Conventional and PSO Based Approaches for Model Reduction of SISO Discrete Systems

S. K. Tomar, R. Prasad, S. Panda, C. Ardil

***Abstract*—**Reduction of Single Input Single Output (SISO) discrete systems into lower order model, using a conventional and an evolutionary technique is presented in this paper. In the conventional technique, the mixed advantages of Modified Cauer Form (MCF) and differentiation are used. In this method the original discrete system is, first, converted into equivalent continuous system by applying bilinear transformation. The denominator of the equivalent continuous system and its reciprocal are differentiated successively, the reduced denominator of the desired order is obtained by combining the differentiated polynomials. The numerator is obtained by matching the quotients of MCF. The reduced continuous system is converted back into discrete system using inverse bilinear transformation. In the evolutionary technique method, Particle Swarm Optimization (PSO) is employed to reduce the higher order model. PSO method is based on the minimization of the Integral Squared Error (ISE) between the transient responses of original higher order model and the reduced order model pertaining to a unit step input. Both the methods are illustrated through numerical example.

***Keywords*—**Discrete System, Single Input Single Output (SISO), Bilinear Transformation, Reduced Order Model, Modified Cauer Form, Polynomial Differentiation, Particle Swarm Optimization, Integral Squared Error.

I. Introduction

REDUCTION of high order systems to lower order models has been an important subject area in control engineering for many years [1]. The mathematical procedure of system modeling often leads to detailed description of a process in the form of high order differential equations. These equations in the frequency domain lead to a high order transfer function. Therefore, it is desirable to reduce higher order transfer functions to lower order systems for analysis and design purposes.

The conventional methods of reduction, developed so far, are mostly available in continuous domain [2-5]. However, the high order systems can be reduced in continuous as well as in discrete domain [6-8]. There are two approaches for the reduction of discrete system, namely the direct method and indirect method. The indirect method uses some transformation and then reduction is carried out in the transformed domain. First the z- domain transfer functions are converted into *w*-domain by the bilinear transformation and then after reducing them in *w*-domain, suitably, they are converted back into z-domain.

In recent years, one of the most promising research fields has been “Evolutionary Techniques”, an area utilizing analogies with nature or social systems. Evolutionary techniques are finding popularity within research community as design tools and problem solvers because of their versatility and ability to optimize in complex multimodal search spaces applied to non-differentiable objective functions. Recently, the particle swarm optimization (PSO) technique appeared as a promising algorithm for handling the optimization problems. PSO is a population-based stochastic optimization technique, inspired by social behavior of bird flocking or fish schooling [9]. PSO shares many similarities with the genetic algorithm (GA), such as initialization of population of random solutions and search for the optimal by updating generations. However, unlike GA, PSO has no evolution operators, such as crossover and mutation. One of the most promising advantages of PSO over the GA is its algorithmic simplicity: it uses a few parameters and is easy to implement [10].

In this paper, two methods of model reduction of single-input single-output (SISO) discrete system have been presented. The first method which is based on the conventional approach combines the advantages of Modified Cauer Form and differentiation of the denominator polynomials. In this method the original high order discrete system is transformed into an equivalent continuous system by applying bilinear transformation separately on the numerator and denominator polynomials. This transformation is accomplished using synthetic division [11, 12]. The denominator of reduced continuous system is derived using differentiation, of both, the original and reciprocal polynomials in *w*-domain and multiplying various derivatives of these two polynomials [13] .The numerator is found by matching the quotients of MCF [14]. After obtaining the ROMs of continuous system its conversion into discrete system is accomplished by using inverse bilinear transformation, separately on numerator and denominator polynomials to give the desired result. Since the bilinear transformation is used twice the resulting reduced order model

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will show error in the steady state, hence the steady state correction is applied to match the final values of responses of original and reduced systems. In the second method, PSO is employed for the order reduction where both the numerator and denominator coefficients of LOS are determined by minimizing an ISE error criterion.

II. Statement of The Problem

Given a high order discrete time stable system of order 'n' that is described by the z-transfer function:

$$G_O(z)=\frac{N(z)}{D(z)}=\frac{a_0+a_1z+.........+a_{n-1}z^{n-1}}{b_0+b_1z+.........+b_{n-1}z^{n-1}+b_nz^n} \quad (1)$$

The objective is to find a reduced r^{th} order model that has a transfer function ($r<n$):

$$R(z)=\frac{N_r(z)}{D_r(z)}=\frac{c_0+c_1z+.........+c_{r-1}z^{r-1}}{d_0+d_1z+.........+d_{r-1}z^{r-1}+d_rz^r} \quad (2)$$

The polynomial $D(z)$ is stable, that is all its zeros reside inside the unit circle $|z|=1$. Where, $a_i(0\le i\le n-1)$, $b_i(0\le i\le n)$, $c_i(0\le i\le r-1)$ and $d_i(0\le i\le r)$ are scalar constants.

