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The Elliptic Curves $y^2 = x^3 - t^2x$ over $\mathbf{F}_p$

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***Abstract*—**Let p be a prime number, $\mathbf{F}_p$ be a finite field and $t \in \mathbf{F}_p^* = \mathbf{F}_p - \{0\}$. In this paper we obtain some properties of elliptic curves $E_{p,t} : y^2 = y^2 = x^3 - t^2x$ over $\mathbf{F}_p$. In the first section we give some notations and preliminaries from elliptic curves. In the second section we consider the rational points (x, y) on $E_{p,t}$. We give a formula for the number of rational points on $E_{p,t}$ over $\mathbf{F}_p^n$ for an integer $n \geq 1$. We also give some formulas for the sum of $x-$and $y-$coordinates of the points (x, y) on $E_{p,t}$. In the third section we consider the rank of $E_t : y^2 = x^3 - t^2x$ and its $2-$isogenous curve $\overline{E}_t$ over $\mathbf{Q}$. We proved that the rank of E_t and $\overline{E}_t$ is 2 over $\mathbf{Q}$. In the last section we obtain some formulas for the sums $\sum_{t\in\mathbf{F}_p^*} a_{p,t}^n$ for an integer $n \geq 1$, where $a_{p,t}$ denote the trace of Frobenius.

***Keywords*—**elliptic curves over finite fields, rational points on elliptic curves, rank, trace of Frobenius.

I. Introduction

Mordell began his famous paper [13] with the words *Mathematicians have been familiar with very few questions for so long a period with so little accomplished in the way of general results, as that of finding the rational points on elliptic curves*. The history of elliptic curves is a long one, and exciting applications for elliptic curves continue to be discovered. Recently, important and useful applications of elliptic curves have been found for cryptography [6,11,12], for factoring large integers [9], and for primality proving [1,5].The mathematical theory of elliptic curves was also crucial in the proof of Fermat's Last Theorem [19].

Let q be a positive integer, $\mathbf{F}_q$ be a finite field and let $\overline{\mathbf{F}}_q$ denote the algebraic closure of $\mathbf{F}_q$ with char$(\overline{\mathbf{F}}_q) \neq 2, 3$. An elliptic curve E over $\mathbf{F}_q$ is defined by an equation

$$E_{q,a,b} : y^2 = x^3 + ax + b,$$

where $a, b \in \mathbf{F}_q$ and $4a^3 + 27b^2 \neq 0$. We can view an elliptic curve $E_{q,a,b}$ as a curve in projective plane $\mathbf{P}^2$, with a homogeneous equation $y^2z = x^3 + axz^2 + bz^3$, and one point at infinity, namely $(0, 1, 0)$. This point ∞ is the point where all vertical lines meet. We denote this point by O. Let

$$E_{q,a,b}(\mathbf{F}_q) = \{(x, y) \in \mathbf{F}_q \times \mathbf{F}_q : y^2 = x^3 + ax + b\} \cup \{O\}$$

denote the set of rational points (x, y) on $E_{q,a,b}$. Then it is a subgroup of $E_{q,a,b}$. The order of $E_{q,a,b}(\mathbf{F}_q)$, denoted by $\#E_{q,a,b}(\mathbf{F}_q)$, is defined as the number of the rational points on $E_{q,a,b}$ (for further details see [15,17,18]), and is given by

$$\begin{aligned}\#E_{q,a,b}(\mathbf{F}_q) &= 1 + \sum_{x\in\mathbf{F}_q} \left(1 + \frac{x^3 + ax + b}{\mathbf{F}_q}\right) \quad (1)\\ &= q + 1 + \sum_{x\in\mathbf{F}_q} \left(\frac{x^3 + ax + b}{\mathbf{F}_q}\right),\end{aligned}$$

where $(\frac{\cdot}{\mathbf{F}_q})$ denotes the Legendre symbol.

Let

$$\#E_{q,a,b}(\mathbf{F}_q) = q + 1 - a_{q,a,b}. \quad (2)$$

Then $a_{q,a,b}$ is called the trace of Frobenius and satisfies the inequality

$$|a_{q,a,b}| \leq 2\sqrt{q}$$

known as the Hasse interval [18, p.91]. The formula (1) can be generalized to any field $\mathbf{F}_{q^n}$ for an integer $n \geq 2$ [18, p.97]. Let $\#E_{q,a,b}(\mathbf{F}_q) = q + 1 - a_{q,a,b}$ and let

$$X^2 - a_{q,a,b}X + q = (X - \alpha)(X - \beta). \quad (3)$$

Then the order of $E_{q,a,b}$ over $\mathbf{F}_{q^n}$ is

$$\#E_{q,a,b}(\mathbf{F}_{q^n}) = q^n + 1 - (\alpha^n + \beta^n). \quad (4)$$

II. Rational Points on Elliptic Curves $E_{p,t} : y^2 = x^3 - t^2x$ Over $\mathbf{F}_p$.

In [16], we consider the elliptic curves $E_{p,\lambda} : y^2 = x(x-1)(x - \lambda)$ over $\mathbf{F}_p$ for $\lambda \neq 0, 1$, where p is a prime number and $\mathbf{F}_p$ is a finite field. We consider the rational points on $E_{p,\lambda}$ and also its rank over $\mathbf{Q}$. In the present paper we consider the elliptic curves

$$E_{p,t} : y^2 = x^3 - t^2x \quad (5)$$

over $\mathbf{F}_p$ for an integer $t \in \mathbf{F}_p^*$. This elliptic curve was studied by Lemmermeyer and Mollin [8] in the sense of its Tate-Shafarevich group. Here we only consider its rational points, rank and trace of Forbenius.

