International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:4, No:9, 2010

Assessing the Relation between Theory of Multiple Algebras and Universal Algebras

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***Abstract*—**In this study, we examine multiple algebras and algebraic structures derived from them and by stating a theory on multiple algebras; we will show that the theory of multiple algebras is a natural extension of the theory of universal algebras. Also, we will treat equivalence relations on multiple algebras, for which the quotient constructed modulo them is a universal algebra and will study the basic relation and the fundamental algebra in question.

In this study, by stating the characteristic theorem of multiple algebras, we show that the theory of multiple algebras is a natural extension of the theory of universal algebras.

***Keywords*—** multiple algebras , universal algebras

I. Introduction

THE multiple algebras theory was discussed for the first time in 1931, by Marty, the French Mathematician, in an article presented to the 8th Scandinavian mathematical congress. In that article, the author proposed an extension of groups, called super groups, and stated some characteristics. Considering many other papers by Marty, one can predict that multiple algebras can be used as tools in other fields of mathematics theory.

Gratzer and Pickett have provided important articles on the theory of multiple algebras. In those articles, multiple algebras are seen as relational systems that are an extension of the theory of the study provided by Gratzer , Pickett and Hansoul [1] , [2], Hoft and Howard [3], Schweigert [4], Walicki and Bialasik [5], and, recently, Breaz and Pelea [6] and Pelea [7]-[10] have published articles.

II. The quotient of a global algebra, a theorem specific for multiple algebras

As it was mentioned in the introduction, Geratz showed in [11] that each multiple algebra can be obtained from the quotient of a global algebra to a definite equivalence relation. In this research we mention this matter by expressing the theorem specific for the multiple algebras.

Suppose that $\mathcal{B}=\left(B,\left(f_r\right)_{r<0(r)}\right)$ is a global algebra and ρ is an equivalence relation on B. In $B/\rho=\left\{\rho<b>|b\in B\right\}$ set, Geratz defines the multioperation of f_r for every $r<o(r)$ as follows:

$$f_r\left(\rho<b_0>,\dots,\rho<b_{n_r-1}>\right)=$$
$$\left\{\rho<c>|c=f_r\left(c_0,\dots,c_{n_r-1}\right),c_i\in\rho<b_i>,i=0,\dots,n_r-1\right\}.$$

The B/ρ set together with the above-defined multiple operations form the multiple algebra of B/ρ which is called the multiple concrete algebra.

Theorem 1

Each multiple algebra is a concrete one.

The above theorem can be expressed as follows:
For every multiple algebra $\mathcal{A}$ of type τ there is a universal algebra $\mathcal{B}$ of type τ and the equivalence relation ρ on B such that $\mathcal{A}\cong\mathcal{B}/\rho$ which is proven in [11]

Remark 1

Let $\mathcal{B}=\left(B,\left(f'_r\right)_{r<o(\tau)}\right)$ be a universal algebra and ρ be an equivalence relation on B . $\mathcal{B}/\rho$ is the quotient algebra corresponding to ρ . If $p\in P_A^{(n)}(\wp^*(\mathcal{A}))$ Then,

$$p\left(\rho<b_0>,\dots,\rho<b_{n-1}>\right)\supseteq$$
$$\left\{\rho<c>|c=p\left(c_0,\dots,c_{n-1}\right),c_i\rho b_i,i\in\left\{0,\dots,n-1\right\}\right\}$$

Proof:

Since π_ρ is a homomorphism between B and $\mathcal{B}/\rho$, we have $p\left(\pi_\rho<b_0>,\dots\pi_\rho<b_{n-1}>\right)\supseteq\pi_\rho\left(p\left(b_0,\dots,b_{n-1}\right)\right)$

Therefore, $p\left(\pi_\rho<b_0>,\dots\pi_\rho<b_{n-1}>\right)\supseteq$
$$\left\{\rho<b>|b\in p\left(b_0,\dots,b_{n-1}\right)\right\}.(*)$$

On the other hand, since $\mathcal{B}$ is a universal algebra, therefore, the defined p on the right side of (*) is monovalued, and, consequently,

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$p(\rho < b_0 >, \ldots, \rho < b_{n-1} >) \supseteq \{\rho < b > | b = p(b_0, \ldots, b_{n-1})\}$

Now, if the elements $c_0, \ldots, c_{n-1} \in B$ exist such that $(i \in \{0, \ldots, n-1\}), c_i \rho b_i$, then

$p(b_0, \ldots, b_{n-1}) = p(c_0, \ldots, c_{n-1})$.