The numerator order is given as being one less than that of the denominator, as for the original system. The $R(z)$ approximates $G_0(z)$ in some sense and retains the important characteristics of $G_0(z)$.

III. Reduction by Conventional method

The reduction procedure by conventional method (modified Cauer Form and differentiation) may be described in the following steps:

Step-1

Apply bilinear transformation $z=\frac{1+w}{1-w}$, separately in the numerator and denominator polynomials of Eq. (1) using synthetic division [12]. This converts $G_0(z)$ into $G(w)$ as [15]:

$$N(w)=N(z)\Big|_{z=\frac{1+w}{1-w}}=\frac{N(w)}{(1-w)^{n-1}}=0 \quad (3)$$

$$D(w)=D(z)\Big|_{z=\frac{1+w}{1-w}}=\frac{D(w)}{(1-w)^{n}}=0 \quad (4)$$

From Eqs. (3) and (4) we get $G(w)=\frac{N(w)}{D(w)}$

This can be expressed as:

$$G(w)=\frac{a_{11}+a_{12}w+....................+a_{1n-1}w^{n-1}}{b_{11}+b_{12}w+...................+b_{1n-1}w^{n-1}+b_{1n}w^n} \quad (5)$$

The reciprocal of $D(w)$ is given as:

$$\widetilde{D}(w)=w^nD(1/w)=b_{1n}+b_{1n-1}w+....+b_{12}w^{n-1}+b_{11}w^n \quad (6)$$

Step-2

Compute the quotients of Modified Cauer Form (MCF) $h_1,h_2,......,H_1,H_2....$ using Modified Routh Array [16].

Step-3

Differentiate successively the denominator of the Eq. (5), and its reciprocal, the r^{th} order denominators of reduced order models can be obtained by multiplying various combinations of differentiated polynomials. In particular $D(w)$ is differentiated $(n-r_1)$ times and its reciprocal $\widetilde{D}(w)$ is differentiated $(n-r_2)$ times. The r^{th} order reduced denominator is obtained as:

$$D_r(w)=D_{r1}(w)\cdot D_{r2}(w) \quad (7)$$

where $\widetilde{D}_{r2}(w)$ is the reciprocal of $D_{r2}(w)$ and $r=r_1+r_2$

Step-4

Match the quotients $h_1,h_2,......,H_1,H_2....$ to find out the numerator $N(w)$ of the ROM as given in [14].

The ROM $G(w)$ will be obtained as $G(w)=\frac{N(w)}{D(w)}$

Step-5

Apply the inverse bilinear transformation $w=\frac{z-1}{z+1}$ separately in the $N(w)$ and $D(w)$ to convert $G(w)$ in z domain. Thus the rank of $G_0(z)$ and $G(w)$ will be same. Hence the step responses of $G_0(z)$ and $G(w)$ will match at initial time $t=0$.

Step-6

Remove steady state error by evaluating $K = \frac{Go(z)}{R(z)}\Big|_{z=1}$, and multiply it in the numerator of $R(z)$.

IV. Particle Swarm Optimization Method

In conventional mathematical optimization techniques, problem formulation must satisfy mathematical restrictions with advanced computer algorithm requirement, and may suffer from numerical problems. Further, in a complex system consisting of number of controllers, the optimization of several controller parameters using the conventional optimization is very complicated process and sometimes gets struck at local minima resulting in sub-optimal controller parameters. In recent years, one of the most promising research field has been "Heuristics from Nature", an area utilizing analogies with nature or social systems. Application of these heuristic optimization methods a) may find a global optimum, b) can produce a number of alternative solutions, c) no mathematical restrictions on the problem formulation, d) relatively easy to implement and e) numerically robust. Several modern heuristic tools have evolved in the last two decades that facilitates solving optimization problems that were previously difficult or impossible to solve. These tools include evolutionary computation, simulated annealing, tabu search, genetic algorithm, particle swarm optimization, etc. Among these heuristic techniques, Genetic Algorithm (GA) and Particle Swarm Optimization (PSO) techniques appeared as promising algorithms for handling the optimization problems. These techniques are finding popularity within research community as design tools and problem solvers because of their versatility and ability to optimize in complex multimodal search spaces applied to non-differentiable objective functions.