Let Q_p denote the set of quadratic residues. Let $Q_p^{4,+}$ denote the set of 4th power of elements of $\mathbf{F}_p^*$ and let $Q_p^{4,-} = \mathbf{F}_p^* - Q_p^{4,+}$. Set $Q_p^4 = Q_p^{4,+} \cup Q_p^{4,-}$. Then $\#Q_p^{4,+} = \#Q_p^{4,-} = \frac{p-1}{4}$ and $\#Q_p^4 = \frac{p-1}{2}$. Recall that the order of $E_{p,t} : y^2 = x^3 - t^2x$ over $\mathbf{F}_p$ is given in [18, p.105] by

1. If $p \equiv 3\,(mod\,4)$, then $\#E_{p,t}(\mathbf{F}_p) = p + 1$.
2. If $p \equiv 1\,(mod\,4)$, write $p = a^2 + b^2$, where a and b are integers with b is even and $a + b \equiv 1\,(mod\,4)$, then

$$\#E_{p,t}(\mathbf{F}_p) = \begin{cases} p + 1 - 2a & if\ k \in Q_p^{4,+} \\ p + 1 + 2a & if\ k \in Q_p^{4,-} \\ p + 1 \pm 2b & if\ k \notin Q_p. \end{cases}$$

First we generalize this result to any field $\mathbf{F}_{p^n}$ for an integer $n \geq 2$.

Theorem 2.1: Let $E_{p,t} : y^2 = x^3 - t^2x$ be an elliptic curve over $\mathbf{F}_p$.

1) If $p \equiv 3(mod\,4)$, then

$$\#E_{p,t}(\mathbf{F}_{p^n}) = \begin{cases} (p^{\frac{n}{2}} - 1)^2 & if\ n \equiv 0\,(mod\,4) \\ p^n + 1 & if\ n \equiv 1, 3\,(mod\,4) \\ (p^{\frac{n}{2}} + 1)^2 & if\ n \equiv 2\,(mod\,4). \end{cases}$$

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2) If $p \equiv 1(mod\,4)$, then $\#E_{p,t}(\mathbf{F}_{p^n}) = p^n + 1 -$
$$\begin{cases} (a+ib)^n + (a-ib)^n & if\ t^2 \in Q_p^{4,+} \\ (-a+ib)^n + (-a-ib)^n & if\ t^2 \in Q_p^{4,-}. \end{cases}$$

Proof: 1. Let $p \equiv 3(mod\,4)$. Then $\#E_{p,t}(\mathbf{F}_p) = p+1$. Hence $a_{p,t} = 0$ by (2). Let
$$X^2 + p = (X-\alpha)(X-\beta)$$
for $\alpha = i\sqrt{p}$ and $\beta = -i\sqrt{p}$ by (3).

Let $n \equiv 0\,(mod\,4)$, i.e. $n = 4m$ for an integer $m \geq 1$. Then we get
$$\begin{aligned} \alpha^n + \beta^n &= (i\sqrt{p})^{4m} + (-i\sqrt{p})^{4m} \\ &= i^{4m}(\sqrt{p})^{4m} + (-i)^{4m}(\sqrt{p})^{4m} \\ &= p^{2m} + p^{2m} \\ &= 2p^{2m} \\ &= 2p^{\frac{n}{2}}. \end{aligned}$$
Therefore $\#E_{p,t}(\mathbf{F}_{p^n}) = p^n + 1 - (\alpha^n + \beta^n) = p^n + 1 - 2p^{\frac{n}{2}} = (p^{\frac{n}{2}} - 1)^2$ by (4).

Let $n \equiv 1(mod\,4)$, say $n = 1+4m$. Then we get
$$\begin{aligned} \alpha^n + \beta^n &= (i\sqrt{p})^n + (-i\sqrt{p})^n \\ &= i^{4m+1}(\sqrt{p})^{4m+1} + (-i)^{4m+1}(\sqrt{p})^{4m+1} \\ &= i(\sqrt{p})^{4m+1} + (-i)(\sqrt{p})^{4m+1} \\ &= 0. \end{aligned}$$
Therefore $\#E_{p,t}(\mathbf{F}_{p^n}) = p^n + 1 - (\alpha^n + \beta^n) = p^n + 1$.

Let $n \equiv 2(mod\,4)$, say $n = 2+4m$. Then we get
$$\begin{aligned} \alpha^n + \beta^n &= (i\sqrt{p})^n + (-i\sqrt{p})^n \\ &= i^{4m+2}(\sqrt{p})^{4m+2} + (-i)^{4m+2}(\sqrt{p})^{4m+2} \\ &= (-1)p^{2m+1} + (-1)p^{2m+1} \\ &= -2p^{2m+1} \\ &= -2p^{\frac{n}{2}}. \end{aligned}$$
Therefore $\#E_{p,t}(\mathbf{F}_{p^n}) = p^n + 1 - (\alpha^n + \beta^n) = p^n + 1 + 2p^{\frac{n}{2}} = (p^{\frac{n}{2}} + 1)^2$.

Finally, let $n \equiv 3(mod\,4)$, say $n = 3+4m$. Then we get
$$\begin{aligned} \alpha^n + \beta^n &= (i\sqrt{p})^n + (-i\sqrt{p})^n \\ &= i^{4m+3}(\sqrt{p})^{4m+3} + (-i)^{4m+3}(\sqrt{p})^{4m+3} \\ &= (-i)(\sqrt{p})^{4m+3} + i(\sqrt{p})^{4m+3} \\ &= 0. \end{aligned}$$
Therefore $\#E_{p,t}(\mathbf{F}_{p^n}) = p^n + 1 - (\alpha^n + \beta^n) = p^n + 1$.

