

Action functional of the electromagnetic field: Effect of Gravitation

Arti Vaish and Harish Parthasarathy

Abstract—The scalar wave equation for a potential in a curved space time, i.e., the Laplace-Beltrami equation has been studied in this work. An action principle is used to derive a finite element algorithm for determining the modes of propagation inside a waveguide of arbitrary shape. Generalizing this idea, the Maxwell theory in a curved space time determines a set of linear partial differential equations for the four electromagnetic potentials given by the metric of space-time. Similar to the Einstein's formulation of the field equations of gravitation, these equations are also derived from an action principle. In this paper, the expressions for the action functional of the electromagnetic field have been derived in the presence of gravitational field.

Keywords—General theory of relativity, Electromagnetism, Metric tensor, Maxwells Equations, Test functions, Finite element method.

I. INTRODUCTION

TO study the propagation of electromagnetic waves inside a waveguide in the presence of an external gravitational field, we assume exponential dependence of the four potential on the z and time coordinate and modify the action principle slightly [2]. This modified action will be quadratic in the electromagnetic four potential but will also involve the external gravitational field as a parameter. This action principle can be used to derive a finite element algorithm for obtaining the potentials. The idea is to partition the cross section of the guide into triangular elements, and assume that each component of the four potential inside a triangle can be expressed as a linear combination of the corresponding component at the vertices of the triangle, the interpolation functions being linear functions of the x and y coordinates [3], [4]. Using these interpolation rules, the action functional for the entire field is expressed as a quadratic function of $4N$ variables where N is the number of vertices and the factor of 4 arises due to the presence of four potential components. The modes of propagation can then be derived from the eigenvalues of the $4N \times 4N$ matrix associated with this quadratic form [5], [6].

A. Introduction to general relativity

In the general theory of relativity, gravitation field is regarded as curvature of space-time [7]. The interval between two infinitely close events in curved spacetime is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (1)$$

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where $g_{\mu\nu}$ is the symmetric metric tensor (depending in general on the spacetime coordinates x^0, x^1, x^2 , and x^3). In flat space-time Cartesian coordinate system, the metric is diagonal with $(-1, 1, 1, 1)$ on the diagonal. This is called a Galilean coordinate system [2], [8]. In curved 4-dimensional space-time, the choice of a reference system is arbitrary and therefore the laws of physics must be written in a form that is independent of the reference system. This is achieved by writing the laws of physics in tensor form. Subsection text here.

1) *General Theory of Relativity: Background:* Since 1915, the theory of relativity has been developed extensively among others by Einstein and by the British astronomers James Hopwood Jeans, Arthur Stanley Eddington, and Edward Arthur Milne, by the Dutch astronomer Willem de Sitter, and by the German-American mathematician Hermann Weyl [9], [10]. Most of their efforts were to extend the theory of relativity under electromagnetic phenomena [11], [12]. Although some progress has been made in this area, these efforts have been marked thus far by less success.

In 1928, a relativistic electron theory was developed by the British mathematician and physicist Paul Dirac, and subsequently a satisfactory quantized field theory, called *quantum electrodynamics*, was evolved. It unifies the concepts of relativity and quantum theory in relation to the interaction between electrons, positrons, and electromagnetic radiation [13]. In recent years, Hawking [14] made an attempt of full integration of quantum mechanics with relativity theory. Subsequently, many attempts have been made in this direction, yet very few people have studied the effect of gravitational field on the electromagnetic, i.e., the frequency of propagation of waveguide. Thus, the present work aims to investigate the effects of gravitational field on the frequency of propagation modes of the waveguide using the finite element method.

B. Tensors and curvilinear coordinates

Consider the transformation from one coordinate system x^0, x^1, x^2 , and x^3 to another coordinate system x'^0, x'^1, x'^2 , and x'^3 .

$$x^\alpha = x^\alpha(x'^0, x'^1, x'^2, x'^3) \quad (2)$$

Any collection of four quantities A^α that, under this coordinate transformation, transform according to

$$A^\alpha = \frac{\partial x^\alpha}{\partial x'^\beta} A'^\beta \quad (3)$$

is called a contravariant four-vector [2]. Similarly any collection of four quantities A_α that transform according to

$$A_\alpha = \frac{\partial x'^\beta}{\partial x^\alpha} A'_\beta \quad (4)$$

is called a covariant four vector [2], [5]. These transformation laws can be generalized to tensors of arbitrary rank. The transformation Changing between contravariant and covariant forms of vectors and tensors is accomplished by using the metric tensor as follows

$$A_\alpha = g_{\alpha\beta} A^\beta \quad (5)$$

$$A^\alpha = g^{\alpha\beta} A_\beta \quad (6)$$

where $g^{\alpha\beta}$ is the contravariant metric tensor, defined by the relation

$$g_{\alpha\beta} g^{\beta\delta} = \delta_\alpha^\delta \quad (7)$$

From equations (3) and (4), it is obvious that if a vector (or tensor) vanishes in one coordinate system, it will vanish in any coordinate system. Thus, if the relation holds in one coordinate system it will hold in any coordinate system. The same is true for more complicated tensor equations and therefore if a law of physics can be expressed as a tensor equation in a given coordinate system it will have the same tensor form in any other coordinate system. It is said to have been formulated in covariant form.