The PSO method is a member of wide category of swarm intelligence methods for solving the optimization problems. It is a population based search algorithm where each individual is referred to as particle and represents a candidate solution. Each particle in PSO flies through the search space with an adaptable velocity that is dynamically modified according to its own flying experience and also to the flying experience of the other particles. In PSO each particles strive to improve themselves by imitating traits from their successful peers. Further, each particle has a memory and hence it is capable of remembering the best position in the search space ever visited by it. The position corresponding to the best fitness is known as *pbest* and the overall best out of all the particles in the population is called *gbest* [9].

The modified velocity and position of each particle can be calculated using the current velocity and the distances from the $pbest_{j,g}$ to $gbest_g$ as shown in the following formulas [2-3, 9-10]:

$$v_{j,g}^{(t+1)} = w * v_{j,g}^{(t)} + c_1 * r_1(\ ) * (pbest_{j,g} - x_{j,g}^{(t)}) + c_2 * r_2(\ ) * (gbest_g - x_{j,g}^{(t)}) \quad (8)$$

$$x_{j,g}^{(t+1)} = x_{j,g}^{(t)} + v_{j,g}^{(t+1)} \quad (9)$$

With $j = 1,2,...,n$ and $g = 1,2,...,m$

Where,

n = number of particles in the swarm

m = number of components for the vectors v_j and x_j

t = number of iterations (generations)

$v_{j,g}^{(t)}$ = the g-th component of the velocity of particle j at iteration t, $v_g^{\min} \le v_{j,g}^{(t)} \le v_g^{\max}$;

w = inertia weight factor

c_1, c_2 = cognitive and social acceleration factors respectively

r_1, r_2 = random numbers uniformly distributed in the range (0, 1)

$x_{j,g}^{(t)}$ = the g-th component of the position of particle j at iteration t

$pbest_j$ = *pbest* of particle j

$gbest$ = *gbest* of the group

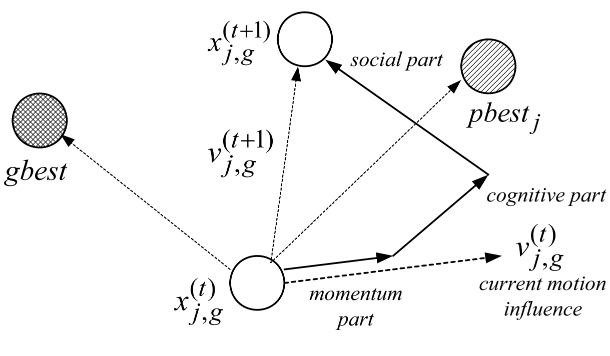

Fig. 1. Description of velocity and position updates in particle swarm optimization for a two dimensional parameter space

The j-th particle in the swarm is represented by a d-dimensional vector $x_j = (x_{j,1}, x_{j,2}, \ldots, x_{j,d})$ and its rate of position change (velocity) is denoted by another d-dimensional vector $v_j = (v_{j,1}, v_{j,2}, \ldots, v_{j,d})$. The best previous position of the j-th particle is represented as $pbest_j = (pbest_{j,1}, pbest_{j,2}, \ldots, pbest_{j,d})$. The index of best particle among all of the particles in the swarm is represented by the $gbest_g$. In PSO, each particle moves in the search space with a velocity according to its own previous best solution and its group's previous best solution. The velocity update in a PSO consists of three parts; namely momentum, cognitive and social parts. The balance among these parts determines the performance of a PSO algorithm. The parameters c_1 and c_2 determine the relative pull of *pbest* and *gbest* and the parameters r_1 and r_2 help in stochastically varying these pulls. In the above equations, superscripts denote the iteration number.

