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I-Vague Groups

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Abstract—The notions of I-vague groups with membership and non-membership functions taking values in an involutary dually residuated lattice ordered semigroup are introduced which generalize the notions with truth values in a Boolean algebra as well as those usual vague sets whose membership and non-membership functions taking values in the unit interval [0, 1]. Moreover, various operations and properties are established.

Keywords—Involutary dually residuated lattice ordered semigroup, I-vague set and I-vague group.

I. Introduction

THE notion of fuzzy groups defined by A. Rosenfeld[13] is the first application of fuzzy set theory in Algebra. Since then a number of works have been done in the area of fuzzy algebra.

M. Demirci[5] studied vague groups. R. Biswas[2] defined the notion of vague groups analogous to the idea of Rosenfeld [13]. H. Khan, M. Ahmad and R. Biswas[8] studied vague groups and made some characterizations. N. Ramakrishna[10] studied vague groups and vague weights.

The vague sets of W. L. Gau and D. J. Buehrer[6] and Atanassov's[1] intuitionistic fuzzy sets are mathematically equivalent objects[3]. In this paper we prefer the terminology of vague sets as the algebraic study intiated by Biswas[2] follows the terminology of vague sets.

K. L. N. Swamy[14], [15], [16] introduced the concept of dually residuated lattice ordered semigroup(in short DRL-semigroup) which is a common abstraction of Boolean algebras and lattice ordered groups. The subclass of DRL-semigroups which are bounded and involutary(i.e having 0 as least, 1 as greatest and satisfying 1-(1-x) = x) which is categorically equivalent to the class of MV-algebras of C. C. Chang[4] and well studied offer a natural generalization of the closed unit interval [0, 1] of real numbers as well as Boolean algebras. Thus, the study of vague sets $(t_A,\ f_A)$ with values in an involutary DRL-semigroup promises a unified study of real valued vague sets and also those Boolean valued vague sets[11].

In his thesis T. Zelalem[19] studied the concept of I-vague sets. In this paper using the definition of I-vague sets, we defined and studied I-vague groups where I is an involutary DRL-semigroup. In this paper we shall recall some basic results in [14], [15], [19] without proof. Moreover, notation, terminology and results of [19] are used in this paper. Throughout this paper, we shall denote the identity element of a group (G, .) by e and the order of an element x of G by $O(x)$. Moreover, for $x \in G$, $< x >$ denotes the cyclic group generated by x.

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II. Preliminaries

Definition 2.1: [14] A system $A = (A, +, \leq, -)$ is called a dually residuated lattice ordered semigroup(in short DRL-semigroup) if and only if

i) $A = (A,\ +)$ is a commutative semigroup with zero"0";

ii) $A = (A,\ \leq)$ is a lattice such that
$a + (b \cup c) = (a + b) \cup (a + c)$ and $a + (b \cap c) = (a + b) \cap (a + c)$ for all $a,\ b,\ c \in$ A;

iii) Given $a,\ b \in$ A, there exists a least x in A such that $b + x \geq a$, and we denote this x by a - b (for a given a, b this x is uniquely determined);

iv) (a-b) $\cup$ 0 + b $\leq a \cup b$ for all a, b $\in$ A;

v) a - a $\geq$ 0 for all $a \in$ A.

Theorem 2.2: [14] Any DRL-semigroup is a distributive lattice.

Definition 2.3: [19] A DRL-semigroup A is said to be involutary if there is an element $1 (\neq 0)$(0 is the identity w.r.t. +) such that

(i) $a + (1 - a) = 1 + 1$;

(ii) $1 - (1 - a) = a$ for all $a \in$ A.

Theorem 2.4: [15] In a DRL-semigroup with 1, 1 is unique.

Theorem 2.5: [15] If a DRL-semigroup contains a least element x, then x = 0. Dually, if a DRL-semigroup with 1 contains a largest element α, then α = 1.

Throughout this paper let $I = (I, +, -, \vee, \wedge, 0, 1)$ be a dually residuated lattice ordered semigroup satisfying $1-(1-a) = a$ for all $a \in$ I.

Lemma 2.6: [19] Let 1 be the largest element of I. Then for $a,\ b \in$ I

(i) a + (1- a) = 1 .

(ii) 1 - a = 1 - b $\Longleftrightarrow$ a = b.

(iii)1 - (a $\cup$ b) = (1- a) $\cap$ (1- b).

Lemma 2.7: [19] Let I be complete. If $a_\alpha \in$ I for every $\alpha \in \Delta$, then

(i) $1 - \bigvee_{\alpha\in\Delta} a_\alpha = \bigwedge_{\alpha\in\Delta} (1 - a_\alpha)$.

(ii) $1 - \bigwedge_{\alpha\in\Delta} a_\alpha = \bigvee_{\alpha\in\Delta} (1 - a_\alpha)$.

Definition 2.8: [19] An I-vague set A of a non-empty set G is a pair $(t_A,\ f_A)$ where $t_A : G \to I$ and $f_A : G \to I$ with $t_A(x) \leq 1 - f_A(x)$ for all $x \in G$.

Definition 2.9: [19] The interval $[t_A(x),\ 1 - f_A(x)]$ is called the I-vague value of $x \in G$ and is denoted by $V_A(x)$.

