessel function of the first kind of order k. Differentiating both sides of (A4) with respect to z gives

$$\left(\lambda - \mu z^{-2}\right)t \exp\left[\left(\lambda z + \mu z^{-1}\right)t\right] = \sum_{k=-\infty}^{\infty} k z^{k-1} \left(\frac{\lambda}{\mu}\right)^{k/2} I_k\left(2t\sqrt{\mu\lambda}\right)$$

(A5)

Replacing the exponential on the left with its power series (A4), and reorganizing, gives

$$\frac{k}{at} I_k(at) = \frac{1}{2}\left[I_{k-1}(at) - I_{k+1}(at)\right] \quad \text{(A6)}$$

This result is available in many texts. The derivation is given here in anticipation of the results for the general case where B is greater than one. The same steps will be followed later for the general case of B >1. The Laplace transform of $\exp[(\lambda z + \mu z^{-1})t]$ is

$$L\left\{\exp\left[\left(\lambda z + \mu z^{-1}\right)t\right]\right\} = \frac{z}{sz + \lambda z^2 + \mu} = \frac{z}{\lambda(z - z_1)(z - z_2)} \quad \text{(A7)}$$

This can be expanded as a power series in z to give

$$L\left\{\exp\left[\left(\lambda z + \mu z^{-1}\right)t\right]\right\} = \frac{1}{\lambda(z_2 - z_1)}\left(\sum_{k=1}^{\infty} z_1^k z^{-k} + \sum_{k=0}^{\infty} z_2^{-k} z^k\right) \quad \text{(A8)}$$

Substituting for Z_1 and Z_2 we obtain

$$L\left\{\exp\left[\left(\lambda z + \mu z^{-1}\right)t\right]\right\} = \frac{1}{\sqrt{s^2 - 4\mu\lambda}} \sum_{k=1}^{\infty}\left(\frac{2\mu}{s + \sqrt{s^2 - 4\mu\lambda}}\right)^k z^{-k} + \frac{1}{\sqrt{s^2 - 4\mu\lambda}} \sum_{k=0}^{\infty}\left(\frac{s + \sqrt{s^2 - 4\mu\lambda}}{2\lambda}\right)^{-k} z^k \quad \text{(A9)}$$

which can be re-written as

$$L\left\{\exp\left[\left(\lambda z + \mu z^{-1}\right)t\right]\right\} = \frac{1}{\sqrt{s^2 - 4\mu\lambda}} \sum_{k=-\infty}^{-1}\left(\frac{s + \sqrt{s^2 - 4\mu\lambda}}{2\mu}\right)^{-k} z^k + \frac{1}{\sqrt{s^2 - 4\mu\lambda}} \sum_{k=0}^{\infty}\left(\frac{s + \sqrt{s^2 - 4\mu\lambda}}{2\lambda}\right)^{-k} z^k \quad \text{(A10)}$$

Combining (A4) and (A10) gives

$$L\left\{I_k\left(2t\sqrt{\mu\lambda}\right)\right\} = \frac{1}{\sqrt{s^2 - 4\mu\lambda}}\left(\frac{s + \sqrt{s^2 - 4\mu\lambda}}{2\lambda}\right)^{-k}\left(\frac{\mu}{\lambda}\right)^{k/2} \quad \text{(A11)}$$

From (A6) it is evident that

$$L\left\{\frac{k}{2t\sqrt{\mu\lambda}} I_k\left(2t\sqrt{\mu\lambda}\right)\right\} = \frac{1}{2\sqrt{s^2 - 4\mu\lambda}}\left(\frac{s + \sqrt{s^2 - 4\mu\lambda}}{2\lambda}\right)^{-k+1}\left(\frac{\mu}{\lambda}\right)^{(k-1)/2} - \frac{1}{2\sqrt{s^2 - 4\mu\lambda}}\left(\frac{s + \sqrt{s^2 - 4\mu\lambda}}{2\lambda}\right)^{-k-1}\left(\frac{\mu}{\lambda}\right)^{(k+1)/2}$$