Thus, the right side of the inclusion relation (*) is equal to the set $\{\rho < c > | c = p(c_0, \ldots, c_{n-1}), c_i \rho b_i, i \in \{0, \ldots, n-1\}\}$.

The defined inclusion relation (*) is not necessarily an equality.

III. A Class of Equivalence Relations on Multiple Algebras

Let $\mathcal{A} = \left(A, \left(f_r\right)_{r<o(\tau)}\right)$ be a multiple algebra of type. τ

Among the equivalence relations defined on $\mathcal{A}$, there are relations that the made quotients of the multiple algebra $\mathcal{A}$ Modulo them is a universal algebra. We proceed by examination of conditions equivalent to the universality of the quotient $\mathcal{A} / \rho$.

Let ρ be an equivalence relation on A. We define the relation $\overset{=}{\rho}$ on $P^*(A)$ as follows:

for every $X, Y \subseteq A$

$$X \overset{=}{\rho} Y \Leftrightarrow \forall x \in X, \forall y \in Y \ x\rho y \quad \text{Or}$$

$$X \overset{=}{\rho} Y \Leftrightarrow X \times Y \subseteq \rho$$

the relation $\overset{=}{\rho}$ is translational and symmetric, but not necessarily reflexive.

Lemma 1

Let $\mathcal{A} = \left(A, \left(f_r\right)_{r<o(\tau)}\right)$ be a multiple algebra. Let ρ be an equivalence relation on A and $\mathcal{A} / \rho$ be a universal algebra. For every $n \in \mathbb{N}$, $p \in P^{(n)}\left(\wp^*(\mathcal{A})\right)$ and $a_\circ, \ldots, a_{n-1} \in A$. if $x, y \in p(a_\circ, \ldots, a_{n-1})$ then $x\rho y$.

Proof:

We prove the lemma by induction on the process of constructing n - nomial functions.

If $p = c_a^n$ and $\chi, y \in c_i^n(a_\circ, \ldots, a_{n-1})$, we conclude that $x = y = a$. Therefore, $x\rho y$.

If $p = e_i^n, i \in \{\circ, \ldots, n-1\}$ we infer from $x, y \in e_i^n(a_\circ, \ldots, a_{n-1})$ that $x = y = a_i$ and therefore, $.x\rho y$.

Now assume that the lemma is satisfied for the n- nomial functions $p_\circ, \ldots, p_{n_r-1} \in P^{(n)}\left(\wp^*(\mathcal{A})\right)$. we prove the result for the function $f_r(p_\circ, \ldots, p_{n-1})$.If

$x, y \in f_r(p_\circ, \ldots, p_{n-1})(a_\circ, \ldots, a_{n-1}) = f_r\left(p_\circ(a_\circ, \ldots, a_{n-1}), \ldots, p_{n_r-1}(a_\circ, \ldots, a_{n-1})\right)$

Then there are elements $(i \in \{\circ, \ldots, n_r-1\}), x_i, y_i \in p_i(a_\circ, \ldots, a_{n-1})$ such that $x \in f_r(x_\circ, \ldots, x_{n-1})$ and $y \in f_r(y_\circ, \ldots, y_{n-1})$. since the lemma is satisfied for every p_i, $i \in \{\circ, \ldots, n_r-1\}$, therefore, for every $i \in \{\circ, \ldots, n_r-1\}, x_i \rho y_i$. as a result, by the definition of multioperations of quotient multiple algebras, we have

$\rho < x >, \rho < y > \in f_r(\rho < x_\circ >, \ldots, \rho < x_{n_r-1} >) \cap f_r(\rho < y_\circ >, \ldots, \rho < y_{n_r-1} >)$.

By assumption, for every $r < o(\tau)$ the defined f_r on $\mathcal{A} / \rho$ is mono-valued. Therefore, $\rho < x >= \rho < y >$.

The following theorem states equivalent conditions under which the multiple algebra $\mathcal{A} / \rho$ is an algebra.