C. The covariant derivative

From the transformation law (equation 4), it follows that the differentials of a covariant vector transform as

$$dA_\alpha = \frac{\partial x'^\beta}{\partial x^\alpha} dA'_\beta + A'_\beta \frac{\partial^2 x'^\beta}{\partial x^\alpha \partial x^\gamma} \quad (8)$$

In case of the non-linear transformation, the second term on the right hand side of Eq. (8) will be non-zero, which means that dA_α is not a vector. Therefore, ordinary differentiation can not be used in the covariant formulation of the laws of physics. In curved space-time coordinates, the subtraction of the two vectors A_α and $A_\alpha + dA_\alpha$ at points x^α and $x^\alpha + dx^\alpha$ is not a vector. To get a vector, it needs one intermediate step, i.e., parallel translation of A_α to $x^\alpha + dx^\alpha$ and then do the subtraction. This is done using the following definition of the derivative of covariant and contravariant tensors;

$$A_{\alpha;\beta} = \frac{\partial A_\alpha}{\partial x^\beta} - \Gamma_{\alpha\beta}^\gamma A_\gamma \quad (9)$$

$$A^\alpha_{;\beta} = \frac{\partial A^\alpha}{\partial x^\beta} + \Gamma_{\beta\gamma}^\alpha A^\gamma \quad (10)$$

where $\Gamma_{\beta\gamma}^\alpha$ are the Christoffel symbols defined as:

$$\Gamma_{\beta\gamma}^\alpha = \frac{1}{2} g^{\alpha\delta} \left(\frac{\partial g_{\delta\beta}}{\partial x^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta} \right) \quad (11)$$

It is shown elsewhere [15], [16] that it is always possible to choose a coordinate system in which $\Gamma_{\beta\gamma}^\alpha$ is zero at the same time as the metric is brought to Galilean form at a given point. This is called a locally inertial coordinate system [15]. At this point, the covariant derivative reduces to the ordinary derivative and any equation between tensors will reduce to

the special relativistic form. The general relativistic forms of the laws of physics can thus be obtained from the special relativistic ones by exchanging the ordinary derivatives with the covariant derivatives [16].

II. MAXWELL'S EQUATIONS IN RELATIVITY

The Static electric field and Static magnetic field due to charges at rest and steady currents, respectively, are given by Maxwell's equations, as:

$$E = -\frac{1}{c} \frac{\partial A}{\partial t} - \text{grad}\phi \quad (12)$$

$$H = \text{curl} A \quad \text{and} \quad \text{div} H = 0 \quad (13)$$

$$\frac{1}{c} \frac{\partial H}{\partial t} = -\text{curl} E \quad (14)$$

$$\frac{1}{c} \frac{\partial E}{\partial t} = \text{curl} H - 4\pi j \quad (15)$$

$$\text{div} E = 4\pi\rho \quad (16)$$

Now for special relativity, we will put them in four-dimensional form. The potential A and ϕ form a four vector k^μ given by,

$$k^0 = \phi, k^m = A^m; (m = 1, 2, 3) \quad (17)$$

Define

$$F_{\mu\nu} = k_{\mu,\nu} - k_{\nu,\mu} \quad (18)$$

From equation 12

$$E^1 = -\frac{\partial \kappa^1}{\partial x^0} - \frac{\partial \kappa^0}{\partial x^1} = \frac{\partial \kappa_1}{\partial x^0} - \frac{\partial \kappa_0}{\partial x^1} = F_{10} = -F^{10} \quad (19)$$

and from equation 13

$$H^1 = -\frac{\partial \kappa^3}{\partial x^2} - \frac{\partial \kappa^2}{\partial x^3} = -\frac{\partial \kappa_3}{\partial x^2} - \frac{\partial \kappa_2}{\partial x^3} = F_{23} = F^{23} \quad (20)$$

From equation 20

$$F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0 \quad (21)$$

This gives the Maxwell equations 14 and 15. We have

$$F^{0\nu}_{;\nu} = F^{0m}_{;m} = -F^{m0}_{;m} = \text{div} E = 4\pi\rho \quad (22)$$

From equation 17, again

$$F^{1\nu}_{;\nu} = F^{10}_{;0} + F^{12}_{;2} + F^{13}_{;3} = -\frac{\partial E^1}{\partial x^0} + \frac{\partial H^3}{\partial x^2} - \frac{\partial H^2}{\partial x^3} = 4\pi j^1 \quad (23)$$

The charge density ρ and current j^m form a four vector J^μ in accordance with

$$J^m = \rho \quad J^0 = j^m \quad (24)$$

From equation 23 and 24, we can write

$$F^{\mu\nu}_{;\nu} = 4\pi J^\mu \quad (25)$$

In this way the Maxwell equations are put into the four dimensional form required by special relativity. It should be noted here that Equation 25 is not valid in general relativity.