Definition 2.10: [19] Let $B_1 = [a_1,\ b_1]$ and $B_2 = [a_2,\ b_2]$ be two I-vague values. We say $B_1 \geq B_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

Definition 2.11: [19] An I-vague set $A = (t_A,\ f_A)$ of G is said to be contained in an I-vague set $B = (t_B, f_B)$ of G written as $A \subseteq B$ if and only if $t_A(x) \leq t_B(x)$ and $f_A(x) \geq f_B(x)$ for all $x \in$ G. A is said to be equal to B written as A = B if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 2.12: [19] An I-vague set A of G with $V_A(x) = V_A(y)$ for all $x,\ y \in$ G is called a constant I-vague set of G.

Definition 2.13: [19] Let A be an I-vague set of a non empty set G. Let $A_{(\alpha,\ \beta)} = \{x \in G : V_A(x) \geq [\alpha,\ \beta]\}$ where $\alpha,\ \beta \in I$ and $\alpha \leq \beta$. Then $A_{(\alpha,\ \beta)}$ is called the $(\alpha,\ \beta)$ cut of the I-vague set A.

Definition 2.14: [19] Let S $\subseteq$G. The characteristic function of S denoted as $\chi_S = (t_{\chi_S},\ f_{\chi_S})$, which takes values in I is defined as follows:

$$t_{\chi_S}(x) = \begin{cases} 1 & \text{if } x \in S \text{ ;} \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{\chi_S}(x) = \begin{cases} 0 & \text{if } x \in S \text{ ;} \\ 1 & \text{otherwise.} \end{cases}$$

χ_S is called the I-vague characteristic set of S in I. Thus

$$V_{\chi_S}(x) = \begin{cases} [1,\ 1] & \text{if } x \in S \text{ ;} \\ [0,\ 0] & \text{otherwise.} \end{cases}$$

Definition 2.15: [19] Let $A = (t_A,\ f_A)$ and $B = (t_B,\ f_B)$ be I-vague sets of a set G.
(i) Their union $A \cup B$ is defined as $A \cup B = (t_{A\cup B},\ f_{A\cup B})$ where $t_{A\cup B}(x) = t_A(x) \vee t_B(x)$ and
$f_{A\cup B}(x) = f_A(x) \wedge f_B(x)$ for each $x \in$ G.
(ii) Their intersection $A \cap B$ is defined as $A \cap B = (t_{A\cap B},\ f_{A\cap B})$ where $t_{A\cap B}(x) = t_A(x) \wedge t_B(x)$ and $f_{A\cap B}(x) = f_A(x) \vee f_B(x)$ for each $x \in$ G.

Definition 2.16: [19] Let $B_1 = [a_1,\ b_1]$ and $B_2 = [a_2,\ b_2]$ be I-vague values. Then
(i) $\text{isup}\{B_1,\ B_2\} = [\sup\{a_1,\ a_2\}, \sup\{b_1,\ b_2\}]$.
(ii) $\text{iinf}\{B_1,\ B_2\} = [\inf\{a_1,\ a_2\}, \inf\{b_1,\ b_2\}]$.

Lemma 2.17: [19] Let A and B be I-vague sets of a set G. Then $A \cup B$ and $A \cap B$ are also I-vague sets of G.
Let $x \in$ G. From the definition of $A \cup B$ and $A \cap B$ we have
(i) $V_{A\cup B}(x) = \text{ isup}\{V_A(x),\ V_B(x)\}$;
(ii) $V_{A\cap B}(x) = \text{ iinf}\{V_A(x),\ V_B(x)\}$.

Definition 2.18: [19] Let I be complete and $\{A_i$: i $\in \triangle\}$ be a non empty family of I-vague sets of G where $A_i = (t_{A_i},\ f_{A_i})$. Then
(i) $\bigcap\limits_{i\in\triangle} A_i = (\bigwedge\limits_{i\in\triangle} t_{A_i},\ \bigvee\limits_{i\in\triangle} f_{A_i})$
(ii) $\bigcup\limits_{i\in\triangle} A_i = (\bigvee\limits_{i\in\triangle} t_{A_i},\ \bigwedge\limits_{i\in\triangle} f_{A_i})$

Lemma 2.19: [19] Let I be complete. If $\{A_i$: i $\in \triangle\}$ is a non empty family of I-vague sets of G, then $\bigcap\limits_{i\in\triangle} A_i$ and $\bigcup\limits_{i\in\triangle} A_i$ are I-vague sets of G.

Definition 2.20: [19] Let I be complete and
$\{A_i = (t_{A_i},\ f_{A_i}) :\ i \in \triangle\}$ be a non empty family of I vague sets of G. Then for each $x \in$ G,
(i) $\text{isup}\{V_{A_i}(x) :\ i \in \triangle\} = [\bigvee\limits_{i\in\triangle} t_{A_i}(x),\ \bigvee\limits_{i\in\triangle} (1 - f_{A_i})(x)]$.
(ii) $\text{iinf}\{V_{A_i}(x) :\ i \in \triangle\} = [\bigwedge\limits_{i\in\triangle} t_{A_i}(x),\ \bigwedge\limits_{i\in\triangle} (1 - f_{A_i})(x)]$.