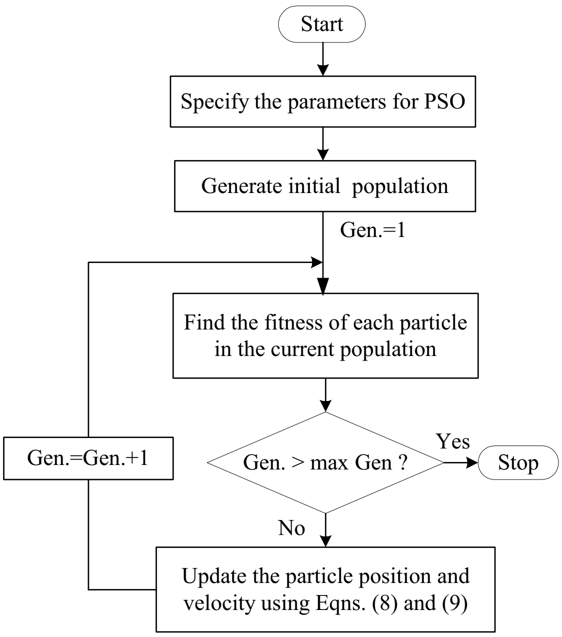

Fig. 2. Flowchart of PSO for order reduction

Fig.1. shows the velocity and position updates of a particle for a two-dimensional parameter space. The computational flow chart of PSO algorithm employed in the present study for the model reduction is shown in Fig. 2.

V. Numerical Examples

Let us consider the discrete system described by the transfer function [17]:

$$G_O(z) = \frac{N(z)}{D(z)} = \frac{0.54377\,z^3 - 0.40473\,z^2 + 0.319216\,z - 0.216608}{z^4 - 1.361178\,z^3 + 0.875599\,z^2 - 0.551205\,z + 0.282145} \tag{10}$$

For which a second order reduced model $R_2(z)$ is desired.

A. Conventional Method

Applying bilinear transformation separately on numerator and denominator, using synthetic division, the equivalent continuous system becomes:

$$G(w) = \frac{1.4879w^3 - 1.077w^2 + 1.568w + 0.24526}{4.0707w^4 + 4.492w^3 + 5.9417w^2 + 1.2503w + 0.24476} \tag{11}$$

The values of the quotients of MCF are obtained as:

$h_1 = 0.99796$ and $H_1 = 0.366$

Using the proposed steps the 2^{nd} order continuous system is obtained as:

$$R(w) = \frac{0.366w + 2.61}{w^2 + 9.7246w + 2.2606} \tag{12}$$

Using synthetic division, the inverse bilinear transformation is obtained as:

$$R(z) = \frac{2.976z + 2.244}{13.3306z^2 + 3.022z - 5.9286} \tag{13}$$

After steady state correction the final reduced order model is

$$R(z) = \frac{5.876z + 4.431}{13.3306z^2 + 3.022z - 5.9286} \tag{14}$$

B. Particle Swarm Optimization Method

For the implementation of PSO, several parameters are required to be specified, such as c_1 and c_2 (cognitive and social acceleration factors, respectively), initial inertia weights, swarm size, and stopping criteria. These parameters should be selected carefully for efficient performance of PSO. The constants c_1 and c_2 represent the weighting of the stochastic acceleration terms that pull each particle toward *pbest* and *gbest* positions. Low values allow particles to roam far from the target regions before being tugged back. On the other hand, high values result in abrupt movement toward, or past, target regions. Hence, the acceleration constants were often set to be 2.0 according to past experiences. Suitable selection of inertia weight, w, provides a balance between global and local explorations, thus requiring less iteration on average to find a sufficiently optimal solution. As originally developed, w often decreases linearly from about 0.9 to 0.4 during a run [10]. One more important point that more or less affects the optimal solution is the range for unknowns. For the very first execution of the program, wider solution space can be given, and after getting the solution, one can shorten the solution space nearer to the values obtained in the previous iterations.

The objective function J is defined as an integral squared error of difference between the responses given by the expression:

International Journal of Electrical, Electronic and Communication Sciences
ISSN: 2517-9438
Vol:7, No:6, 2013

$$J = \int_0^{t_\infty} [y(t) - y_r(t)]^2 \, dt \tag{15}$$

Where

$y(t)$ and $y_r(t)$ are the unit step responses of original and reduced order systems.

The reduced 2nd order model employing PSO technique is obtained as follows:

$$R_2(z) = \frac{-27.5858z + 18.3556}{-56.7977z^2 + 85.7487z - 5.9286} \tag{16}$$

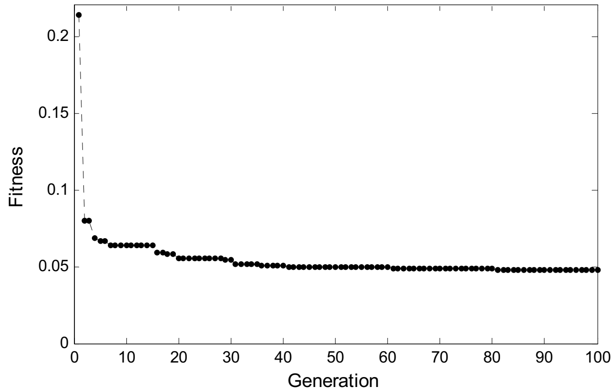

Fig. 3. Convergence of fitness function

The convergence of objective function with the number of generations is shown in Fig. 3. The unit step responses of original and reduced systems by both the methods are shown in Fig. 4. It can be seen that the steady state responses of both the proposed reduced order models are exactly matching with that of the original model. However, compared to conventional method of reduced models, the transient response of evolutionary reduced model by PSO is very close to that of original model.