(A12)

Therefore

$$L^{-1}\left(\frac{s + \sqrt{s^2 - 4\mu\lambda}}{2\lambda}\right)^{-k} = \left(\frac{\lambda}{\mu}\right)^{k/2}\left\{\frac{k}{t} I_k\left(2t\sqrt{\mu\lambda}\right)\right\} \quad \text{(A13)}$$

Using the fact that $|z_1| < 1$ in (A3) gives

$$P_0^*(s) = \frac{1}{\mu}\sum_{m=i+1}^{\infty} z_1^m \quad \text{(A14)}$$

which is easily inverted using (A14) to give

$$P_0(t) = \exp[-(\lambda + \mu)t]\left(\frac{1}{\mu}\right)\sum_{m=i+1}^{\infty}\left(\frac{\lambda}{\mu}\right)^{(m-1)/2}\frac{m}{t} I_m\left(2t\sqrt{\mu\lambda}\right) \quad \text{(A15)}$$

Using (A6) in (A15) finally yields

$$P_0(t) = \exp[-(\lambda + \mu)t]\sum_{m=i+1}^{\infty}\left(\frac{\lambda}{\mu}\right)^{m/2}\left[I_{m-1}\left(2t\sqrt{\mu\lambda}\right) - I_{m+1}\left(2t\sqrt{\mu\lambda}\right)\right]$$

(A16)

This result is available in other places, e.g. [11], and is derived here in preparation for the general case to be addressed in Appendix B.

APPENDIX B. RELATIONS BETWEEN BATCH ARRIVAL FUNCTIONS AND THE MODIFIED BESSEL FUNCTIONS

For B > 1, the exponential (3) now takes the form

$$\exp\left[\left(\lambda z^B + \mu z^{-1}\right)t\right] = \sum_{k=-\infty}^{\infty} z^k \left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} V_k^{(B)}(\alpha t) \quad \text{(B1)}$$

where

$$\alpha = \left(\mu^B \lambda\right)^{\frac{1}{B+1}} = \left(\frac{\lambda}{\mu}\right)^{\frac{1}{B+1}} \mu, \quad \text{(B2)}$$

the functions $V_k^{(B)}(x)$ have been introduced, and are defined as

$$V_{-k}^{(B)}(x) = \sum_{l=0}^{\infty} \frac{x^{l(B+1)+k}}{l!(lB+k)!} \quad \text{(B3)}$$

$$V_k^{(B)}(x) = \sum_{l=\sigma_k}^{\infty} \frac{x^{l(B+1)-k}}{l!(lB-k)!} \quad \text{(B4)}$$

and for k > 0 the variable σ_k is defined by $\sigma_k = \lceil k/B \rceil$, *the smallest integer not less than k/B.* It is shown in the sequel that the functions in (B3) and (B4) reduce to the modified Bessel functions and the relations among them also reduce to equivalent relations among the Bessel functions.

A. Coalescence to Modified Bessel Functions

When B=1 the functions in (B3) and (B4) coalesce to $I_k(2x)$ where $I_k(.)$ is the modified Bessel function of the first kind of order k. This can be seen by directly substituting B = 1 above. Specifically for B=1 gives $\sigma_k = k$, and so

$$V_k^{(1)}(x) = V_{-k}^{(1)}(x) = \sum_{l=0}^{\infty} \frac{x^{2l+k}}{l!(l+k)!} = I_k(2x) \quad \text{(B5)}$$

This is the first case of the relations between the functions $V_k^{(B)}(\cdot)$ and $I_k(\cdot)$, the kth order modified Bessel function of the first kind.