III. I-Vague Groups

Definition 3.1: Let G be a group. An I-vague set A of a group G is called an I-vague group of G if
(i) $V_A(xy) \geq \text{ iinf}\{V_A(x), V_A(y)\}$ for all $x,\ y \in G$ and
(ii)$V_A(x^{-1}) \geq V_A(x)$ for all $x \in G$.

Lemma 3.2: If A is an I-vague group of a group G, then $V_A(x) = V_A(x^{-1})$ for all $x \in G$.
Proof: Since A is an I-vague group of G, $V_A(x^{-1}) \geq V_A(x)$ for all $x \in G$. $V_A(x) = V_A((x^{-1})^{-1}) \geq V_A(x^{-1})$. Hence the lemma follows.

Lemma 3.3: If A is an I-vague group of a group G, then $V_A(e) \geq V_A(x)$ for all $x \in G$.
Proof: Let $x \in G$.
$V_A(e) = V_A(xx^{-1}) \geq \text{ iinf}\{V_A(x),\ V_A(x^{-1})\} = V_A(x)$ for all $x \in G$. Therefore $V_A(e) \geq V_A(x)$ for all $x \in G$.

Lemma 3.4: Let $m \in Z$. If A is an I-vague group of a group G, then $V_A(x^m) \geq V_A(x)$ for all $x \in G$.
Proof: Let $m \in Z$. We prove that $V_A(x^m) \geq V_A(x)$ for all $x \in G$. Since $V_A(e) \geq V_A(x)$ for all $x \in G$ by lemma 3.3, the statement is true for m = 0.
First we prove that the lemma is true for positive integers by induction.
Since $V_A(x) \geq V_A(x)$, it is true for m = 1.
Assume it is true for m.
$V_A(x^{m+1}) = V_A(x^m x) \geq \text{iinf}\{V_A(x^m), V_A(x)\} = V_A(x)$.
Thus $V_A(x^{m+1}) \geq V_A(x)$. Hence the statement is true for non-negative integers.
Suppose that m is a negative integer.
$V_A(x^m) = V_A((x^{-1})^{-m}) \geq V_A(x^{-1}) = V_A(x)$. We have $V_A(x^m) \geq V_A(x)$.
Consequently, $V_A(x^m) \geq V_A(x)$ for all $x \in G$ and for every integer m. Hence the lemma follows.

Lemma 3.5: A necessary and sufficient condition for an I-vague set A of a group G to be an I-vague group of G is that $V_A(xy^{-1}) \geq \text{ iinf}\{V_A(x),\ V_A(y)\}$ for all $x,\ y \in G$.
Proof: Let A be an I-vague set of G.
Suppose that $V_A(xy^{-1}) \geq\text{iinf}\{V_A(x), V_A(y)\}$ for all
$x,\ y \in G$. Let $x \in$ G.
Then $V_A(e) = V_A(xx^{-1}) \geq\text{iinf}\{V_A(x), V_A(x)\} = V_A(x)$.
Thus $V_A(e) \geq V_A(x)$ for all $x \in G$.
$V_A(x^{-1}) = V_A(ex^{-1}) \geq \text{ iinf}\{V_A(e), V_A(x)\} = V_A(x)$.
Thus $V_A(x^{-1}) \geq V_A(x)$ for each $x \in$ G.
Let $x,\ y \in$ G. Then
$V_A(xy) = V_A(x(y^{-1})^{-1}) \geq\text{iinf}\{V_A(x), V_A(y^{-1})\}$
$\geq\text{iinf}\{V_A(x), V_A(y)\}$. Hence
$V_A(xy) \geq\text{iinf}\{V_A(x), V_A(y)\}$ for each $x, y \in$ G, so A is an I-vague group of G.
Conversely, suppose that A is an I-vague group of G. Let $x,\ y \in$ G. Then
$V_A(xy^{-1}) \geq \text{ iinf}\{V_A(x),\ V_A(y^{-1})\} = \text{ iinf}\{V_A(x),\ V_A(y)\}$.
Therefore $V_A(xy^{-1}) \geq\text{iinf}\{V_A(x),\ V_A(y)\}$ for all $x,\ y \in$ G.
Hence the theorem follows.

Lemma 3.6: Let H be a subgroup of G and
$[\gamma,\ \delta] \leq [\alpha,\ \beta]$ with $\alpha, \beta, \gamma, \delta \in$ I where $\alpha \leq \beta$ and $\gamma \leq \delta$. Then the I-vague set A of G defined by

$$V_A(x) = \begin{cases} [\alpha,\ \beta] & \text{if } x \in H \text{ ;} \\ [\gamma,\ \delta] & \text{otherwise} \end{cases}$$

is an I-vague group of G.
Proof: Let H be a subgroup of G. We prove that the I-vague set A defined as above is an I-vague group of G.

Let $x,\ y \in G$. If $xy^{-1} \in H$, then $V_A(xy^{-1}) = [\alpha,\ \beta]$.
Hence $V_A(xy^{-1}) \geq$ iinf$\{V_A(x), V_A(y)\}$.
If $xy^{-1} \notin H$, then either $x \notin H$ or $y \notin H$.
Thus, iinf$\{V_A(x),\ V_A(y)\} = [\gamma,\ \delta]$. It follows that $V_A(xy^{-1}) \geq$ iinf$\{V_A(x),\ V_A(y)\}$. Hence $V_A(xy^{-1}) \geq$ iinf$\{V_A(x),\ V_A(y)\}$ for every $x,\ y \in$ G.
Therefore A is an I-vague group of G.