For general relativity, rewrite the equations in covariant form as:

$$F_{\mu\nu} = k_{\mu;\nu} - k_{\nu;\mu} \quad (26)$$

This gives us the covariant definition of the quantities $F_{\mu\nu}$. We have further

$$F_{\mu\nu;\sigma} = F_{\mu\nu,\sigma} - \Gamma_{\mu\sigma}^{\alpha} F_{\alpha\nu} - \Gamma_{\nu\sigma}^{\alpha} F_{\mu\alpha} \quad (27)$$

Now making cyclic permutation of μ, ν and σ and adding the three equations, we get Maxwell equation to the covariant form.

$$F_{\mu\nu;\sigma} + F_{\nu\sigma;\mu} + F_{\sigma\mu;\nu} = F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} \quad (28)$$

Thus, the Maxwell equation in general relativity can be written in covariant form as:

$$F_{;\nu}^{\mu\nu} = 4\pi J^{\mu} \quad (29)$$

Equation 14 can be applied to any antisymmetric two suffix tensor, we get

$$(F^{\mu\nu}\sqrt{g})_{;\nu} = 4\pi J^{\mu}\sqrt{g}. \quad (30)$$

it leads to

$$(J^{\mu}\sqrt{g})_{;\mu} = (4\pi)^{-1}(F^{\mu\nu}\sqrt{g})_{;\mu\nu} = 0 \quad (31)$$

This equation gives us the law of conservation of electricity. The conservation of electricity holds accurately, undisturbed by the curvature of space.

A. Action principle for the Maxwell equation

In classical electromagnetic theory in flat space-time, the action for the free space electromagnetic field is given by:

$$S = \int F_{\mu\nu} F^{\mu\nu} d^4x \quad (32)$$

Where $F_{\mu\nu}$ is given by equation 20. The action principle $\delta S = 0$ leads to the free space field equations.

$$F_{;\nu}^{\mu\nu} = 0 \quad (33)$$

This follows from,

$$\begin{aligned} \delta(F_{\mu\nu} F^{\mu\nu}) &= 2F^{\mu\nu} \delta F_{\mu\nu} - 2F^{\mu\nu} (\delta A_{\nu,\mu} - \delta A_{\mu,\nu}) \\ &= 4(F^{\mu\nu} \delta A_{\nu,\mu}) \\ &= 4(F^{\mu\nu} \delta A_{\nu})_{;\mu} - 4F_{;\nu}^{\mu\nu} \delta A_{\nu} \end{aligned} \quad (34)$$

The integral $(F^{\mu\nu} \delta A_{\nu})_{;\mu} d^4x = 0$ (by Gauss theorem), so

$$\delta S = -4 \int F_{;\nu}^{\mu\nu} \delta A_{\nu} d^4x = 0 \quad (35)$$

gives

$$F_{;\nu}^{\mu\nu} = 0 \quad (36)$$

Taking $\mu = 0, 1, 2, 3$, we get the Maxwell equations;

$$\text{div } \vec{E} = 0 \quad (37)$$

and

$$\text{curl } \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t} \quad (38)$$

The action functional apart from a proportionality constant may be shown to be equal to $\int (E^2 - H^2) d^4x$. In general

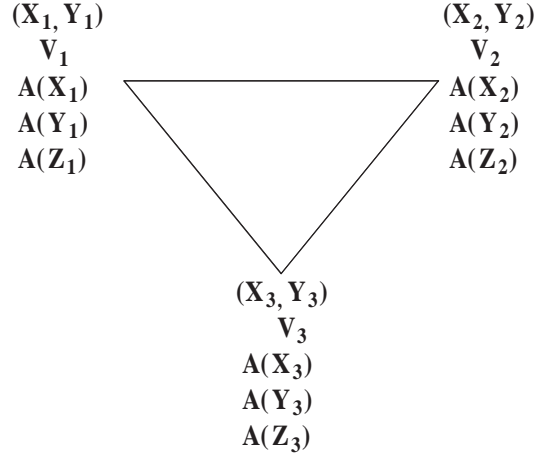


Fig. 1. Anisotropic waveguide cross section for two regions

relativity, the action functional should be a scalar. This is generated by replacing d^4x with $\sqrt{-g}d^4x$. The Maxwell equations for a fixed metric $g_{\mu\nu}$ are derived from

$$\delta \int F_{\mu\nu} F^{\mu\nu} \sqrt{-g} d^4x = 0 \quad (39)$$

Which yields,

$$(F^{\mu\nu} \sqrt{-g})_{;\nu} = 0 \quad (40)$$

or

$$F_{;\nu}^{\mu\nu} = 0 \quad (41)$$

III. FINITE ELEMENT FORMULATION

A schematics of a triangular finite element in the rectangular waveguide is shown in Figure 1.