Theorem 2

Let $\mathcal{A} = \left(A, \left(f_r\right)_{r<o(\tau)}\right)$ be a multiple algebra of type τ and ρ be an equivalence relation on A. All the conditions states below are equivalent.

1 $\mathcal{A} / \rho$ is a universal algebra.

2 If $r < o(\tau), a, b, x_i \in A(i \in \{\circ, \ldots, n_r-1\})$ *and* apb, then

$$f_r(x_\circ, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{n_r-1}) \overset{=}{\rho} f_r(x_\circ, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_{n_r-1})$$

For every $i \in \{\circ, \ldots, n_r - 1\}$

3- if $r < o(\tau)$ and $x_i, y_i \in A$ such that $x_i \rho y_i \ (i \in \{\circ, \ldots, n_r-1\})$,then

$$f_r(x_\circ, \ldots, x_{i-1}) \overset{=}{\rho} f_r(y_\circ, \ldots, y_{i-1}).$$

4- if $n \in \mathbb{N}$, $p \in P_A^n\left(\wp^*(\mathcal{A})\right)$, $x_i, y_i \in A$ such that, for every $i \in \{\circ, \ldots, n_r - 1\}$ $x_i \rho y_i$ then

$$p(x_\circ, \ldots, x_{n-1}) \overset{=}{\rho}\, p(y_\circ, \ldots, y_{n-1}).$$

Proof:

(2) $\Leftrightarrow$ (1) we have ,from $a\rho b$, that $\rho < a >= \rho < b >$. As a result, since fr is well- defined, for every $r < o(\tau)$,

$$f_r\left(\rho < x_\circ >, \ldots, \rho < a > \ldots, \rho < x_{n_r-1} >\right) = f_r\left(\rho < x_\circ >, \ldots, \rho < b >, \ldots, \rho < x_{n_r-1} >\right).$$

On the other hand, by the definition of multiple operations on the quotients of multiple algebras, for every

$x \in f_r\left(x_\circ, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{n_r-1}\right)$

And for every $y \in f_r\left(x_\circ, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_{n_r-1}\right)$

$\rho < x > \rho < y > \in f_r\left(\rho < x_\circ >, \ldots, \rho < a >, \ldots, \rho < x_{n_r-1} >\right)$.

According to assumption, $\mathcal{A} / \rho$ is a universal algebra. Therefore, for every $r < o(\tau)$, the defined f_r on A/ρ is mono-valued. Thus, $\rho < x >= \rho < y >$ and, consequently,

$f_r\left(x_\circ, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_{n_r-1}\right) \overset{=}{\rho} f_r \left(x_\circ, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_{n_r-1}\right)$

(2) $\Rightarrow$ (1) Let $\rho < x >, \rho < y > \in f_r\left(\rho < x_\circ >, \ldots, \rho < x_i >, \ldots, \rho < x_{n_r-1} >\right)$

Where $x_\circ, \ldots, x_{n_r-1} \in A$ are arbitrary. By the definition of f_r on A / ρ, there are elements $b_\circ, \ldots, b_{n_r-1}, a_\circ, \ldots, a_{n_r-1} \in A$ such that, for every $i \in \{\circ, \ldots, n_r - 1\}$, $b_i \rho x_i \rho a_i$, $y \in f_r\left(b_\circ, \ldots, b_{n_r-1}\right)$ and $x \in f_r\left(a_\circ, \ldots, a_{n_r-1}\right)$. As a result we have, by the hypothesis,

$$f_r\left(b_\circ, \ldots, b_{n_r-1}\right) \overset{=}{\rho} f_r\left(a_\circ, b_1, \ldots, b_{n_r-1}\right),$$

$$f_r\left(a_\circ, b_1, b_2, \ldots, b_{n_r-1}\right) \overset{=}{\rho} f_r\left(a_\circ, a_1, b_2, \ldots, b_{n_r-1}\right),$$

$$f_r\left(a_\circ, \ldots, a_{n_r-2}, b_{n_r-1}\right) \overset{=}{\rho} f_r\left(a_\circ, \ldots, a_{n_r-2},\ a_{n_r-1}\right),$$

On the other hand, $\overset{=}{\rho}$ is translational, Therefore,

$$f_r\left(b_\circ, \ldots, b_{n_r-1}\right) \overset{=}{\rho} f_r\left(a_\circ, \ldots, a_{n_r-1}\right),$$

And thus $\rho < x >, \rho < y >$. Therefore, for every $r < o(\tau)$, f_r is mono-valued. As a result, $\mathcal{A} / \rho$ is a universal algebra.

