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Haar wavelet method for solving Initial and Boundary Value Problems of Bratu-type

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Abstract—In this paper, we present a framework to determine Haar solutions of Bratu-type equations that are widely applicable in fuel ignition of the combustion theory and heat transfer. The method is proposed by applying Haar series for the highest derivatives and integrate the series. Several examples are given to confirm the efficiency and the accuracy of the proposed algorithm. The results show that the proposed way is quite reasonable when compared to exact solution.

Keywords—Haar wavelet method; Bratu's problem; Boundary value problems; Initial value problems; Adomain decomposition method

I. Introduction

This paper is concerned with boundary value problems and an initial value problem of the Bratu-type. It is well known that Bratu's boundary value problem in one-dimensional planar coordinates is of the form [1-5].

$$U'' + \lambda e^{U} = 0, 0 < x < 1, U(0) = U(1) = 0 \tag{1}$$

The standard Bratu problem (1) was used to model a combustion problem in a numerical slab. Bratu's problem [6-9] is also used in a large variety of applications such as the fuel ignition model of the thermal combustion theory, the model of the thermal reaction process, the Chandrasekhar model of the expansion of the universe, questions in geometry and relativity concerning the Chandrasekhar model, chemical reaction theory, radiative heat transfer and nanotechnology [10-14,15,17].

A substantial amount of research work has been directed for the study of the Bratu problem [1-5]. Several numerical techniques, such as the finite difference method, finite element approximation, weighted residual method, and the shooting method, have been implemented independently to handle the Bratu model numerically. In addition, Boyd [2,3] employed Chebyshev polynomial expansions and the Gegenbauer as base functions. Muhammed I. Syam and Abdelrahem Hamdan [16] presented the Laplace Adomain decomposition method (LADM) for solving Bratu's problem.

The exact solution to (1) is given in [1] and [2,3] and given by

$$U(x) = -2ln\left[\frac{cosh\left(\left(x-\frac{1}{2}\right)\frac{\theta}{2}\right)}{cosh\left(\frac{\theta}{4}\right)}\right] \tag{2}$$

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where θ satisfies

$$\theta = \sqrt{2\lambda}cosh\left(\frac{\theta}{4}\right) \tag{3}$$

The Bratu problem has zero, one or two solutions when $\lambda > \lambda_c, \lambda = \lambda_c, or \lambda < \lambda_c$ respectively, where the critical value satisfies the equation

$$1 = \frac{1}{4}\sqrt{2\lambda_c}sinh\left(\frac{\theta_c}{4}\right) \tag{4}$$

It was evaluated in [1,2,3,4,11,18] that the critical value is given by

$$\lambda_c = 3.513830719 \tag{5}$$

The main goal of this problem is to introduce a new Haar wavelet treatment of two boundary value problems of Bratu-type model, given by

$$U'' - \pi^2 e^{U} = 0, 0 < x < 1, U(0) = U(1) = 0 \tag{6}$$

and

$$U'' + \pi^2 e^{-U} = 0, 0 < x < 1, U(0) = U(1) = 0 \tag{7}$$

In addition, an initial value problem of the Bratu-type

$$U'' - 2e^{U} = 0, 0 < x < 1, U(0) = U'(0) = 0 \tag{8}$$

will be investigated. In this paper, our work stems mainly from the Haar wavelet method. The Haar wavelet method, which will exhibit several advantageous features:

i) Very high accuracy fast transformation and possibility of implementation of fast algorithms compared with other known methods. ii) The simplicity and small computation costs, resulting from the sparsity of the transform matrices and the small number of significant wavelet coefficients. iii) The method is also very convenient for solving the boundary value problems, since the boundary conditions are taken care of automatically.

Haar wavelets (which are Daubechies of order 1) consists of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. The main advantage of the Haar wavelet is that simplicity gets to some extent lost. In solving ordinary differential equations by using Haar wavelet related method, Chen and Hsiao [6] had derived an operational matrix of integration based on Haar wavelet. Lepik [12,13] had solved higher order as well as nonlinear ODEs by using Haar wavelet method. The paper is organized in the following way. For completeness sake the Haar wavelet method is presented in Section 2. Function approximation is presented in Section 3. Procedure of Haar wavelet method for ODE is presented in Section 4. The method of solution of the first Bratu-type problem is proposed in Section 5. The method of solution of the second Bratu-type problem is proposed in Section 6. The method of solution of the initial value problem of Bratu-type equation is proposed in Section 7. Concluding remarks are given in Section 8.