B. Recursive Relations

Starting with the definition (B1) and differentiating both sides with respect to z gives

$$\left(B\lambda z^{B-1} - \mu z^{-2}\right)t \exp\left[\left(\lambda z^B + \mu z^{-1}\right)t\right] = \sum_{k=-\infty}^{\infty} k z^{k-1}\left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} V_k^{(B)}(\alpha t) \quad \text{(B6)}$$

which can be re-written as

$$\sum_{k=-\infty}^{\infty} z^k \left\{ B\lambda\, t \left(\frac{\lambda}{\mu}\right)^{\frac{k-B+1}{B+1}} V_{k-B+1}^{(B)}(\alpha\, t) - \mu\, t \left(\frac{\lambda}{\mu}\right)^{\frac{k+2}{B+1}} V_{k+2}^{(B)}(\alpha\, t) \right\} \quad \text{(B7)}$$

$$= \sum_{k=-\infty}^{\infty} z^k (k+1) \left(\frac{\lambda}{\mu}\right)^{\frac{k+1}{B+1}} V_{k+1}^{(B)}(\alpha\, t)$$

which then gives the relations

$$B\left(\frac{\lambda}{\mu}\right)^{\frac{-B}{B+1}} \lambda\, t\, V_{k-B}^{(B)}(\alpha\, t) - \left(\frac{\lambda}{\mu}\right)^{\frac{1}{B+1}} \mu\, t\, V_{k+1}^{(B)}(\alpha\, t) = k V_k^{(B)}(\alpha\, t) \quad \text{(B8)}$$

Using the definition (B2) of α gives

$$\frac{k}{\alpha\, t} V_k^{(B)}(\alpha\, t) = B\, V_{k-B}^{(B)}(\alpha\, t) - V_{k+1}^{(B)}(\alpha\, t) \quad \text{(B9)}$$

Incidentally for B=1, and substituting $V_k^{(1)}(x) = I_k(2x)$ in (B9) gives

$$\frac{2k}{2x} I_k(2x) = I_{k-1}(2x) - I_{k+1}(2x) \quad \text{(B10)}$$

Replacing $2x$ by x gives

$$\frac{k}{x} I_k(x) = \frac{1}{2}\left[I_{k-1}(x) - I_{k+1}(x) \right] \quad \text{(B11)}$$

which is one of the functional relations [10] for $I_k(x)$, the modified Bessel functions of the first kind of order k.

C. Differentiation with Respect to Argument

Differentiating both sides of (B1) with respect to t, and letting $x = \alpha t$, gives fhe following result

$$\left(\lambda z^B + \mu z^{-1}\right) \exp\left[\left(\lambda z^B + \mu z^{-1}\right) t\right] = \sum_{k=-\infty}^{\infty} z^k \left(\frac{\lambda}{\mu}\right)^{\frac{k}{B+1}} \alpha \frac{d}{dx}\left[V_k^{(B)}(x)\right] \quad \text{(B12)}$$

Substituting for the exponential on the left hand side and reorganizing gives

$$\frac{d}{dx}\left[V_k^{(B)}(x)\right] = V_{k-B}^{(B)}(x) + V_{k+1}^{(B)}(x) \quad \text{(B13)}$$

As a check, setting B=1, this becomes

$$\frac{d}{dx}\left[V_k^{(1)}(x)\right] = V_{k-1}^{(B)}(x) + V_{k+1}^{(1)}(x) \quad \text{(B14)}$$

Recalling that $V_k^{(1)}(x) = I_k(2x)$ the above becomes

$$2\frac{d}{d(2x)}\left[I_k(2x)\right] = I_{k-1}(2x) + I_{k+1}(2x) \quad \text{(B15)}$$

which finally gives

$$\frac{d}{dx} I_k(x) = \frac{1}{2}\left[I_{k-1}(x) + I_{k+1}(x)\right] \quad \text{(B16)}$$

and this is another one of the functional relations [10] of the modified Bessel functions of the first kind of order k. Other forms of correspondence may be obtained by applying these relations and other operations to the defining equations. There is sufficient correspondence to take it that the functions are indeed generations of the modified Bessel functions of the first kind of the appropriate order. The batch size B plays the role of the generalizing parameter. It remains to be shown how the development presented here can be used to obtain equivalent generalizations of the Bessel functions. This task can be explored elsewhere; the main concern of this paper is to use the set of functions already presented to solve the transient queuing problem.

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