Consider a triangle having vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. The expansion of the four functions A_x, A_y, A_z and V as a linear combination of their vertex values can be describes as:

$$A_i = A_i(1)\phi_1 + A_i(2)\phi_2 + A_i(3)\phi_3, \quad (42)$$

where $i = x, y, z$ and

$$V = V_1 \cdot \phi_1(x, y) + V_2 \cdot \phi_2(x, y) + V_3 \cdot \phi_3(x, y). \quad (43)$$

Here, $A_i(j)$ and $V(j)$ are the nodal potential at node $j = 1, 2, 3$ or at points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) .

Here we are taking $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) as $(0.0, 0.0), (0.5, 0.5)$ and $(0.0, 0.5)$. Two vectors, \vec{u} and \vec{v} have been drawn by joining the vertices $[(x_1, y_1), (x_2, y_2)]$ and $[(x_1, y_1) \text{ and } (x_3, y_3)]$ [17], respectively.

Let

$$d_1 = |u| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = .5 \quad (44)$$

and

$$d_2 = |v| = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2} = .25 \quad (45)$$

The unit vector along the two directions u and v are

$$\hat{u} = \frac{u}{|u|} = \frac{(x_2 - x_1, y_2 - y_1)}{d_1} \quad (46)$$

and

$$\hat{v} = \frac{v}{|v|} = \frac{(x_3 - x_1, y_3 - y_1)}{d_2} \quad (47)$$

any point (x, y) inside this triangle can be represented as

$$\begin{aligned} (x, y) &= (x_1, y_1) + u\hat{u} + v\hat{v} \\ &= (x_1, y_1) + \frac{u(x_2 - x_1, y_2 - y_1)}{d_1} + \frac{v(x_3 - x_1, y_3 - y_1)}{d_2} \end{aligned}$$

so

$$x = x_1 + \frac{u(x_2 - x_1)}{d_1} + \frac{v(x_3 - x_1)}{d_2} \quad (48)$$

and

$$y = y_1 + \frac{u(y_2 - y_1)}{d_1} + \frac{v(y_3 - y_1)}{d_2} \quad (49)$$

The solution of these two linear equations results in the variables u, v as linear functions of x, y . The area measure is given by

$$ds(u, v) = |\vec{u} \times \vec{v}| du dv$$

where

$$|\vec{u} \times \vec{v}| = \sin \alpha$$

Here, α , the angle between the vectors u and v , is defined as

$$\begin{aligned} \cos \alpha &= \frac{u \cdot v}{d_1 d_2} \\ &= \frac{(x_2 - x_1)(x_3 - x_1) + (y_2 - y_1)(y_3 - y_1)}{d_1 d_2} \end{aligned} \quad (50)$$

where

$$\Delta = x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1 = .25 \quad (51)$$

where

$$\begin{aligned} \phi_1(x, y) &= \frac{(y_2 - y_3)x + (x_2 - x_3)y + (x_2 y_3 - x_3 y_2)}{\Delta} \\ &= 2y + 1, \end{aligned} \quad (52)$$

$$\begin{aligned} \phi_2(x, y) &= \frac{(y_3 - y_1)x + (x_3 - x_1)y + (x_3 y_1 - x_1 y_3)}{\Delta} \\ &= 2x, \end{aligned} \quad (53)$$

$$\begin{aligned} \phi_3(x, y) &= \frac{(y_1 - y_2)x + (x_1 - x_2)y + (x_1 y_2 - x_2 y_1)}{\Delta} \\ &= -2x - 2y + 2. \end{aligned} \quad (54)$$

$$A_{x(x_i, y_i)} = A_{x_i} \quad (55)$$

And

$$\phi_i(x_i, y_i) = \delta_{ij} \quad (56)$$

A_x evaluated at (x_1, y_1) is A_{x1} . Now the derivative of ϕ_i with respect to x is given as:

$$\begin{aligned} \phi_{1,x} &= \frac{(y_2 - y_3)}{\Delta} = 0, & \phi_{2,x} &= \frac{(y_3 - y_1)}{\Delta} = 2, \\ \phi_{3,x} &= \frac{(y_1 - y_2)}{\Delta} = 2 \end{aligned} \quad (57)$$

and the derivative of ϕ_i with respect to y is given as:

$$\begin{aligned} \phi_{1,y} &= \frac{(x_2 - x_3)}{\Delta} = 2, & \phi_{2,y} &= \frac{(x_3 - x_1)}{\Delta} = 0, \\ \phi_{3,y} &= \frac{(x_1 - x_2)}{\Delta} = 2, \end{aligned} \quad (58)$$