II. Haar wavelet preliminaries

Haar wavelet is the simplest wavelet. Haar transform or Haar wavelet transform has been used as an earliest example for orthonormal wavelet transform with compact support. The Haar wavelet transform is the first known wavelet and was proposed in 1910 by Alfred Haar. They are step functions (piecewise constant functions) on the real line that can take only three values. Haar wavelets, like the well-known Walsh functions (Rao 1983), form an orthogonal and complete set of functions representing discretized functions and piecewise constant functions. A function is said to be piecewise constant if it is locally constant in connected regions.

The Haar transform is one of the earliest examples of what is known now as a compact, dyadic, orthonormal wavelet transform. The Haar function, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. In the mean time, several definitions of the Haar functions and various generalizations have been published and used. They were intended to adopt this concept to some practical applications as well as to extend its applications to different classes of signals. Haar functions appear very attractive in many applications as for example, image coding, edge extraction, and binary logic design.

After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.

The previous work in system analysis via Haar wavelets was led by Chen and Hsiao [6], who first derived a Haar operational matrix for the integrals of the Haar function vector and put the application for the Haar analysis into the dynamical systems. Then, the pioneer work in state analysis of linear time delayed systems via Haar wavelets was laid down by Hsiao [10], who first proposed a Haar product matrix and a coefficient matrix. Hsiao and Wang proposed a key idea to transform the time-varying function and its product with states into a Haar product matrix.

The Haar wavelet family for is defined as follows.

$$h_i(t) = \begin{cases} 1 & for\ t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right) \\ -1 & for\ t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right) \\ 0 & elsewhere \end{cases} \tag{9}$$

Integer $m = 2^j \, (j = 0, 1, 2, \ldots J)$ indicates the level of the wavelet; k = 0,1,2, $\cdots$, m-1 is the translation parameter. Maximal level of resolution is J. The index i is calculated according to the formula i = m + k + 1 ; in the case of minimal values m=1,k=0, we have i=2, the maximal value of i is $2M = 2^{(J+1)}$. It is assumed that the value i=1 corresponds to the scaling function for which $h_1 \equiv 1$ in $[0, 1]$. Let us define the collocation points $t_l = (l - 0.5)/2M, (1, 2..., 2M)$ and discretise the Haar function $h_i(t)$: In this way we get the coefficient matrix , which has the dimension . The operational matrix of integration P, which is a 2M square matrix, is defined by the equation

$$(PH)_{il} = \int_0^{t_l} h_i(t)\, dt \tag{10}$$

$$(QH)_{il} = \int_0^{t_l} dt \int_0^{t} h_i(t)\, dt \tag{11}$$

The elements of the matrices H, P and Q can be evaluated according to (1), (2) and (3).

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$P_2 = \frac{1}{4}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$P_4 = \frac{1}{16}\begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$P_8 = \frac{1}{64}\begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chen and Hsiao[6] showed that the following matrix equation for calculating the matrix P of order m holds

$$P_{(m)} = \frac{1}{2m}\begin{bmatrix} 2mP_{(m/2)} & -H_{(m/2)} \\ H^{-1}_{(m/2)} & O \end{bmatrix}$$

where O is a null matrix of order $\frac{m}{2} \times \frac{m}{2}$

$$H_{m\times m} \underline{\underline{\Delta}} \left[h_m(t_0)\ \ h_m(t_1) - - - h_m(t_{m-1})\right] \tag{12}$$

and $\frac{i}{m} \leq t < i + \frac{1}{m}$ and $H^{-1}_{m\times m} = \frac{1}{m} H^T_{m\times m} diag\,(r)$

It should be noted that calculations for P_m and H_m must be carried out only once; after that they will be applicable for solving whatever differential equations. Since H and H^{-1} contain many zeros, this phenomenon makes the Haar transform much faster than the Fourier Transform and it is even faster than the Walsh transform. This is one of the reasons for rapid convergence of the Haar wavelet series.

III. Function Approximation

Any function $y(x) \in L^2[0,1)$ can be decomposed as

$$y(x) = \Sigma c_n h_n(x) \tag{13}$$

where the coefficients c_n are determined by

$$c_n = 2^j \int_0^1 y(x)\, h_n(x)\, dx \tag{14}$$

where $n = 2^j + k, j \geq 0, 0 \leq k < 2^j$. Especially $c_0 = \int_0^1 y(x)\, dx$.