2. Let $p \equiv 1(mod\,4)$, and let $t^2 \in Q_p^{4,+}$. Then $\#E_{p,t}(\mathbf{F}_p) = p + 1 - 2a$ and hence $a_{p,t} = 2a$ by (2). Let
$$\begin{aligned} X^2 - 2aX + p &= (X-\alpha)(X-\beta) \\ &= X^2 - X(\alpha+\beta) + \alpha\beta. \end{aligned}$$
Then $2a = \alpha + \beta$ and $p = \alpha\beta$. Hence we get
$$\begin{aligned} 2a = \alpha + \frac{p}{\alpha} &\Leftrightarrow \alpha^2 - 2a\alpha + p = 0 \\ &\Leftrightarrow \alpha_{1,2} = \frac{2a \pm \sqrt{4a^2 - 4p}}{2} \\ &\Leftrightarrow \alpha_{1,2} = a \pm ib. \end{aligned}$$

Therefore
$$\alpha_1 = a + ib \Rightarrow \beta_1 = \frac{p}{\alpha_1} = a - ib$$
or
$$\alpha_2 = a - ib \Rightarrow \beta_2 = \frac{p}{\alpha_2} = a + ib.$$
Consequently in both cases, the order of $E_{p,t}$ over $\mathbf{F}_{p^n}$ is
$$\begin{aligned} \#E_{p,t}(\mathbf{F}_{p^n}) &= p^n + 1 - (\alpha^n + \beta^n) \\ &= p^n + 1 - [(a+ib)^n + (a-ib)^n]. \end{aligned}$$

Let $t^2 \in Q_p^{4,-}$. Then $\#E_{p,t}(\mathbf{F}_p) = p + 1 + 2a$ and hence $a_{p,t} = -2a$ by (2). Let
$$\begin{aligned} X^2 + 2aX + p &= (X-\alpha)(X-\beta) \\ &= X^2 - X(\alpha+\beta) + \alpha\beta. \end{aligned}$$
Then $-2a = \alpha + \beta$ and $p = \alpha\beta$. Hence we get
$$\begin{aligned} -2a = \alpha + \frac{p}{\alpha} &\Leftrightarrow \alpha^2 + 2a\alpha + p = 0 \\ &\Leftrightarrow \alpha_{1,2} = \frac{-2a \pm \sqrt{4a^2 - 4p}}{2} \\ &\Leftrightarrow \alpha_{1,2} = -a \pm ib. \end{aligned}$$
Therefore
$$\alpha_1 = -a + ib \Rightarrow \beta_1 = \frac{p}{\alpha_1} = -a - ib$$
or
$$\alpha_2 = -a - ib \Rightarrow \beta_2 = \frac{p}{\alpha_2} = -a + ib.$$
Consequently the order of $E_{p,t}$ over $\mathbf{F}_{p^n}$ is
$$\begin{aligned} \#E_{p,t}(\mathbf{F}_{p^n}) &= p^n + 1 - (\alpha^n + \beta^n) \\ &= p^n + 1 - [(-a+ib)^n + (-a-ib)^n]. \end{aligned}$$
This completes the proof. ■

In the following table some values of p, a and b is given.

p	a	b	p	a	b
5	1	2	229	15	2
13	3	2	233	13	8
17	1	4	241	15	4
29	5	2	257	1	16
37	1	6	269	13	10
41	5	4	277	9	14
53	7	2	281	5	16
61	5	6	293	17	2
73	3	8	313	13	12
89	5	8	317	11	14
97	9	4	337	9	16
101	1	10	349	5	18
109	3	10	353	17	8
113	7	8	373	7	18
137	11	4	389	17	10
149	7	10	397	19	6
157	11	6	401	1	20
173	13	2	409	3	20
181	9	10	421	15	14
193	7	12	433	17	12
197	1	14	449	7	20

In the following examples the orders of $E_{p,t} : y^2 = x^3 - t^2x$ over $\mathbf{F}_{p^n}$ are given for $2 \leq n \leq 15$.

Example 2.1: Let $p = 23$ and $t = 2$. Then the order of $E_{23,2} : y^2 = x^3 - 4x$ over $\mathbf{F}_{23^n}$ is

n	$\mathbf{F}_{23^n}$
2	576
3	12168
4	278784
5	6436344
6	148060224
7	3404825448
8	78310425600
9	1801152661464
10	41426524086336
11	952809757913928
12	21914624135948544
13	504036361936467384
14	11592836331348400704
15	266635235464391245608

Example 2.2: Let $p = 13$. Then $a = 3$ and $b = 2$. Let $t = 4$. Then $t^2 \equiv 3(mod\,13)$. So $t^2 \in Q_{13}^{4,+} = \{1, 3, 9\}$. Then the order of $E_{13,4} : y^2 = x^3 - 3x$ over $\mathbf{F}_{13^n}$ is

n	$\mathbf{F}_{13^n}$
2	160
3	2216
4	28800
5	372488
6	4830880
7	62757416
8	815731200
9	10604386564
10	137857808810
11	1792157762000
12	23298078210000
13	302875099300000
14	3937376432000000
15	51185893380000000

Similarly let $p = 13$ and $t = 11$. Then $t^2 \equiv 4(mod\,13)$. So $t^2 \in Q_{13}^{4,-}$. Therefore the order of $E_{13,11} : y^2 = x^3 - 4x$ over $\mathbf{F}_{13^n}$ is

n	$\mathbf{F}_{13^n}$
2	160
3	2180
4	28800
5	370100
6	4830880
7	62739620
8	815731200
9	106041612184
10	137857808810
11	1792163026000
12	23298078210000
13	302875113900000
14	3937376432000000
15	51185892640000000

Now we consider some properties of rational points on elliptic curve $E_{p,t}$.