Example: Consider the group $(Z,\ +)$. Let I be the unit interval [0, 1] of real numbers. Let a $\oplus$ b = min $\{1,\ a+b\}$. With the usual ordering $(I, \oplus, \leq, -)$ is an involutary DRL-semigroup.
Define the I-vague set A of G as follows:

$$V_A(x) = \begin{cases} [a_1,\ b_1] & \text{if } x \in 4Z; \\ [a_2,\ b_2] & \text{if } x \in 2Z - 4Z; \\ [a_3,\ b_3] & \text{otherwise} \end{cases}$$

where $[a_3,\ b_3] \leq [a_2,\ b_2] \leq [a_1,\ b_1]$ and $a_i,\ b_i \in$[0, 1] for i = 1, 2, 3. Then A is an I-vague group of G.
We prove that $V_A(xy^{-1}) \geq$ iinf$\{V_A(x),\ V_A(y)\}$ for all $x,\ y \in$G.
(i) If $xy^{-1} \in 4Z$, then $V_A(xy^{-1})$ $= [a_1,\ b_1] \geq$ iinf$\{V_A(x),\ V_A(y)\}$.
(ii) If $xy^{-1} \in 2Z - 4Z$, then there exist $x, y \in Z$ such that $x \notin 4Z$ or $y \notin 4Z$. This implies iinf$\{V_A(x),\ V_A(y)\} \leq [a_2,\ b_2] = V_A(xy^{-1})$.
(iii) If $xy^{-1} = x - y$ is odd, then one of them must be odd. Hence iinf$\{V_A(x),\ V_A(y)\} = [a_3,\ b_3] \leq V_A(xy^{-1})$.
Therefore A is an I-vague group of G.

Lemma 3.7: Let H $\neq \emptyset$ and $H \subseteq G$. The I-vague characterstic set of H, χ_H is an I-vague group of G iff H is a subgroup of G.

Proof: Suppose that H is a subgroup of G. By Lemma 3.6, χ_H is an I-vague group of G.
Conversely, suppose that χ_H is an I-vague group of G.
We show that H is a subgroup of G. Let $x,\ y \in$ H. Then $V_{\chi_H}(xy^{-1}) \geq$ iinf$\{V_{\chi_H}(x), V_{\chi_H}(y)\} = [1,\ 1]$. Hence $V_{\chi_H}(xy^{-1}) = [1,\ 1]$, so $xy^{-1} \in$ H. Therefore H is a subgroup of G. Hence the lemma follows.

Lemma 3.8: If A and B are I-vague groups of a group G, then $A \cap B$ is also an I-vague group of G.

Proof: Let A and B are I-vague groups of G. Then $A \cap B$ is an I-vague set of G by lemma 2.17. Now we show that $V_{A\cap B}(xy^{-1}) \geq$ iinf$\{V_{A\cap B}(x), V_{A\cap B}(y)\}$ for each $x, y \in$ G.
Let $x,\ y \in$ G. Then

$$\begin{aligned} V_{A\cap B}(xy^{-1}) &= \text{iinf}\{V_A(xy^{-1}),\ V_B(xy^{-1})\} \\ &\geq \text{iinf}\{\text{iinf}\{V_A(x),\ V_A(y)\},\ \text{iinf}\{V_B(x),\ V_B(y)\}\} \\ &= \text{iinf}\{\text{iinf}\{V_A(x),\ V_B(x)\},\ \text{iinf}\ \{V_A(y),\ V_B(y)\}\} \\ &= \text{iinf}\{V_{A\cap B}(x),\ V_{A\cap B}(y)\}. \end{aligned}$$

Thus $V_{A\cap B}(xy^{-1}) \geq$ iinf$\{V_{A\cap B}(x),\ V_{A\cap B}(y)\}$ for every $x,\ y \in$ G. Therefore $A \cap B$ is an I-vague group of G.

Lemma 3.9: Let I be complete. If $\{A_i$: i $\in \triangle\}$ is a non empty family of I-vague groups of G, then $\bigcap_{i\in\triangle} A_i$ is an I-vague group of G.

Proof: Let A = $\bigcap_{i\in\triangle} A_i$. Then A is an I-vague set of G by lemma 2.19.
Now we prove that $V_A(xy^{-1}) \geq$ iinf$\{V_A(x), V_A(y)\}$ for every $x,\ y \in$ G. Let $x,\ y \in$ G. Then

$$\begin{aligned} V_A(xy^{-1}) &= V_{\bigcap_{i\in\triangle} A_i}(xy^{-1}) \\ &= \text{iinf}\{V_{A_i}(xy^{-1}):\ i \in \triangle\} \\ &\geq \text{iinf}\{\text{iinf}\{V_{A_i}(x), V_{A_i}(y)\}:\ i \in \triangle\} \\ &= \text{iinf}\{\text{iinf}\{V_{A_i}(x):\ i \in \triangle\},\ \text{iinf}\{V_{A_i}(y): i \in \triangle\}\} \\ &= \text{iinf}\{V_A(x),\ V_A(y)\}. \end{aligned}$$

Hence $V_A(xy^{-1}) \geq$ iinf$\{V_A(x),\ V_A(y)\}$ for every $x,\ y \in$ G.
Therefore $\bigcap_{i\in\triangle} A_i$ is an I-vague group of G.