The series expansion of $y(x)$ contains infinite number of terms. If $y(x)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $y(x)$ will be terminated at finite terms, that is

$$y(x) = \Sigma_0^{m-1} c_n h_n(x) = C_{(m)}^T h_{(m)}(x) \tag{15}$$

where the coefficients $C_{(m)}^T$ and the Haar function vector $h_m(x)$ are defined as
$C_{(m)}^T = [c_0, c_1, ..., c_{m-1}]$ and $h_m(x) = [h_0(x), h_1(x), ..., h_{m-1}(x)]^T$ where 'T' means transpose and $m = 2^j$

IV. Haar wavelet method for Ordinary Differential Equations

For solving linear Ordinary differential equation with n^{th} order, say
$a_1 y^{(n)}(x) + a_2 y^{(n-1)}(x) + ... + a_n y(x) = f(x)$, where $x \in [A, B]$ and initial conditions $y^{(n-1)}(a), y^{(n-2)}(a), ..., y(a)$ are known.

We follow the work done by Lepik [13].We intend to do until j^{th} level of resolution, hence we let $m = 2\left(2^j\right)$. The interval $[A, B]$ will be divided in to m subintervals, hence $\Delta x = \frac{B-A}{m}$ and the matrices are in the dimension of $m \times m$. Here we suggest the step-by-step procedures for easy understanding. Mainly, there are 5 steps in the procedure,which are as follows.

Haar wavelet algorithm for solving Ordinary Differential Equation:

Step 1: Let $y^n(x) = \Sigma_{i=1}^m a_i h_i(x)$, where h is Haar matrix and a_i is the wavelet coefficients.

Step 2: Obtain appropriate ν order of $y(x)$ by using $y^{(\nu)}(x) = \Sigma_{i=1}^m a_i P_{n-\nu,i}(x) + \Sigma_{\sigma=0}^{n-\nu-1} \frac{1}{\sigma!}(x-A)^\sigma y_0^{\nu+\sigma}$

Step 3: Replace $y^n(x)$ and all of the values of $y^{(\nu)}(x)$in to the problem.

Step 4: Calculate the wavelet coefficients, a_i.

Step 5: Obtain the numerical solution for $y(x)$.

Step 2 is the key procedure where the matrix $P_{n-\nu,i}(x)$ will be counted.If we intend to do the calculation until level j of resolution , we will obtain the matrix $P_{n-\nu}(x)\,(let\ n-\nu=\alpha)$ as in the pattern, where $C = B - A$.

V. Method of solution of first Bratu-type problem

We consider the Bratu-type model

$$u'' - \pi^2 e^u = 0, 0 < x < 1, u(0) = u(1) = 0 \tag{16}$$

As indicated before, (16)differs from Bratu problem by the value λ given by $\lambda = -\pi^2 < 0$ However, in (1), $\lambda = \lambda_c = 3.513830719 > 0$. The effect of this change in λ will be examined.

Any function $u(x) \in L^2[0,1)$ can be decomposed as

$$u(x) = \Sigma_{i=1}^m a_i h_i(x) \tag{17}$$

$$u(x) = \Sigma_{i=1}^m a_i P_{2,i}(x) + \Sigma_{\sigma=0}^1 \frac{1}{\sigma!}(x-0)^\sigma y_0^{(\sigma)} \tag{18}$$

$$u(x) = \Sigma_{i=1}^n a_i P_{2,i}(x) + 1 + x \tag{19}$$

By Carry out steps 1 to 3, we obtain

$$\Sigma_{i=1}^m a_i h_i(x) - \pi^2 exp\left[\Sigma_{i=1}^m a_i P_{2,i}(x) + 1 + x\right] = 0 \tag{20}$$

Solving the system of linear equations, we obtain wavelet coefficients, a_i and we obtain the numerical solution of $u(x)$.

Using Adomain decomposition method, the exact solution can be obtained by considering the boundary condition $u(1) = 0$ to obtain $a = \pi$, and consequently the closed form of the solution

$$u(x) = -ln\left[1 + cos\left(\left(\frac{1}{2} + x\right)\pi\right)\right] \tag{21}$$

which is in full agreement with the results in [18].