Theorem 2.2: Let $[x]$ denote the $x-$coordinates of (x, y) on $E_{p,t}$. Then sum of $[x]$ on $E_{p,t}$ is

$$\sum\nolimits_{[x]} E_{p,t}(\mathbf{F}_p) = \sum \left(1 + \left(\frac{x^3 - t^2x}{\mathbf{F}_p}\right)\right).x$$

for all primes p

Proof: We know that

$$\left(\frac{x^3 - t^2x}{\mathbf{F}_p}\right) = \begin{cases} 0 & if\ \ x^3 - t^2x\ is\ zero \\ 1 & if\ \ x^3 - t^2x\ is\ a\ square \\ -1 & if\ \ x^3 - t^2x\ is\ not\ a\ square. \end{cases}$$

Let $\left(\frac{x^3 - t^2x}{\mathbf{F}_p}\right) = 0$. Then $x^3 - t^2x = 0$, and hence this equation has three solutions $x = 0, x = t$ and $x = -t$. Then $y^2 \equiv 0\,(mod\,p) \Leftrightarrow y \equiv 0\,(mod\,p)$. So for such a point x, we have a point $(x, 0)$ on $E_{p,t}$. Therefore we get $(x + 0).x = x$ is added to the sum.

Let $\left(\frac{x^3 - t^2x}{\mathbf{F}_p}\right) = 1$. Then $x^3 - t^2x$ is a square in $\mathbf{F}_p$. Let $x^3 - t^2x = k^2$ for any $k \in \mathbf{F}_p^*$. Then $y^2 \equiv k^2\,(mod\,p) \Leftrightarrow y = \pm k$, that is, for any point (x, k) on $E_{p,t}$, the point $(x, -k)$ is also on $E_{p,t}$. Therefore for each point x we have $(1 + 1).x = 2x$ is added to the sum.

Finally, let $\left(\frac{x^3 - t^2x}{\mathbf{F}_p}\right) = -1$. Then $x^3 - t^2x$ is not a square in $\mathbf{F}_p$. Therefore the equation $y^2 \equiv x^3 - t^2x(mod\,p)$ has no solution. Therefore for each point x, we have $(1 + (-1)).x = 0$ as we claimed. ■

Theorem 2.3: Let $[y]$ denote the $y-$coordinates of (x, y) on $E_{p,t}$.

1) If $p \equiv 3(mod\,4)$, then the sum of $[y]$ on $E_{p,t}$ is

$$\sum\nolimits_{[y]} E_{p,t}(\mathbf{F}_p) = \frac{p^2 - 3p}{2}.$$

2) If $p \equiv 1(mod\,4)$, then the sum of $[y]$ on $E_{p,t}$ is

$$\sum\nolimits_{[y]} E_{p,t}(\mathbf{F}_p) = \begin{cases} \frac{p^2-(2a+3)p}{2} & if\ t^2 \in Q_p^{4,+} \\ \frac{p^2+(2a-3)p}{2} & if\ t^2 \in Q_p^{4,-}. \end{cases}$$

Proof: 1. Let $p \equiv 3(mod\,4)$. Note that the cubic equation $x^3 - t^2x = 0$ has three solutions $x = 0, x = t$ and $x = -t$. For the other values of x, we have both x and $-x$. One of these gives two points. The one makes $x^3 - t^2x$ a square. So there are two values of y since $y^2 = x^3 - t^2x$ is square. Let $x^3 - t^2x = k^2$ for any $k \in \mathbf{F}_p^*$. Then we have $y^2 = k^2$ if and only if $y = k$ and $y = -k = p - k$. So the sum of these values of y is $k + (p - k) = p$. We know that there are $\frac{p-3}{2}$ points x such that $y^2 = x^3 - t^2x$ is a square. Therefore the sum of $y-$coordinates of all points (x, y) is

$$p\left(\frac{p-3}{2}\right) = \frac{p^2 - 3p}{2}.$$

2. Let $p \equiv 3(mod\,4)$. If $t^2 \in Q_p^{4,+}$, then $E_{p,t}(\mathbf{F}_p) = p+1-2a$. We know that the cubic equation $x^3 - t^2x = 0$ has three solutions $x = 0, x = t$ and $x = -t$, that is, there are three points $(0, 0), (t, 0), (-t, 0)$ on $E_{p,t}$. The sum of y-coordinates of these points is 0. Further we have to disregard the point ∞. Then there are $(p + 1 - 2a) - 4 = p - 2a - 3$ points (x, y) on

$E_{p,t}$ such that $y \neq 0$. Half of these points make $x^3 - t^2x$ a square, that is, there are $\frac{p-2a-3}{2}$ points x such that $x^3 - t^2x$ is a square. Let $x^3 - t^2x = k^2$ for any $k \in \mathbf{F}_p^*$. Then we have $y^2 = k^2$ if and only if $y = k$ and $y = -k = p - k$. So the sum of these values of y is $k + (p - k) = p$. Hence the sum of $y-$coordinates of all points (x, y) on $E_{p,t}$ is