(2) $\Leftrightarrow$ (3) since we assumed that (2) is satisfied, therefore, $\mathcal{A} / \rho$ is universal algebra. Thus, like the proof of (1) $\Rightarrow$ (2) , we have

$$f_r\left(x_\circ, \ldots, x_{n_r-1}\right) \overset{=}{\rho} f_r\left(y_\circ, \ldots, y_{n_r-1}\right)$$

(1) $\Rightarrow$ (3) the stages of proof are ((1) $\Rightarrow$ (2) $\Rightarrow$ (3))

(1) $\Leftarrow$ (3) the proof is as (1) $\Leftarrow$ (2)

(4) $\Leftarrow$ (3) if we consider each element as a single element set, in other words, if we consider $\{x_i\}$ instead of x_i , for every $r < o(\tau)$, we can consider f_r as a map from $\left(P^*(A)\right)^{n_r}$ to $P^*(A)$ as follows:

$$f_r = f_r^*\left(e_{x_0}^{n_r}, \ldots, e_{x_{n_{r-1}}}^{n_r}\right) \in P_A^{n_r}\left(\mathcal{P}^*(\mathcal{A})\right)$$

(f_r^* is an operation on the universal algebra $\wp^*(\mathcal{A})$). Therefore for every $r < o(\tau)$, the result is satisfied for f_r.

(1) $\Rightarrow$ (4) if $p = e_i^n$ such that $i \in \{\circ, \ldots, n-1\}$, then

$$p\left(x_\circ, \ldots, x_{n-1}\right) = e_i^n\left(x_\circ, \ldots, x_{n-1}\right) = x_i,$$

$$p\left(y_\circ, \ldots, y_{n-1}\right) = e_i^n\left(y_\circ, \ldots, y_{n-1}\right) = y_i.$$

On the other hand, since for every $i \in \{\circ, \ldots, n-1\}$, $y_i \rho x_i$ therefore,

$$p\left(x_\circ, \ldots, x_{n_r-1}\right) \overset{=}{\rho} p\left(y_\circ, \ldots, y_{n_r-1}\right).$$

Assume that the result is satisfied for polynomial functions $p_\circ, \ldots, p_{n_r-1} \in P_A^n \wp^*(\mathcal{A})$. we prove it for

$p = f_r\left(p_\circ,\ldots,p_{n_r-1}\right)$. for every $x \in p\left(x_\circ,\ldots,x_{n-1}\right)$ and $y \in p\left(y_\circ,\ldots,y_{n-1}\right)$, there are elements $a_i \in p_i\left(x_\circ,\ldots,x_{n-1}\right)$ and $b_i \in p_i\left(y_\circ,\ldots,y_{n-1}\right)\left(i \in \{\circ,\ldots,n_r-1\}\right)$ such that

$$x \in f_r\left(a_\circ,\ldots,a_{n_r-1}\right), y \in f_r\left(b_\circ,\ldots,b_{n_r-1}\right).$$

Since we assume that the result is satisfied for p_i, for every $i \in \{\circ,\ldots,n_r-1\}$, $a_i \rho b_i$, . thus , by (1) $\Rightarrow$ (3) , we have

$$f_r\left(a_\circ,\ldots,a_{n_r-1}\right)\overset{=}{\rho} f_r\left(b_\circ,\ldots,b_{n_r-1}\right).$$

Therefore, $x\rho y$.