VI. Method of solution of second Bratu-type Problem

We next consider the Bratu-type model

$$u'' + \pi^2 e^{-u} = 0, 0 < x < 1, u(0) = u(1) = 0 \tag{22}$$

equation (22) differs from the standard Bratu-type Problem by the term e^{-u} and $\lambda = \pi^2 > \lambda_c$ The effect of these changes will be examined.Using Adomain decomposition method, the exact solution can be obtained by considering the boundary condition $u(1) = 0$ to obtain $a = \pi$, and consequently the closed form of the solution $u(x) = ln[1 + sin(1 + \pi x)]$ which is in full agreement with the results in [18].

Any function $u(x) \in L^2[0,1)$ can be decomposed as

$$u(x) = \Sigma_{i=1}^m a_i h_i(x) \tag{23}$$

$$u(x) = \Sigma_{i=1}^m a_i P_{2,i}(x) + \Sigma_{\sigma=0}^{n-\nu-1} \frac{1}{\sigma!}(x-\nu)^\sigma y_0^{(\sigma)} \tag{24}$$

$$u(x) = \Sigma_{i=1}^{n} a_i P_{2,i}(x) + 1 + x \qquad (25)$$

By Carry out steps 1 to 3, we obtain

$$\Sigma_{i=1}^{m} a_i h_i(x) + \pi^2 exp\left[-\left(\Sigma_{i=1}^{m} a_i P_{2,i}(x) + 1 + x\right)\right] = 0 \qquad (26)$$

By carry out steps 4 to 5, we can obtain the Haar solutions.

VII. Initial value problem of the Bratu-type

We finally cionsider the initial value problem of Bratu-type

$$u'' - 2e^u = 0, 0 < x < 1, u(0) = u'(0) = 0 \qquad (27)$$

Unlike the Bratu-type where boundaru conditiions are used,(21) is an initial value problem where $u(0) = u'(0) = 0$
The exact solution in a closed form is given by

$$u(x) = -2ln\left[cos(x)\right] \qquad (28)$$

It is interesting to point out that the solution (22) is bounded in the domain $0 \leq x \leq 1$.

Any function $u(x) \in L^2[0,1)$ can be decomposed as

$$u(x) = \Sigma_{i=1}^{m} a_i h_i(x) \qquad (29)$$

$$u(x) = \Sigma_{i=1}^{m} a_i P_{2,i}(x) + \Sigma_{\sigma=0}^{n-\nu-1} \frac{1}{\sigma!}(x-\nu)^{\sigma} y_0^{(\sigma)} \qquad (30)$$

$$u(x) = \Sigma_{i=1}^{n} a_i P_{2,i}(x) + 1 + x \qquad (31)$$

By Carry out steps 1 to 3, we obtain

$$\Sigma_{i=1}^{m} a_i h_i(x) - 2exp\left[\left(\Sigma_{i=1}^{m} a_i P_{2,i}(x) + 1 + x\right)\right] = 0 \qquad (32)$$

By carry out steps 4 to 5, we can obtain the Haar solutions.

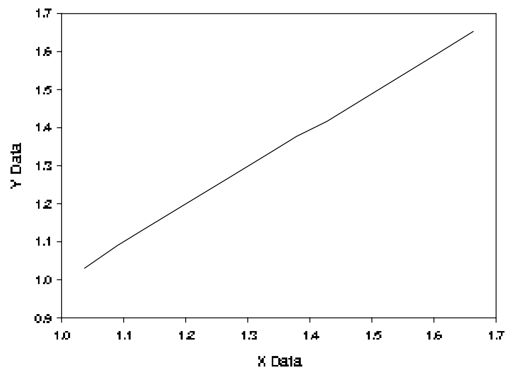

Fig. 1. Haar and Exact solutions for the initial value problem of Bratu-Type equation(27)

Fig 1 shows the comparison between the Haar and exact solutions of initial value problem of Bratu-type equation (27). In this example,m=16 is used.

VIII. Conclusion

The main goal of this paper is to demonstrate that Haar wavelet method is a powerful tool for solving initial and boundary value problems of Bratu-type. The algotithm and procedure have been applied to use Haar wavelet method in solving ODE's. The result is compared with the exact solutions. The method with far less degrees of freedom and with smaller CPU time provides better solutions than classical ones. It is worthmentioning that Haar solution provides excellent results even for small values of m(m=16). For larger values of m (m=32,m=64), we can obtain the results closer to exact values. The main advantages of this method are its simplicity and small computation costs: it is due to the sparcity of the transform matrices and to the small number of significant wavelet coefficients. The method is also very convenient for solving the boundary value problems, since the boundary conditions are taken care of automatically. In our opinion the method is wholly competetive in comparison with the classical methods.

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