$$p\left(\frac{p-2a-3}{2}\right) = \frac{p^2-(2a+3)p}{2}.$$

If $t^2 \in Q_p^{4,-}$, then $E_{p,t}(\mathbf{F}_p) = p + 1 + 2a$. The cubic equation $x^3 - t^2x = 0$ has three solutions $x = 0, x = t$ and $x = -t$, that is, there are three points $(0, 0), (t, 0), (-t, 0)$ on $E_{p,t}$ and the sum of y-coordinates of these points is 0. Further we have to disregard the point ∞. Then there are $(p+1+2a)-4 = p+2a-3$ points (x, y) on $E_{p,t}$ such that $y \neq 0$. Half of these points make $x^3 - t^2x$ a square, that is, there are $\frac{p+2a-3}{2}$ points x such that $x^3 - t^2x$ is a square. Let $x^3 - t^2x = k^2$ for any $k \in \mathbf{F}_p^*$. Then we have $y^2 = k^2$ if and only if $y = k$ and $y = -k = p-k$. So the sum of these values of y is $k + (p - k) = p$. Hence the sum of $y-$coordinates of all points (x, y) on $E_{p,t}$ is

$$p\left(\frac{p+2a-3}{2}\right) = \frac{p^2+(2a-3)p}{2}.$$

■

Theorem 2.4: Let $\mathbf{E}_{p,t} = \left\{E_{p,t} : t \in \mathbf{F}_p^*\right\}$ denote the set of all elliptic curves $E_{p,t}$ over $\mathbf{F}_p$. Then

$$\sum\nolimits_{t\in\mathbf{F}_p^*} \#\mathbf{E}_{p,t}(\mathbf{F}_p) = \frac{p^2-1}{2}$$

for all primes p.

Proof: Note that there are $\frac{p-1}{2}$ elliptic curves $E_{p,t}$ in $\mathbf{E}_{p,t}$ over $\mathbf{F}_p$. We know that the order of $E_{p,t}$ over $\mathbf{F}_p$ is $p+1$ when $p \equiv 3(mod\,4)$. Therefore the total number of the points (x, y) on all elliptic curves $E_{p,t}$ in $\mathbf{E}_{p,t}$ over $\mathbf{F}_p$ is

$$(p+1)\left(\frac{p-1}{2}\right) = \frac{p^2-1}{2}.$$

Let $p \equiv 1(mod\,4)$. If $t^2 \in Q_p^{4,+}$, then the order of $E_{p,t}$ over $\mathbf{F}_p$ is $p + 1 - 2a$, and if $t^2 \in Q^{4,-}$, then the order of $E_{p,t}$ over $\mathbf{F}_p$ is $p + 1 + 2a$. Further the order of $Q_p^{4,+}$ and $Q_p^{4,-}$ is $\frac{p-1}{4}$. Therefore the total number of the points (x, y) on all elliptic curves $E_{p,t}$ in $\mathbf{E}_{p,t}$ over $\mathbf{F}_p$ is

$$\begin{aligned}&\frac{p-1}{4}(p+1-2a) + \frac{p-1}{4}(p+1+2a)\\ &= \frac{p-1}{4}(p+1-2a+p+1+2a)\\ &= \frac{p-1}{4}(2p+2)\\ &= \frac{p^2-1}{2}.\end{aligned}$$

as we claimed. ■

Theorem 2.5: The sum of $[y]$ in $\mathbf{E}_{p,t}(\mathbf{F}_p)$ is

$$\sum\nolimits_{t\in\mathbf{F}_p^*} \mathbf{E}_{p,t}(\mathbf{F}_p) = \frac{p^3-4p^2+3p}{4}$$

for all primes p.

Proof: Let $p \equiv 3(mod\,4)$. We know that the sum of $[y]$ is $\frac{p^2-3p}{2}$. Further there are $\frac{p-1}{2}$ elliptic curves in $\mathbf{E}_{p,t}$. Therefore the sum of $[y]$ of all points (x, y) on all elliptic curves $E_{p,t}$ in $\mathbf{E}_{p,t}(\mathbf{F}_p)$ is

$$\left(\frac{p-1}{2}\right)\left(\frac{p^2-3p}{2}\right) = \frac{p^3-4p^2+3p}{4}.$$

Let $p \equiv 1(mod\,4)$. We know that there are $\frac{p-1}{4}$ elements in both $Q_p^{4,+}$ and $Q_p^{4,-}$. Further by Theorem 2.3, if $t^2 \in Q_p^{4,+}$, then the the sum of $[y]$ of all points on elliptic curves $E_{p,t}$ is $\frac{p^2-(2a+3)p}{2}$, and if $t^2 \in Q_p^{4,-}$, then the the sum of $[y]$ of all points on elliptic curves $E_{p,t}$ is $\frac{p^2+(2a-3)p}{2}$. Therefore the sum of $[y]$ of all points on elliptic curves $E_{p,t}$ is

$$\begin{aligned}&\left(\frac{p-1}{4}\right)\left[\frac{p^2-(2a+3)p}{2} + \frac{p^2+(2a-3)p}{2}\right]\\ &= \left(\frac{p-1}{4}\right)\left(\frac{2p^2-6p}{2}\right)\\ &= \frac{p^3-4p^2+3p}{4}.\end{aligned}$$

■

III. RANK OF $E_t : y^2 = x^3 - t^2x$ OVER $\mathbf{Q}$.

Let E be an elliptic curve over $\mathbf{Q}$. By Mordell's theorem, we know that $E(\mathbf{Q})$ is a finitely generated abelian group, that is, $E(\mathbf{Q}) = E(\mathbf{Q})_{tors} \times \mathbf{Z}^r$. Further by Mazur's theorem,

$$E(Q)_{tors} \cong \mathbf{Z}/n\mathbf{Z} \;\; for \;\; 1 \leq n \leq 10 \;\; or \;\; n = 12$$

or

$$E(Q)_{tors} \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z} \;\; for \;\; 1 \leq n \leq 4.$$

On the other hand, it is not known that what values of rank r are possible for elliptic curves over $\mathbf{Q}$. The main idea is that a rank can be arbitrary large. The current record is an example of elliptic curve with rank ≥ 28, found by Elkies [3] in 2006. The previous record one with rank ≥ 24, found by Martin and McMillen [10] in 2000. The highest rank of an elliptic curve which is known exactly (not only a lower bound for rank) is equal to 18, and it was found by Elkies [3] in 2006. It improves previous records due to Kretschmer [7](rank = 10), Schneiders-Zimmer [14](rank = 11), Fermigier [4](rank = 14), Dujella [2](rank = 15) and Elkies [3](rank = 17).