IV. ELECTROMAGNETIC FIELD EQUATIONS IN A WAVEGUIDE IN THE PRESENCE OF A GRAVITATIONAL FIELD:

The four potentials is are given by [18], [19]

$$(A^\mu) = (V, A_x, A_y, A_z) \quad (59)$$

The metric of space-time is assumed to be Newtonian with the gravitational potential U independent of z

$$d\tau^2 = \left(1 + \frac{2U(x, y)}{c^2}\right) dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \quad (60)$$

so that

$$g_{00} = \left(1 + \frac{2U}{c^2}\right), \quad \text{and} \quad g_{11} = g_{22} = g_{33} = -\frac{1}{c^2} \quad (61)$$

The electromagnetic field tensor is given by [19]

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad (62)$$

The Lagrangian density is given by

$$L = F_{\mu\nu} F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} \quad (63)$$

The derivatives with respect to z are replaced by multiplication with $-\gamma$ and the derivatives with respect to time are replaced by multiplication by $i\omega$ or $j\omega$ and a negative sign is added before these. To see the reason for the negative sign, we consider a simple case in which one of the components of the action functional is,

$$\int V_{,z}^2 dx dy dz dt \quad (64)$$

The variation of this gives on integration by parts,

$$\int V_{,z} \delta V_{,z} dx dy dz dt = - \int V_{,zz} \delta V dx dy dz dt \quad (65)$$

The coefficient of δV is $-V_{,zz} = -\gamma^2 V$. In the original action, if we replace $V_{,z}$ by $-\gamma V$, then we get the term $\gamma^2 \int V^2 dx dy dz dt$ which on variation gives $\gamma^2 \int V \delta V dx dy dz dt$ and the coefficient of δV is $\gamma^2 V$. This shows that in the replacement of $V_{,z}$ by γV in the original action, a minus sign must be introduced to obtain the correct field equations. The same goes for derivative with respect to time [6]. This can also be seen to be true for terms of the form $\int \psi V_{,z} dx dy dz dt$. Taking the variation with respect to V , integrating by parts and selecting the coefficient of δV gives us $-\psi_{,z}$ which is $\gamma \psi$. On the other hand, if we directly make the replacement $V_{,z} \rightarrow -\gamma V$ in the action, we get $-\gamma \int \psi V dx dy dz dt$. Taking the variation with respect to V and selecting the coefficient of δV gives us $-\gamma \psi$ and to get

agreement with the former, a negative sign must be introduced. Now,

$$L = 2g^{00}g^{11}F_{01}^2 + 2g^{00}g^{22}F_{02}^2 + 2g^{00}g^{33}F_{03}^2 + 2g^{11}g^{22}F_{12}^2 + 2g^{22}g^{33}F_{23}^2 + 2g^{11}g^{33}F_{31}^2 \quad (66)$$

The first term contains

$$F_{01}^2 = (A_{1,0} - A_{0,1})^2 = A_{1,0}^2 + A_{0,1}^2 - 2A_{1,0}A_{0,1} \quad (67)$$

As per the above rule, this term is to be replaced by

$$\omega^2 A_1^2 + A_{0,1}^2 + 2j\omega.A_1A_{0,1} = \omega^2(g_{11}A_x)^2 + (g_{00}V)_{,x}^2 + 2j\omega.g_{11}A_x(g_{00}V)_{,x} \quad (68)$$

Similarly, the second and third terms which contain F_{02}^2 and F_{03}^2 must be replaced by $\omega^2(g_{22}A_y)^2 + (g_{00}V)_{,y}^2 + 2j\omega.g_{11}A_y(g_{00}V)_{,y}$ and $\omega^2(g_{33}A_z)^2 - \gamma^2 g_{00}V^2 - 2j\omega\gamma.g_{33}g_{00}A_zV$, respectively. The sign of the last term can be understood as follows. Consider the variation of $A_{3,0}A_{0,3}$ with respect to A_3 . On integrating by part, it gives $-A_{0,30}\delta A_3$ and the coefficient of δA_3 here is $-A_{0,30} = j\omega\gamma A_0$. On the other hand, replacing $A_{3,0}$ and $A_{0,3}$ respectively by $j\omega A_3$ and $-\gamma A_0$ reduces the term $A_{3,0}A_{0,3}$ to $-j\omega\gamma A_3A_0$ and the variation with respect to A_3 gives $-j\omega\gamma A_0$. To get agreement, we must thus introduce an extra negative sign here. The fourth term contains F_{12}^2 . This evaluates to

$$F_{12}^2 = (A_{2,1} - A_{1,2})^2 = ((g_{22}A_y)_{,x} + (g_{11}A_x)_{,y} - 2A_{y,x}A_{x,y})^2 \quad (69)$$