Example: Let I = The positive divisors of 30 = $\{1, 2, 3, 5, 6, 10, 15, 30\}$ in which
$x \vee y$ = The least common multiple of x and y.
$x \wedge y$ = The greatest common divisor of x and y.
$x' = \frac{30}{x}$. Then I = $(I, \vee, \wedge, ', 1, 30)$ is a Boolean algebra. Hence it is an involutary DRL-semigroup.
Consider the group G = $(Z,\ +)$. Then H = $(2Z,\ +)$ and K = $(3Z,\ +)$ are subgroups of G. Define the I-vague groups A and B of G as follows:

$$V_A(x) = \begin{cases} [15,\ 30] & \text{if } x \in H\ ; \\ [5,\ 10] & \text{otherwise} \end{cases}$$

and

$$V_B(x) = \begin{cases} [15,\ 30] & \text{if } x \in K; \\ [5,\ 10] & \text{otherwise.} \end{cases}$$

Let $x = 2$ and $y = 3$. $xy = x + y = 5$.
$V_{A\cup B}(xy) = V_{A\cup B}(5)$ = isup$\{V_A(5),\ V_B(5)\} = [5,\ 10]$.
$V_{A\cup B}(x) = V_{A\cup B}(2)$ = isup$\{V_A(2),\ V_B(2)\} = [15,\ 30]$.
$V_{A\cup B}(y) = V_{A\cup B}(3)$ = isup$\{V_A(3),\ V_B(3)\} = [15,\ 30]$.
iinf$\{V_{A\cup B}(x),\ V_{A\cup B}(y)\} = [15,\ 30]$.
But $V_{A\cup B}(xy) = [5, 10] < [15, 30]$ = iinf$\{V_{A\cup B}(x),\ V_{A\cup B}(y)\}$. Therefore $A \cup B$ is not an I-vague group of G.
The above example shows that the union of two I-vague groups of G is not an I-vague group of G.
However we have the following.

Lemma 3.10: Let A be an I-vague group of G and B be a constant I-vague group of G. Then A∪B is an I-vague group of G.

Proof: Let A be an I-vague group of G and B be a constant I-vague group of G. Then A∪B is an I-vague set of G by lemma 2.17.
We prove that A∪B is an I-vague group of G.
Since B is a constant I-vague group of G, $V_B(x) = V_B(y)$ for all $x,\ y \in$G. Let $x,\ y \in$ G. Then

$$\begin{aligned} V_{A\cup B}(xy^{-1}) &= \text{isup}\{V_A(xy^{-1}),\ V_B(xy^{-1})\ \} \\ &\geq\ \text{isup}\{\text{iinf}\{V_A(x),\ V_A(y)\},\ V_B(x)\} \\ &= \text{iinf}\{\text{isup}\{V_A(x),\ V_B(x)\}, \text{isup}\{V_A(y),\ V_B(x)\}\} \\ &= \text{iinf}\{\text{isup}\{V_A(x),\ V_B(x)\}, \text{isup}\{V_A(y),\ V_B(y)\}\} \\ &= \text{iinf}\{V_{A\cup B}(x),\ V_{A\cup B}(y)\}. \end{aligned}$$

Thus $V_{A\cup B}(xy^{-1}) \geq$ iinf$\{V_{A\cup B}(x),\ V_{A\cup B}(y)\}$ for all $x,\ y \in$G. Hence A∪B is an I-vague group of G.

Theorem 3.11: An I-vague set A of a group G is an I-vague group of G if and only if for all $\alpha,\ \beta \in$ I with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha,\ \beta)}$ is a subgroup of G whenever it is non empty.

Proof: Let A be an I-vague set of G.
Suppose that A is an I-vague group of G. We prove that $A_{(\alpha,\ \beta)}$ is a subgroup of G whenever it is non empty.
Let $x,\ y \in A_{(\alpha,\ \beta)}$. Then $V_A(x) \geq [\alpha,\ \beta]$ and

$V_A(y) \geq [\alpha,\ \beta]$. Since A is an I-vague group of G, $V_A(xy^{-1}) \geq$ iinf$\{V_A(x),\ V_A(y)\} \geq [\alpha,\ \beta]$. Hence $xy^{-1} \in A_{(\alpha,\ \beta)}$, so $A_{(\alpha,\ \beta)}$ is a subgroup of G.
Conversely, suppose that for all $\alpha,\ \beta \in$ I with $\alpha \leq \beta$, the non empty set $A_{(\alpha,\ \beta)}$ is a subgroup of G. We prove that A is an I-vague group of G.
Let $x,\ y \in$ G. Suppose that $V_A(x) = [\alpha,\ \beta]$ and $V_A(y) = [\gamma,\ \delta]$. Then $x \in A_{(\alpha,\ \beta)}$ and $y \in A_{(\gamma,\ \delta)}$.
Let iinf$\{V_A(x),\ V_A(y)\} = [\alpha \wedge \gamma,\ \beta \wedge \delta] = [\eta,\ \zeta]$. It follows that $x,\ y \in A_{(\eta,\ \zeta)}$. Since $A_{(\eta,\ \zeta)}$ is a subgroup of G, $xy^{-1} \in A_{(\eta,\ \zeta)}$. Thus $V_A(xy^{-1}) \geq [\eta,\ \zeta]$. As a result we have $V_A(xy^{-1}) \geq$ iinf$\{V_A(x),\ V_A(y)\}$. Therefore A is an I-vague group of G. Hence the theorem follows.