Recall that the $2-$isogenous curve of an elliptic curve

$$E_{a,b} : y^2 = x^3 + ax^2 + bx$$

is given by

$$\overline{E}_{a,b} : y^2 = x^3 + \overline{a}x^2 + \overline{b}x, \tag{6}$$

where $\overline{a} = -2a$ and $\overline{b} = a^2 - 4b$. Then there exists a $2-$isogeny ϕ from $E_{a,b}$ to $\overline{E}_{a,b}$ given by

$$\phi : E_{a,b} \rightarrow \overline{E}_{a,b}, \quad \phi(x, y) = \left(\frac{y^2}{x^2}, \frac{y(b-x^2)}{x^2}\right).$$

Conversely, there exists a dual isogeny ψ from $\overline{E}_{a,b}$ to $E_{a,b}$ given by

$$\psi : \overline{E}_{a,b} \to E_{a,b}, \quad \psi(x, y) = \left(\frac{y^2}{4x^2}, \frac{y(a^2 - 4b - x^2)}{8x^2} \right).$$

Let

$$2^r = \frac{\#\alpha(E_{a,b}(\mathbf{Q}))\#\overline{\alpha}(\overline{E}_{a,b}(\mathbf{Q}))}{4}, \tag{7}$$

where α is a homomorphism

$$\alpha : E_{a,b}(\mathbf{Q}) \to \mathbf{Q}^*/\mathbf{Q}^{*2}$$

such that

$$\begin{aligned} &0 \to 1\,(mod\mathbf{Q}^{*2}) \\ &(0,0) \to b\,(mod\mathbf{Q}^{*2}) \\ &(x,y) \to x\,(mod\mathbf{Q}^{*2}), \end{aligned}$$

where $\mathbf{Q}^*$ is the multiplicative group of rational units, and $\mathbf{Q}^{*2}$ is the subgroup consisting of perfect squares. So $\mathbf{Q}^*/\mathbf{Q}^{*2}$ is like the non-zero rational numbers, with two elements identified if their quotient is the square of a rational number. We shall call α the Weil map (in fact it is actually a group homomorphism). We found the Weil map from the group of rational points on $E_{a,b}$ to the group $\mathbf{Q}^*/\mathbf{Q}^{*2}$ by studying the rational points on torsors

$$T^{(\psi)}(b_1) : N^2 = b_1 M^4 + aM^2e^2 + b_2 e^4, \tag{8}$$

where b_1 runs through the square free divisors of $b = b_1 b_2$. Then $\alpha(E_{a,b}(\mathbf{Q}))$ consists of $b(mod\,\mathbf{Q}^{*2})$, together with those $b_1(mod\,\mathbf{Q}^{*2})$ such that (8) has a solution (N, M, e).

Similarly, $\overline{\alpha}$ is an Weil map, which is from the group of rational points on $\overline{E}_{a,b}$ to the group $\mathbf{Q}^*/\mathbf{Q}^{*2}$ by studying the rational points on torsors

$$T^{(\phi)}(\overline{b}_1) : N^2 = \overline{b}_1 M^4 + \overline{a}M^2e^2 + \overline{b}_2 e^4, \tag{9}$$

where $\overline{b}_1$ runs through the square free divisors of $\overline{b} = \overline{b}_1\overline{b}_2$. Then $\overline{\alpha}(\overline{E}_{a,b}(\mathbf{Q}))$ consists of $\overline{b}(mod\,\mathbf{Q}^{*2})$, together with those $\overline{b}_1(mod\,\mathbf{Q}^{*2})$ such that (9) has a solution (N, M, e).

Note that the $2-$isogenous curve of our curve $E_t : y^2 = x^3 - t^2x$ is

$$\overline{E}_t : y^2 = x^3 + 4t^2x \tag{10}$$

if t is odd, or

$$\overline{E}_t : y^2 = x^3 + \frac{t^2}{4}x \tag{11}$$

if t is even by (6). Now we can consider the rank of E_t and $\overline{E}_t$ over $\mathbf{Q}$.

Theorem 3.1: The rank of E_t and $\overline{E}_t$ over $\mathbf{Q}$ is 2.

Proof: Elliptic curves with a rational point of order 2 like our curves $E_t : y^2 = x^3 - t^2x$ come attached with a $2-$isogeny $\phi : E_t \to \overline{E}_t$ (depending of choice of point if E_t has three rational points of order 2) as we mentioned above.

Now consider the our elliptic curve $E_t : y^2 = x^3 - t^2x$. Then there are four possibilities for $b_1 = -t^2$ which are ± 1 and $\pm t$.

If $b_1 = 1$, then the equation

$$N^2 = M^4 - t^2e^4$$

has a solution $(N, M, e) = (t^2, t, 0)$. If $b_1 = -1$, then the equation

$$N^2 = -M^4 + t^2e^4$$

has a solution $(N, M, e) = (t, 0, -1)$. If $b_1 = t$, then the equation

$$N^2 = tM^4 - te^4$$

has a solution $(N, M, e) = (0, t^2, t^2)$ and if $b_1 = -t$, then the equation

$$N^2 = -tM^4 + te^4$$

has a solution $(N, M, e) = (0, t^2, -t^2)$. So

$$\begin{gathered} \alpha(E_t(\mathbf{Q})) = \{\pm 1, \pm t \ \ (mod\mathbf{Q}^{*2})\} \ \ and \\ \#\alpha(E_t(\mathbf{Q})) = 4 \end{gathered} \tag{12}$$

by (8).