It results in no modification. In the accord to the above rules, the fifth and sixth terms which contain F_{23}^2 and F_{31}^2 must be replaced with $(g_{33}A_z)_{,y}^2 - \gamma^2 A_y^2 - 2\gamma(g_{33}A_z)_{,y}g_{22}A_y$ and $-\gamma^2(g_{11}A_x)_{,x}^2 + (g_{33}A_z)_{,x}^2 - 2\gamma g_{11}A_x(g_{33}A_z)_{,x}$, respectively. Taking $c = 1$ and substituting for metric coefficients, we get the modified Lagrangian density as

$$L = -2(1-2U)(\omega^2(A_x)^2 + ((1+2U)V)_{,x}^2 - 2j\omega.A_x((1+2U)V)_{,x}) - 2(1-2U)(\omega^2(A_y)^2 + ((1+2U)V)_{,y}^2 - 2j\omega.A_y((1+2U)V)_{,y}) - 2(1-2U)(\omega^2(A_z)^2 - \gamma^2(1+2U)V^2 + 2j\omega\gamma.(1+2U)A_zV) + 2(A_{y,x}^2 + A_{x,y}^2 - 2A_{y,x}A_{x,y}) + 2(A_{z,y}^2 - \gamma^2 A_y^2 - 2\gamma A_{z,y}A_y) + 2(-\gamma^2 A_x^2 + A_{z,x}^2 - 2\gamma A_x A_{z,x}) \quad (70)$$

Note that we have assumed the gravitational potential U to be weak, i.e., $|U| \ll 1$ so that g^{00} can be taken as $1 - 2U$ [20], [21]. We further simplify the above expression so that $O(U^2)$ terms do not appear. We then get

$$L = -2\omega^2(1-2U)A_x^2 - 2(1-2U)(V_{,x})^2 - 8(UV)_{,x}V_{,x} + 4j\omega(1-2U)A_xV_{,x} + 8j\omega A_x(UV)_{,x} - 2\omega^2(1-2U)A_y^2 - 2(1-2U)V_{,y}^2 - 8(UV)_{,y}V_{,y} + 4j\omega(1-2U)A_yV_{,y} + 8j\omega A_y(UV)_{,y} - 2\omega^2(1-2U)A_z^2 - 2\gamma^2(1-2U)V^2 + 4\gamma^2 UV^2 - 4j\omega\gamma(1-2U)A_zV - 4j\omega\gamma.U.A_z.V + 2(A_{y,x}^2 + A_{x,y}^2 - 2A_{y,x}A_{x,y}) + 2(A_{z,y}^2 - \gamma^2 A_y^2 - 2\gamma A_{z,y}A_y) + 2(-\gamma^2 A_x^2 + A_{z,x}^2 - 2\gamma A_x A_{z,x}) \quad (71)$$

We can write this equation as

$$L = F + UG \quad (72)$$

where F , G depend only on the four electromagnetic potentials. The Action principle involves

$$\delta \int L \sqrt{-g} dx dy dt = 0 \quad (73)$$

and we have

$$\begin{aligned} L \sqrt{-g} &= L(1+U) \\ &= (F+U.G)(1+U) \\ &= F+U(F+G) \end{aligned} \quad (74)$$

with neglect of $O(U^2)$ terms, and after multiplication L with $(1+U)$ we get,

$$\begin{aligned} L(1+U) &= -2\omega^2(1-U)A_x^2 - 2\omega^2(1-U)A_y^2 - 2\omega^2(1-U)A_z^2 - 2(1-U)V_{,x}^2 - 2(1-U)V_{,y}^2 + 4j\omega(1-U)V_{,x}A_x + 4j\omega(1-U)V_{,y}A_y - 4j\omega\gamma A_zV(1-2U) - 2\gamma^2(1-U)V^2 \\ &\quad + 8(UV)_{,x}V_{,x} + 8j\omega A_x(UV)_{,x} - 8(UV)_{,y}V_{,y} + 8j\omega A_y(UV)_{,y} \\ &\quad + 2(1+U)[A_{y,x}^2 + A_{x,y}^2 - 2A_{y,x}A_{x,y} + A_{z,y}^2 - \gamma^2 A_y^2 - 2\gamma A_{z,y}A_y] \\ &\quad - \gamma^2 A_x^2 + A_{z,x}^2 - 2\gamma A_x A_{z,x} A_{z,y} A_y \end{aligned} \quad (75)$$

We can write above equation as:

$$\begin{aligned} \int L \sqrt{-g} dx dy &= L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8 \\ &\quad + L_9 + L_{10} + L_{11} + L_{12} + L_{13} + L_{14} + L_{15} \\ &\quad + L_{16} + L_{17} + L_{18} + L_{19} + L_{20} + L_{21} + L_{22} \end{aligned} \quad (76)$$