Theorem 3.12: Let A be an I-vague group of a group G. If $V_A(xy^{-1}) = V_A(e)$ for $x,\ y \in G$, then $V_A(x) = V_A(y)$.
Proof: Suppose that $V_A(xy^{-1}) = V_A(e)$ for $x,\ y \in G$.
$V_A(x) = V_A(xe) = V_A(xy^{-1}y) \geq$ iinf$\{V_A(xy^{-1}),\ V_A(y)\}$= iinf$\{V_A(e),\ V_A(y)\} = V_A(y)$. Thus $V_A(x) \geq V_A(y)$.
Since $V_A(xy^{-1}) = V_A(yx^{-1})$, we have $V_A(y) \geq V_A(x)$.
Therefore $V_A(x) = V_A(y)$. Hence the theorem follows.
The following example shows that the converse of the preceeding theorem is not true.
Example: Let I be the unit interval [0, 1] of real numbers. Define a $\oplus$ b = min $\{1,\ a+b\}$. With the usual ordering $(I, \oplus, \leq, -)$ is an involutary DRL-semigroup.
Consider G = $(Z,\ +)$ and H = $(3Z,\ +)$. Let A be the I-vague group of G defined by

$$V_A(x) = \begin{cases} [\frac{1}{2},\ 1] & \text{if } x \in H\ ; \\ [0,\ \frac{3}{4}] & \text{otherwise.} \end{cases}$$

Let x = 2 and y = 1. $V_A(x) = V_A(y) = [0,\ \frac{3}{4}]$ but $V_A(xy^{-1}) = V_A(2-1) = V_A(1) = [0,\ \frac{3}{4}] \neq V_A(0)$.

Theorem 3.13: Let A be an I-vague group of a group G and $x \in G$. Then $V_A(yx) = V_A(xy) = V_A(y)$ for all $y \in G$ iff $V_A(x) = V_A(e)$.
Proof: Let A be an I-vague group of a group G and $x \in G$.
Suppose that $V_A(yx) = V_A(xy) = V_A(y)$ for all $y \in G$. Take $y = e$. It follows that $V_A(x) = V_A(e)$.
Conversely, suppose that $V_A(x) = V_A(e)$.
We prove that $V_A(yx) = V_A(xy) = V_A(y)$ for all $y \in G$.
For any $y \in G,\ V_A(y) \leq V_A(e) = V_A(x)$.
$V_A(xy) \geq$ iinf$\{V_A(x),\ V_A(y)\} = V_A(y)$.
Hence $V_A(xy) \geq V_A(y)$.

$$\begin{aligned} V_A(y) = V_A(ey) &= V_A(x^{-1}xy) \\ &\geq \text{iinf}\{V_A(x^{-1}),\ V_A(xy)\} \\ &= \text{iinf}\{V_A(x),\ V_A(xy)\} \\ &= \text{iinf}\{V_A(e),\ V_A(xy)\} \\ &= V_A(xy). \end{aligned}$$

Thus $V_A(y) \geq V_A(xy)$. Hence we have $V_A(xy) = V_A(y)$
Similarly, $V_A(yx) = V_A(y)$. Therefore $V_A(yx) = V_A(xy) = V_A(y)$. Hence the theorem follows.

Lemma 3.14: Let A be an I-vague group of a group G. Then $GV_A = \{x \in G :\ V_A(x) = V_A(e)\}$ is a subgroup of G.
Proof: Let A be an I-vague group of G. Since $e \in GV_A$, $GV_A \neq \emptyset$ and $GV_A \subseteq G$. Let $x,\ y \in GV_A$. We prove that $xy^{-1} \in GV_A$.
$V_A(xy^{-1}) \geq$ iinf$\{V_A(x),\ V_A(y)\} = V_A(e)$. Since $V_A(e) \geq V_A(xy^{-1})$ for all $x,\ y \in GV_A$, $V_A(xy^{-1}) = V_A(e)$. Thus $xy^{-1} \in GV_A$. Therefore GV_A is a subgroup of G.

Lemma 3.15: Let A be an I-vague group of a group G. If $< x > \subseteq < y >$ then $V_A(y) \leq V_A(x)$.
Proof: Suppose that $< x > \subseteq < y >$. Then $x \in < y >$. It follows that $x = y^m$ for some $m \in Z$.
$V_A(x) = V_A(y^m) \geq V_A(y)$. Therefore $V_A(x) \geq V_A(y)$.
The following example shows that the converse of lemma 3.15 is not true.
Example: Let I be the unit interval [0, 1] of real numbers. Let a $\oplus$ b = min $\{1, a+b\}$. With the usual ordering $(I, \oplus, \leq, -)$ is an involutary DRL-semigroup. Let G = The klein-4-group = $\{e,\ a,\ b,\ c\}$.
Define the I-vague set A of G by

$$V_A(x) = \begin{cases} [\frac{1}{2},\ 1] & \text{if } x \in < a >\ ; \\ [0,\ \frac{3}{4}] & \text{otherwise.} \end{cases}$$

Then $V_A(c) = [0,\ \frac{3}{4}] \leq [\frac{1}{2},\ 1] = V_A(a)$ but $< a >$ is not a subset of $< c >$.