Now we consider the $2-$isogeny of E_t. If t is odd, then the $2-$isogenous curve of E_t is $\overline{E}_t : y^2 = x^3 + 4t^2x$ by (10). Then there are four possibilities for $\overline{b}_1 = 4t^2$ which are ± 1 and $\pm 2t$.

If $\overline{b}_1 = 1$, then the equation

$$N^2 = M^4 + 4t^2e^4$$

has a solution $(N, M, e) = (2t, 0, 1)$. If $\overline{b}_1 = -1$, then the equation

$$N^2 = -M^4 - 4t^2e^4$$

has no solution (N, M, e) since its right-hand side is strictly negative. If $\overline{b}_1 = 2t$, then the equation

$$N^2 = 2tM^4 + 2te^4$$

has no solution (N, M, e) and if $\overline{b}_1 = -2t$, then the equation

$$N^2 = -2tM^4 - 2te^4$$

has no solution (N, M, e) since its right-hand side is strictly negative. Hence

$$\overline{\alpha}(\overline{E}_t(\mathbf{Q})) = \{1 \ \ (mod\mathbf{Q}^{*2})\} \ \ and \ \ \#\overline{\alpha}(\overline{E}_t(\mathbf{Q})) = 1$$

by (9).

If t is even, then the $2-$isogenous curve of E_t is $\overline{E}_t : y^2 = x^3 + \frac{t^2}{4}x$ by (11). Let $t = 2k$ for integers $k \geq 1$. Then $\overline{E}_t$ becomes an elliptic curve has the form $\overline{E}_t : y^2 = x^3 + k^2x$. Then there are four possibilities for $\overline{b}_1 = k^2$ which are ± 1 and $\pm k$.

If $\overline{b}_1 = 1$, then the equation

$$N^2 = M^4 + k^2e^4$$

has a solution $(N, M, e) = (k, 0, 1)$. If $\overline{b}_1 = -1$, then the equation

$$N^2 = -M^4 - k^2e^4$$

has no solution (N, M, e) since its right-hand side is strictly negative. If $\overline{b}_1 = k$, then the equation

$$N^2 = kM^4 + ke^4$$

has no solution and if $\bar{b}_1 = -k$, then the equation

$$N^2 = -kM^4 - ke^4$$

has no solution since its right-hand side is strictly negative. Hence

$$\overline{\alpha}(\overline{E}_t(\mathbf{Q})) = \{1 \ (mod\, \mathbf{Q}^{*2})\} \ \ and \ \ \#\overline{\alpha}(\overline{E}_t(\mathbf{Q})) = 1$$

by (9). So in both cases, i.e. whether t is even or odd, we have

$$\begin{aligned} \overline{\alpha}(\overline{E}_t(\mathbf{Q})) = \{1 \ (mod\, \mathbf{Q}^{*2})\} \ \ and \\ \#\overline{\alpha}(\overline{E}_t(\mathbf{Q})) = 1. \end{aligned} \tag{13}$$

Applying (12) and (13), we get

$$\begin{aligned} 2^r &= \frac{\#\alpha(E_t(\mathbf{Q})).\#\overline{\alpha}(\overline{E}_t(\mathbf{Q}))}{4} \\ &= \frac{4.1}{4} \\ &= 4 \\ &\Leftrightarrow r = 2. \end{aligned}$$

Consequently, the rank of $E_t(\mathbf{Q})$ and $\overline{E}_t(\mathbf{Q})$ over $\mathbf{Q}$ is 2 by (7) as we claimed. ■

IV. Trace of Frobenius of Elliptic Curves $E_{p,t} : y^2 = x^3 - t^2x$.

Let $a_{p,t}$ denote the trace of Frobenius of elliptic curve $E_{p,t} : y^2 = x^3 - t^2x$. Then by (2), we get $\#E_{p,t}(\mathbf{F}_p) = p+1-a_{p,t}$. In this section we will obtain some relations on the sums

$$\sum_{t\in\mathbf{F}_p^*} a_{p,t}^n$$

for an integer $n \geq 1$.

Theorem 4.1: Let $a_{p,t}$ denote the trace of Frobenius of elliptic curve $E_{p,t}$.

1) If $p \equiv 3(mod\, 4)$, then

$$\sum_{t\in\mathbf{F}_p^*} a_{p,t}^n = 0$$

for all integers $n \geq 1$.

2) Let $p \equiv 1(mod\, 4)$, write $p = a^2 + b^2$.

i. If $a + b \equiv 1(mod\, 4)$, then

$$\sum_{t^2\in Q^{4,+}} a_{p,t}^n = 2^{n-2}a^n(p-1)$$

and

$$\sum_{t^2\in Q^{4,-}} a_{p,t}^n = (-1)^n 2^{n-2}a^n(p-1).$$

ii. If $a + b \equiv 3(mod\, 4)$, then

$$\sum_{t^2\in Q^{4,+}} a_{p,t}^n = (-1)^n 2^{n-2}a^n(p-1)$$

and

$$\sum_{t^2\in Q^{4,-}} a_{p,t}^n = 2^{n-2}a^n(p-1).$$

for all integers $n \geq 1$.

Proof: 1. Let $p \equiv 3(mod\, 4)$. Then $E_{p,t}(\mathbf{F}) = p + 1$. So $a_{p,t} = 0$ by (2). Consequently all powers of sums of $a_{p,t} = 0$ is 0, that is

$$\sum_{t\in\mathbf{F}_p^*} a_{p,t}^n = 0$$

for all integers $n \geq 1$.