Where

$$L_{i1} = -2\omega^2 \int_{\Delta} (1-U)A_{j1}^2 dx dy \quad \text{where } i1 = 1, 2, 3 \text{ and } j1 = x, y, z \quad (77)$$

$$L_{i2} = -2 \int (1-U)V_{,j2}^2 dx dy \quad \text{where } i2 = 4, 5 \text{ and } j2 = x, y \quad (78)$$

$$L_{i3} = 4j\omega \int (1-U)V_{,j3}A_{j3} dx dy \quad \text{where } i3 = 6, 7 \text{ and } j3 = x, y \quad (79)$$

$$L_{i4} = 2 \int (1+U)A_{j4}^2 dx dy \quad \text{where } i4 = 8, \dots, 11 \text{ and } j3 = x, y; y, x; z, y; z, x \quad (80)$$

$$L_{i5} = -2\gamma^2 \int (1+U)A_{j5} dx dy \quad \text{where } i5 = 12, 13 \text{ and } j5 = x, y \quad (81)$$

$$L_{i6} = 8j\omega \int A_{j6}(UV)_{,j6} dx dy \quad \text{where } i6 = 14, 15 \text{ and } j6 = x, y \quad (82)$$

$$L_{i7} = -4\gamma \int (1+U)A_{j7}A_{k7} dx dy \quad \text{where } i7 = 16, 17 \text{ and } j7 = x, y; k7 = z, x; z, y \quad (83)$$

$$L_{18} = -4j\omega \gamma \int (1-2U)VA_z dx dy \quad (84)$$

$$L_{19} = -2\gamma^2 \int_{\Delta} (1-U)V^2 dx dy \quad (85)$$

$$L_{20} = 8 \int (UV)_{,x} V_x dx dy \quad (86)$$

$$L_{21} = -8 \int (UV)_{,y} V_y dx dy \quad (87)$$

$$L_{22} = -4 \int_{\Delta} (1+U)A_{y,x}A_{x,y} dx dy \quad (88)$$

$$U(x, y) = -\frac{GM}{C^2((x-R)^2 + y^2)^{1/2}} \quad (89)$$

Let $GM = 1$ and velocity of light, C is given by $C = 1$ for simplification of calculation. Now

$$\begin{aligned} \int L\sqrt{-g} dx dy &= L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 \\ &+ L_8 + L_9 + L_{10} + L_{11} + L_{12} + L_{13} \\ &+ L_{14} + L_{15} + L_{16} + L_{17} + L_{18} + L_{19} \\ &+ L_{20} + L_{21} + L_{22} \end{aligned} \quad (90)$$

V. INTEGRAL CALCULATION

Now, we are taking triangular element and expand the four functions A_x, A_y, A_z, V as linear combination of their vertex values. Now the integration L_1 can be calculated as:

A. Calculation of first integral

$$\int L_1 dx dy = -2\omega^2 \int_{\Delta} (1-U)A_x^2 dx dy \quad (91)$$

Now, expand each term in the expression and finally we get

$$\begin{aligned} \int L_1 dx dy &= -\omega^2 \sin \alpha \int_0^{d_1} \int_0^{d_2 - \frac{d_2 u}{d_1}} \left(1 + \frac{1}{\sqrt{(.7u-5)^2 + (.7u+v)^2}} \right) \\ &\left[A_x^2(1)(1.96u^2 + 4v^2 + 5.6uv + 1 + 5.6u + 8v) \right. \\ &+ .49u^2 A_x^2(2) + A_x^2(3) \\ &(7.84u^2 + 4v^2 + 5.6uv + 4 + 5.6uv) \\ &+ A_x(1)A_x(2)(3.92u^2 + 5.6uv + 2.8u) + A_x(2)A_x(3) \\ &(-7.84u^2 - 5.6uv + 5.6) \\ &\left. + A_x(1)A_x(3)(-7.84u^2 - 8v^2 - 16.8uv + 4v + 4) \right] \\ &dudv \end{aligned} \quad (92)$$

All other integrals can be calculations in the similar way.

VI. RESULT AND DISCUSSION

Resultant integral is given by:

$$\begin{aligned} \int L\sqrt{-g} dx dy &= \\ &(3.6 - 1.74\omega^2 - 1.64\gamma^2)A_x^2(1) - (1.6\omega^2 + 4.0\gamma^2)A_x^2(2) + \\ &(4.2 - 2.6\gamma^2 - 2.2\omega^2)A_x^2(3) - (1.65\omega^2 + 6.2\gamma^2)A_y^2(1) + \\ &(2.62 - 1.4\omega^2 - 8.5\gamma^2)A_y^2(2) + (4.8 - 6.6\gamma^2 + 1.9\omega^2)A_y^2(3) + \\ &(3.7 - 1.4\omega^2)A_z^2(1) + (4.3 - 1.9\omega^2)A_z^2(2) + (4.8 - 2.84\omega^2) \\ &A_z^2(3) - (1.58\omega^2 + 4.8\gamma^2)A_x(1)A_x(2) - (1.84 + 4.56\gamma^2) \\ &A_x(2)A_x(3) + (2.6 - 4.7\gamma^2)A_x(1)A_x(3) - (4.8\omega^2 + 8.7\gamma^2) \\ &A_y(1)A_y(2) + (1.7 - 2.2\omega^2 - 3.4\gamma^2)A_y(2)A_y(3) - (2.0 + \\ &1.9\gamma^2)A_y(1)A_y(3) + (4.2 - 1.9\omega^2)A_z(1)A_z(2) + (4.4 - \\ &1.09\omega^2)A_z(3)A_z(2) + (4.9 - 1.84\omega^2)A_z(1)A_z(3) - (4.1259 \\ &+ 1.27\gamma^2)V_1^2 - (3.3 + 3.2\gamma^2)V_2^2 - (4.683 + 2.424\gamma^2)V_3^2 \\ &+ (4.7 - 3.0\gamma^2)V_2V_3 + (4.08 - 2.518\gamma^2)V_1V_2 + (2.875 - \\ &3.125\gamma^2)V_1V_3 - (1.8566j\omega\gamma)V_1A_z(1) - (1.678j\omega\gamma)V_1A_z(2) \\ &- (3.424j\omega\gamma)V_1A_z(3) - (1.974j\omega\gamma)V_2A_z(1) - (3.43j\omega\gamma)V_2 \\ &A_z(2) - (6.353j\omega\gamma)V_2A_z(3) - (4.757j\omega\gamma)V_3A_z(1) - (8.7 \\ &j\omega\gamma)V_3A_z(2) - (4.7j\omega\gamma)V_3A_z(3) + 1.5j\omega V_1A_x(1) + 7.5j\omega \\ &V_2A_x(1) + 3.7j\omega V_1A_x(2) + 4.3j\omega V_1A_x(3) + 8.9j\omega V_1A_y(1) + \\ &4.4j\omega V_2A_y(1) + .09j\omega V_3A_y(2) + 3.7j\omega V_2A_y(2) + 6.8j\omega V_3 \\ &A_y(3) + 4.7j\omega V_2A_y(3) + 9.7j\omega V_1A_y(3) + 3.6j\omega V_2A_x(2) \\ &+ 6.7j\omega V_2A_x(3) + 4.0j\omega V_1A_y(2) - 4.4j\omega V_3A_x(1) - 3.2j\omega \\ &V_3A_x(2) - 5.8j\omega V_3A_x(3) - 3.4j\omega V_3A_y(1) - .7A_x(1)A_y(2) \\ &- 1.4A_x(2)A_y(3) - 8.41A_x(1)A_y(3) - 5.4A_x(3)A_y(3) \\ &- 6.42\gamma A_z(1)A_y(1) - 1.4\gamma A_z(1)A_y(2) - 3.2\gamma A_z(1)A_y(3) \\ &- 3.6\gamma A_z(3)A_y(1) - 6.0\gamma A_z(3)A_y(2) - 2.4\gamma A_z(3)A_y(3) - \\ &4.4\gamma A_z(1)A_x(1) - 1.5\gamma A_z(1)A_x(3) - 3.4\gamma A_z(2)A_x(2) \\ &- 2.0\gamma A_z(3)A_x(1) - 3.4\gamma A_z(3)A_x(3) - 1.4\gamma A_z(1)A_x(2) - \\ &.4\gamma A_z(2)A_x(1) - .4\gamma A_z(2)A_x(3) - 4.5\gamma A_z(3)A_x(2) \end{aligned} \quad (93)$$

Equation 93 results in a matrix of size 12 X 12. In the general formulation of waveguide problem, using a variational

principle applied to four potential, we still get a term linear in γ , i.e. the eigen problem is of the form

$$\det(A + \gamma B + \gamma^2 C) = 0$$

However, if we take in to account the Lorentz Gauge Condition, this linear term disappears. It is evident from the wave-equations:

$$\nabla^2 \vec{A} + \omega^2 \mu \epsilon \vec{A} = 0$$

$$\nabla^2 \vec{V} + \omega^2 \mu \epsilon \vec{V} = 0$$

Which yields

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \omega^2 \mu \epsilon + \gamma^2 \right) \vec{A} = 0$$

In the presence of the gravitational field, however, the term linear in γ can not be made to vanish even after applying the general relativity gauge condition.

$$(A^\mu \sqrt{-g})_{,\mu} = 0$$

VII. CONCLUSION

In this paper, the expressions for the action functional of the electromagnetic field have been written in the presence of a weak gravitational field. The cross-section of the waveguide has been partitioned in to small triangles and the fields inside each triangle have been expressed as linear combinations of the vertex fields using linear test functions. The action integral reduces to a quadratic function of the vertex fields. In this function, first and second powers of the frequency and propagation constant appear. For a fixed frequency, minimization of the action function with respect to the vertex field values leads to an eigen equation of the form

$$\det(A + \gamma B + \gamma^2 C) = 0$$

Where γ is the propagation constant. In the absence of gravity, the term γB is absent. Numerical techniques to solve the later problem need to be investigated. Sample matrices A , B , and C have been computed for special typical example in this paper. This work also gives a path way to the future work in which one can develop a numerical algorithm and/or software to solve this problem fully (up to the eigen value).

ACKNOWLEDGMENT

The authors would like to thank Prof. Raj Senani for his constant encouragement and provision of facilities for this research work.

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