Definition 3.16: Let A be an I-vague group of a group G. Image of A is defined as $ImA = \{V_A(x) :\ x \in G\}$.
Since $V_A(e) \geq V_A(x)$ for all $x \in G,\ V_A(e)$ is the greatest element of ImA.

Theorem 3.17: Let A be an I-vague group of a group G.
(i) If G is cyclic then ImA has a least element.
(ii) If $V_A(x) \leq V_A(y)$ then $< x > \ \supseteq < y >$ and ImA has a least element then G is cyclic.
Proof: Let A be an I-vague group of G.
(i) Suppose that G is cyclic. Then G = $< x >$ for some $x \in$ G. We prove that $V_A(x)$ is the least element of ImA.
Let $y \in$ G. Then $y = x^m$ for some $m \in Z$. $V_A(y) = V_A(x^m) \geq V_A(x)$. We have $V_A(x) \leq V_A(y)$ for every $y \in$ G. Thus $V_A(x)$ is the least element of image of A. Hence ImA has a least element.
(ii) Suppose that ImA has a least element say $V_A(x)$ for some $x \in G$. Let $y \in G$. Thus $V_A(y) \geq V_A(x)$ for all $y \in$ G. By our condition we have $< y > \subseteq < x >$. Since $y \in < y >,\ y \in < x >$. Hence G $\subseteq < x >$. Consequently, we have G = $< x >$. Therefore G is cyclic.

Lemma 3.18: Let A be an I-vague group of G. Let $x,\ y \in$ G. The two conditions
i) $V_A(x) = V_A(y) \Rightarrow < x > = < y >$
ii) $V_A(x) > V_A(y) \Rightarrow < x > \subseteq < y >$ are equivalent to the condition $V_A(x) \geq V_A(y) \Rightarrow < x > \subseteq < y >$.
Proof: Assume that the two conditions are given.
We prove that $V_A(x) \geq V_A(y) \Rightarrow < x > \subseteq < y >$.
If $V_A(x) > V_A(y)$, then $< x > \subseteq < y >$ by (ii).
If $V_A(x) = V_A(y)$, then $< x > = < y >$ by (i).
We have $< x > \subseteq < y >$.
Conversely, assume that $V_A(x) \geq V_A(y) \Rightarrow < x > \subseteq < y >$.
(i) Suppose that $V_A(x) = V_A(y)$.

$$\begin{aligned} V_A(x) = V_A(y) &\Rightarrow V_A(x) \geq V_A(y) \text{ and } V_A(y) \geq V_A(x). \\ &\Rightarrow\ < x > \subseteq < y > \text{ and } < y > \subseteq < x >. \\ &\Rightarrow\ < x > = < y >. \end{aligned}$$

$$\begin{aligned} \text{(ii) } V_A(x) > V_A(y) &\Rightarrow V_A(x) \geq V_A(y) \\ &\Rightarrow < x > \subseteq < y >. \end{aligned}$$

Thus $V_A(x) > V_A(y) \Rightarrow < x > \subseteq < y >$. Therefore (i) and (ii) are equivalent to $V_A(x) \geq V_A(y) \Leftrightarrow < x > \subseteq < y >$.

Theorem 3.19: Let A be an I-vague group of a group G such that the image set of A is given by ImA = $\{I_0 > I_1 > ... > I_n\}$ and such that
(i) $V_A(x) = V_A(y) \Rightarrow < x >=< y >$;
(ii) $V_A(x) < V_A(y) \Rightarrow < x > \supseteq < y >$.
Then G is a cyclic group of prime power order.
Proof: Let A be an I-vague group of a group G. Since ImA =$\{I_0 > I_1 > ... > I_n\}$, Im A has a least element. By theorem 3.17, G is cyclic. It follows that $G \cong Z$ or $G \cong Z_m$ for some $m \in N$. Suppose that $G \cong Z$. Consider $V_A(2)$ and $V_A(3)$.
If $V_A(2) = V_A(3)$, then $< 2 >=< 3 >$ by (i). But this is not true since $2 \notin < 3 >$. So either $V_A(2) > V_A(3)$ or $V_A(3) > V_A(2)$.
If $V_A(2) > V_A(3)$, then $< 2 > \subseteq < 3 >$ by (ii). But this is not true since $2 \notin < 3 >$.
If $V_A(3) > V_A(2)$, then $< 3 > \subseteq < 2 >$ by (ii). But this is not true since $3 \notin < 2 >$. Therefore G is not isomorphic to Z. Thus $G \cong Z_m$ for some $m \in N$.
Suppose that m is not a prime power. Then there exist prime numbers p and q such that $p \neq q$ which are factors of m. Consider $V_A(p)$ and $V_A(q)$.
Since ImA = $\{I_0 > I_1 > ... > I_n\}$, either $V_A(p) \geq V_A(q)$ or $V_A(p) < V_A(q)$. It follows that $< p > \subseteq < q >$ or $< q > \subseteq < p >$, a contradiction.
Thus our supposition is false. Therefore m is prime power. Hence the theorem follows.