VI. Comparison of Methods

The performance comparison of both the proposed algorithm for order reduction techniques is given in Table I. The comparison is made by computing the error index known as integral square error ISE [2-5] in between the transient parts of the original and reduced order model, is calculated to measure the goodness/quality of the [i.e. the smaller the ISE, the closer is $R(s)$ to $G(s)$, which is given by:

$$ISE = \int_0^{t_\infty} [y(t) - y_r(t)]^2 \, dt \tag{17}$$

Where $y(t)$ and $y_r(t)$ are the unit step responses of original and reduced order systems for a second- order reduced respectively. This error index is calculated for both reduced order models.

Table I: Comparison of Methods

Method	Reduced model	ISE
PSO method	$\frac{-27.5858z + 18.3556}{-56.7977z^2 + 85.7487z - 5.9286}$	0.0131
Conventional method	$\frac{5.876z + 4.431}{13.3306z^2 + 3.022z - 5.9286}$	44.874

VII. Conclusion

In this paper, two methods for reducing a high order discrete system into a lower order system have been proposed. In the first method, a conventional technique has been proposed which uses the advantages of both modified Cauer Form (MCF) and differentiation. In this method, first the original discrete system is converted into equivalent continuous system by applying bilinear transformation. Then the denominator of the equivalent continuous system and its reciprocal are differentiated successively and the reduced denominator of the desired order is obtained by combining the

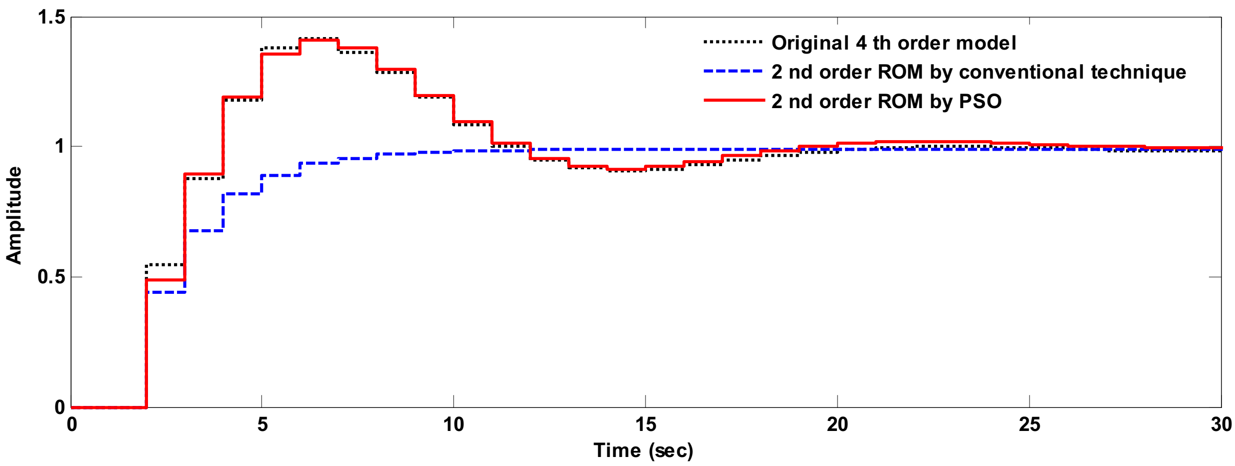

Fig. 4. Step Responses of original system and reduced model

differentiated polynomials. The numerator is obtained by matching the quotients of MCF. Finally the reduce continuous system is converted back into discrete system using inverse bilinear transformation. In the second method, an evolutionary swarm intelligence based method known as Particle Swarm Optimization (PSO) is employed to reduce the higher order model. PSO method is based on the minimization of the Integral Squared Error (ISE) between the transient responses of original higher order model and the reduced order model pertaining to a unit step input. Both the methods are illustrated through a numerical example. Also, a comparison of both the proposed methods has been presented. It is observed that both the proposed methods preserve model stability and the time domain characteristics of the original system. However, PSO method seems to achieve better results in view of its simplicity, easy implementation and better response.

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