Corollary 1

Let $\mathcal{A}$ be a multiple algebra of type τ and ρ is an equivalence relation on $\mathcal{A}$. $\mathcal{A}/\rho$ Is a universal algebra, if and only if for every a $,b \in A$, where $a\rho b$ and for every $r < o(\tau)$ and $x_\circ,\ldots,x_{n_r-1} \in A$ We have

$$f_r\left(x_\circ,\ldots,x_{i-1},a,x_{i+1},\ldots,x_{n_r-1}\right)\overset{=}{\rho}$$
$$f_r\left(x_\circ,\ldots,x_{i-1},b,x_{i+1},\ldots,x_{n_r-1}\right)$$

For every $i \in \{\circ,\ldots,n_r-1\}$

Corollary 2

If ρ is an equivalence relation on A and $\mathcal{A}/\rho$ is a universal algebra, then

$$f_r\left(\rho<a_\circ>,\ldots,\rho<a_{n_r-1}>\right)=$$
$$\left\{\rho<b>|b \in f_r\left(a_\circ,\ldots,a_{n_r-1}\right)\right\}.$$

Also we can write

$$f_r\left(\rho<a_\circ>,\ldots,\rho<a_{n_r-1}>\right)=$$
$$\rho<b>,b \in f_r\left(a_\circ,\ldots,a_{n_r-1}\right).$$

Proof:

In the general case, we have

$$f_r\left(\rho<a_\circ>,\ldots,\rho<a_{n_r-1}>\right)=$$
$$\left\{\rho<b>|b \in f_r\left(b_\circ,\ldots,b_{n_r-1}\right),b_i\rho a_i,i \in \{\circ,\ldots,n_r-1\}\right\}$$

Since $\mathcal{A}/\rho$ is a universal algebra, by theorem 2 for every $r < o(\tau)$

$$f_r\left(b_\circ,\ldots,b_{n_r-1}\right)\overset{=}{\rho} f_r\left(a_\circ,\ldots,a_{n_r-1}\right).$$

Consequently, by the definition of $\overset{=}{\rho}$ for every $a \in f_r\left(a_\circ,\ldots,a_{n_r-1}\right)$ and $b \in f_r\left(b_\circ,\ldots,b_{n_r-1}\right)$ we have $a\rho b$. therefore, $\rho<a>=\rho<b>$. So

$$f_r\left(\rho<a_\circ>,\ldots,\rho<a_{n_r-1}>\right)=$$
$$\left\{\rho<b>|a \in f_r\left(a_\circ,\ldots,a_{n_r-1}\right)\right\}.$$

Also, by lemma 1 , for every $a,b \in f_r\left(a_\circ,\ldots,a_{n_r-1}\right) a\rho b$. thus, $\rho<a>=\rho<b>$. Therefore, we can write

$$f_r\left(\rho<a_\circ>,\ldots,\rho<a_{n_r-1}>\right)=$$
$$\rho<a>,a \in f_r\left(a_\circ,\ldots,a_{n_r-1}\right).$$

let $(H,.)$ be a hyper groupoid (a semi- hyper group, a hyper group). The equivalence relation ρ over H is called a strongly regular relation if for every $a,b,x \in H$.

$$a\rho b \Rightarrow a.x\overset{=}{\rho} b.x, x.a\overset{=}{\rho} x.b.$$

let $\mathcal{A}=\left(A,(f_r)_{r<o(\tau)}\right)$ be a multiple algebra of type τ . the equivalence relation ρ over A is called strongly regular if for every $a_i b_i \in A$ where $a_i\rho b_i\left(i \in \{\circ,\ldots,n_r-1\}\right)$ and every $r < o(\tau)$, the following relation is satisfied.

$$f_r\left(a_\circ,\ldots,a_{n_r-1}\right)\overset{=}{\rho} f_r\left(b_\circ,\ldots,b_{n_r-1}\right).$$

Considering the above definitions, if $\mathcal{A}=\left(A,\left(f_r\right)_{r<o(\tau)}\right)$ is a universal algebra, the congruence relation and the strongly regular relations on $\mathcal{A}$ are equivalent.

Remark 2

If ρ is an equivalence relation on the multiple algebra $\mathcal{A}$ such that $\mathcal{A}/\rho$ is universal, then ρ is a strongly regular relation on the multiple algebra $\mathcal{A}$ and vice versa.

Remark 3

Let $\mathcal{A}=\left(A,\left(f_r\right)_{r<o(\tau)}\right)$ be a multiple algebra of type τ . if ρ is an equivalence relation on A and $\mathcal{A}/\rho$ is a universal algebra, then p is an ideal equivalence relation. Therefore, the map π_ρ is an ideal homomorphism and , therefore, the inclusion stated in remark 1 turns into an equality.

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