Theorem 3.20: Let G be a cyclic group of prime power order then there is an I and an I-vague group A of G such that for all $x, \ y \in G$
(i) $V_A(x) = V_A(y) \Rightarrow < x > = < y >$;
(ii) $V_A(x) > V_A(y) \Rightarrow < x > \subseteq < y >$.
Proof: Suppose that G is a cyclic group of order p^n where p is prime and $n \in N \cup \{0\}$. We find an I and an I-vague group A of G satisfying (i) and (ii).
Step(1) We construct an I and an I-vague set of G.
Let I be the unit interval [0, 1] of real numbers. Define
$a \oplus b$ = min $\{1, \ a + b\}$
With the usual ordering $(I, \oplus, \leq, -)$ is an involutary DRL-semigroup.
Now we construct our I-vague set of G.
Let $z \in$ G. Then O(z) = p^i where $i = 0, 1, 2, ..., n$.
Define A=$(t_A, \ f_A)$ where $t_A : \ G \rightarrow I$ and $f_A : G \rightarrow I$ such that $t_A(z) = a_i, \ f_A(z) = b_i$ where $a_i, \ b_i \in$[0, 1] satisfying $a_i \leq 1 - b_i$ for $i = 0, 1, 2, ..., n$. Choose the intervals $I_0, \ I_1, \ ..., \ I_n$ in such a way that $I_0 > I_1 > ... > I_n$ where $I_i = [a_i, 1 - b_i]$. Then $V_A(z) = I_i$. Hence A is an I-vague set of G. We have $V_A(e) = I_0$.
Step(2) We show that A is an I-vague group of G.
Let $x \in$ G. $O(x) = O(x^{-1})$ implies $V_A(x) = V_A(x^{-1})$.
To show A is an I-vague group of G it remains to prove that $V_A(xy) \geq$ iinf$\{V_A(x), \ V_A(y)\}$ for every $x, \ y \in$ G.
Let $x, \ y \in$ G. Since G is a cyclic group of order p^n and the order of the subgroup divides the order of the group, $O(< x >) = p^j, \ O(< y >) = p^k$ and $O(< xy >) = p^m$ for some $j, \ k, \ m \in \{0, 1, ..., n\}$ say.
Therefore $V_A(x) = I_j, \ V_A(y) = I_k$ and $V_A(xy) = I_m$. Moreover, since G is a cyclic group of prime power order, $< x > \subseteq < y >$ or $< y > \subseteq < x >$.
If $< x > \subseteq < y > \ then \ x, \ y \in < y >$. Hence $< xy > \subseteq < y >$.
If $< y > \subseteq < x > \ then \ x, \ y \in < x >$. Hence $< xy > \subseteq < x >$.
Therefore $< xy > \subseteq < y >$ or $< xy > \subseteq < x >$.
Assume that $< xy > \subseteq < x >$. It follows that $O(< xy >) < O(< x >)$ or $O(< xy >) = O(< x >)$.
If $O(< xy >) < O(< x >)$ then $m < j$. It follows that $I_m > I_j$.
Hence $V_A(xy) = I_m \geq$ iinf$\{I_j, \ I_k\}$= iinf$\{V_A(x), \ V_A(y)\}$.
Thus $V_A(xy) \geq$ iinf$\{V_A(x), \ V_A(y)\}$.
If $O(< xy >) = O(< x >)$ then $m = j$. Hence $I_m = I_j$.
$V_A(xy) = I_m \geq$ iinf$\{I_m, \ I_k\}$ =iinf$\{V_A(x), \ V_A(y)\}$.
Thus $V_A(xy) \geq$ iinf$\{V_A(x), \ V_A(y)\}$.
In both cases $V_A(xy) \geq$ iinf$\{V_A(x), \ V_A(y)\}$ and $V_A(x) \geq V_A(x^{-1})$ for all $x, \ y \in$ G.
Thus A is an I-vague group of G.
Step(3) We show that A satisfies the conditions (i) and (ii) of the theorem.
(a) Suppose that $V_A(x) = V_A(y)$ for $x, \ y \in$ G.
By the definition of A we have $O(< x >) = O(< y >)$.
Since G is a cyclic group of prime power order, $O(< x >) = O(< y >)$ implies $< x >=< y >$.
Hence $V_A(x) = V_A(y) \Rightarrow < x > = < y >$.
(b) Suppose that $V_A(x) > V_A(y)$ for $x, \ y \in$ G. Then $I_j > I_k$.
It follows that $j < k$.
Hence $p^j < p^k$, so $O(< x >) < O(< y >)$.
Since G is a cyclic group of order p^n and $O(< x >) < O(< y >)$, $< x > \subseteq < y >$.
Thus $V_A(x) > V_A(y) \Rightarrow < x > \subseteq < y >$.
Therefore A satisfies (i) and (ii).
Hence the theorem follows.

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