2. Let $p \equiv 1(mod\, 4)$ and let $a+b \equiv 1(mod\, 4)$. If $t^2 \in Q_p^{4,+}$, then $a_{p,t} = 2a$ and hence the sum of $a_{p,t}^n$ over $t^2 \in Q_p^{4,+}$ is

$$\begin{aligned} \sum_{t^2\in Q^{4,+}} a_{p,t}^n &= \#Q_p^{4,+}.\sum_{t^2\in Q^{4,+}} a_{p,t}^n \\ &= \#Q_p^{4,+}.(2a)^n \\ &= \frac{p-1}{4}.2^n a^n \\ &= 2^{n-2}(p-1)a^n. \end{aligned}$$

If $t^2 \in Q_p^{4,-}$, then $a_{p,t} = -2a$ and hence the sum of $a_{p,t}^n$ over $t^2 \in Q_p^{4,-}$ is

$$\begin{aligned} \sum_{t^2\in Q^{4,-}} a_{p,t}^n &= \#Q_p^{4,-}.\sum_{t^2\in Q^{4,-}} a_{p,t}^n \\ &= \#Q_p^{4,-}.(-2a)^n \\ &= \frac{p-1}{4}.(-1)^n 2^n a^n \\ &= (-1)^n 2^{n-2}(p-1)a^n. \end{aligned}$$

Let $a + b \equiv 3(mod\, 4)$. If $t^2 \in Q_p^{4,+}$, then $a_{p,t} = -2a$ and hence the sum of $a_{p,t}^n$ over $t^2 \in Q_p^{4,+}$ is

$$\begin{aligned} \sum_{t^2\in Q^{4,+}} a_{p,t}^n &= \#Q_p^{4,+}.\sum_{t^2\in Q^{4,+}} a_{p,t}^n \\ &= \#Q_p^{4,+}.(-2a)^n \\ &= \frac{p-1}{4}.(-1)^n 2^n a^n \\ &= (-1)^n 2^{n-2}(p-1)a^n. \end{aligned}$$

If $t^2 \in Q_p^{4,-}$, then $a_{p,t} = 2a$ and hence the sum of $a_{p,t}^n$ over $t^2 \in Q_p^{4,-}$ is

$$\begin{aligned} \sum_{t^2\in Q^{4,-}} a_{p,t}^n &= \#Q_p^{4,-}.\sum_{t^2\in Q^{4,-}} a_{p,t}^n \\ &= \#Q_p^{4,-}.(2a)^n \\ &= \frac{p-1}{4}.2^n a^n \\ &= 2^{n-2}(p-1)a^n. \end{aligned}$$

■

Form above theorem we can give the following theorem.

Theorem 4.2: If $p \equiv 1(mod\, 4)$, then

$$\sum_{t\in\mathbf{F}_p^*} a_{p,t}^n = \begin{cases} 0 & if \ \ n \ is \ odd \\ 2^{n-1}a^n(p-1) & if \ \ n \ is \ even \end{cases}$$

for all integers $n \geq 1$.

Proof: Let $p \equiv 1(mod\, 4)$ and let $a+b \equiv 1(mod\, 4)$. Then we know that

$$\sum_{t^2\in Q^{4,+}} a_{p,t}^n = 2^{n-2}a^n(p-1)$$

and

$$\sum_{t^2 \in Q^{4,-}} a_{p,t}^n = (-1)^n 2^{n-2} a^n (p-1).$$

If n is odd, then

$$\begin{aligned}\sum_{t \in \mathbf{F}_p^*} a_{p,t}^n &= \sum_{t^2 \in Q^{4,+}} a_{p,t}^n + \sum_{t^2 \in Q^{4,-}} a_{p,t}^n \\ &= 2^{n-2} a^n (p-1) - 2^{n-2} a^n (p-1) \\ &= 0.\end{aligned}$$

If n is even, then

$$\begin{aligned}\sum_{t \in \mathbf{F}_p^*} a_{p,t}^n &= \sum_{t^2 \in Q^{4,+}} a_{p,t}^n + \sum_{t^2 \in Q^{4,-}} a_{p,t}^n \\ &= 2^{n-2} a^n (p-1) + 2^{n-2} a^n (p-1) \\ &= 2(2^{n-2} a^n (p-1)) \\ &= 2^{n-1} a^n (p-1).\end{aligned}$$

Similarly let $a+b \equiv 3(mod\,4)$. Then we know that

$$\sum_{t^2 \in Q^{4,+}} a_{p,t}^n = (-1)^n 2^{n-2} a^n (p-1)$$

and

$$\sum_{t^2 \in Q^{4,-}} a_{p,t}^n = 2^{n-2} a^n (p-1).$$

If n is odd, then

$$\begin{aligned}\sum_{t \in \mathbf{F}_p^*} a_{p,t}^n &= \sum_{t^2 \in Q^{4,+}} a_{p,t}^n + \sum_{t^2 \in Q^{4,-}} a_{p,t}^n \\ &= -2^{n-2} a^n (p-1) + 2^{n-2} a^n (p-1) \\ &= 0.\end{aligned}$$

If n is even, then

$$\begin{aligned}\sum_{t \in \mathbf{F}_p^*} a_{p,t}^n &= \sum_{t^2 \in Q^{4,+}} a_{p,t}^n + \sum_{t^2 \in Q^{4,-}} a_{p,t}^n \\ &= 2^{n-2} a^n (p-1) + 2^{n-2} a^n (p-1) \\ &= 2(2^{n-2} a^n (p-1)) \\ &= 2^{n-1} a^n (p-1).\end{aligned}$$

■

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