

I-Vague Groups

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Abstract—The notions of I-vague groups with membership and non-membership functions taking values in an involutory dually residuated lattice ordered semigroup are introduced which generalize the notions with truth values in a Boolean algebra as well as those usual vague sets whose membership and non-membership functions taking values in the unit interval $[0, 1]$. Moreover, various operations and properties are established.

Keywords—Involutory dually residuated lattice ordered semigroup, I-vague set and I-vague group.

I. INTRODUCTION

THE notion of fuzzy groups defined by A. Rosenfeld[13] is the first application of fuzzy set theory in Algebra. Since then a number of works have been done in the area of fuzzy algebra.

M. Demirci[5] studied vague groups. R. Biswas[2] defined the notion of vague groups analogous to the idea of Rosenfeld [13]. H. Khan, M. Ahmad and R. Biswas[8] studied vague groups and made some characterizations. N. Ramakrishna[10] studied vague groups and vague weights.

The vague sets of W. L. Gau and D. J. Buehrer[6] and Atanassov's[1] intuitionistic fuzzy sets are mathematically equivalent objects[3]. In this paper we prefer the terminology of vague sets as the algebraic study initiated by Biswas[2] follows the terminology of vague sets.

K. L. N. Swamy[14], [15], [16] introduced the concept of dually residuated lattice ordered semigroup(in short DRL-semigroup) which is a common abstraction of Boolean algebras and lattice ordered groups. The subclass of DRL-semigroups which are bounded and involutory(i.e having 0 as least, 1 as greatest and satisfying $1-(1-x) = x$) which is categorically equivalent to the class of MV-algebras of C. C. Chang[4] and well studied offer a natural generalization of the closed unit interval $[0, 1]$ of real numbers as well as Boolean algebras. Thus, the study of vague sets (t_A, f_A) with values in an involutory DRL-semigroup promises a unified study of real valued vague sets and also those Boolean valued vague sets[11].

In his thesis T. Zelalem[19] studied the concept of I-vague sets. In this paper using the definition of I-vague sets, we defined and studied I-vague groups where I is an involutory DRL-semigroup. In this paper we shall recall some basic results in [14], [15], [19] without proof. Moreover, notation, terminology and results of [19] are used in this paper. Throughout this paper, we shall denote the identity element of a group (G, \cdot) by e and the order of an element x of G by $O(x)$. Moreover, for $x \in G$, $\langle x \rangle$ denotes the cyclic group generated by x .

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II. PRELIMINARIES

Definition 2.1: [14] A system $A = (A, +, \leq, -)$ is called a dually residuated lattice ordered semigroup(in short DRL-semigroup) if and only if

- i) $A = (A, +)$ is a commutative semigroup with zero"0";
- ii) $A = (A, \leq)$ is a lattice such that

$$a + (b \cup c) = (a + b) \cup (a + c) \quad \text{and} \quad a + (b \cap c) = (a + b) \cap (a + c) \quad \text{for all } a, b, c \in A;$$

iii) Given $a, b \in A$, there exists a least x in A such that $b + x \geq a$, and we denote this x by $a - b$ (for a given a, b this x is uniquely determined);

- iv) $(a-b) \cup 0 + b \leq a \cup b$ for all $a, b \in A$;
- v) $a - a \geq 0$ for all $a \in A$.

Theorem 2.2: [14] Any DRL-semigroup is a distributive lattice.

Definition 2.3: [19] A DRL-semigroup A is said to be involutory if there is an element $1(\neq 0)$ (0 is the identity w.r.t. $+$) such that

$$(i) \quad a + (1 - a) = 1 + 1;$$

$$(ii) \quad 1 - (1 - a) = a \quad \text{for all } a \in A.$$

Theorem 2.4: [15] In a DRL-semigroup with 1, 1 is unique.

Theorem 2.5: [15] If a DRL-semigroup contains a least element x , then $x = 0$. Dually, if a DRL-semigroup with 1 contains a largest element α , then $\alpha = 1$.

Throughout this paper let $I = (I, +, -, \vee, \wedge, 0, 1)$ be a dually residuated lattice ordered semigroup satisfying $1 - (1 - a) = a$ for all $a \in I$.

Lemma 2.6: [19] Let 1 be the largest element of I. Then for $a, b \in I$

$$(i) \quad a + (1 - a) = 1.$$

$$(ii) \quad 1 - a = 1 - b \iff a = b.$$

$$(iii) \quad 1 - (a \cup b) = (1 - a) \cap (1 - b).$$

Lemma 2.7: [19] Let I be complete. If $a_\alpha \in I$ for every $\alpha \in \Delta$, then

$$(i) \quad 1 - \bigvee_{\alpha \in \Delta} a_\alpha = \bigwedge_{\alpha \in \Delta} (1 - a_\alpha).$$

$$(ii) \quad 1 - \bigwedge_{\alpha \in \Delta} a_\alpha = \bigvee_{\alpha \in \Delta} (1 - a_\alpha).$$

Definition 2.8: [19] An I-vague set A of a non-empty set G is a pair (t_A, f_A) where $t_A : G \rightarrow I$ and $f_A : G \rightarrow I$ with $t_A(x) \leq 1 - f_A(x)$ for all $x \in G$.

Definition 2.9: [19] The interval $[t_A(x), 1 - f_A(x)]$ is called the I-vague value of $x \in G$ and is denoted by $V_A(x)$.

Definition 2.10: [19] Let $B_1 = [a_1, b_1]$ and $B_2 = [a_2, b_2]$ be two I-vague values. We say $B_1 \geq B_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

Definition 2.11: [19] An I-vague set $A = (t_A, f_A)$ of G is said to be contained in an I-vague set $B = (t_B, f_B)$ of G written as $A \subseteq B$ if and only if $t_A(x) \leq t_B(x)$ and $f_A(x) \geq f_B(x)$ for all $x \in G$. A is said to be equal to B written as $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 2.12: [19] An I-vague set A of G with $V_A(x) = V_A(y)$ for all $x, y \in G$ is called a constant I-vague set of G.

Definition 2.13: [19] Let A be an I-vague set of a non empty set G. Let $A_{(\alpha, \beta)} = \{x \in G : V_A(x) \geq [\alpha, \beta]\}$ where $\alpha, \beta \in I$ and $\alpha \leq \beta$. Then $A_{(\alpha, \beta)}$ is called the (α, β) cut of the I-vague set A.

Definition 2.14: [19] Let $S \subseteq G$. The characteristic function of S denoted as $\chi_S = (t_{\chi_S}, f_{\chi_S})$, which takes values in I is defined as follows:

$$t_{\chi_S}(x) = \begin{cases} 1 & \text{if } x \in S; \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{\chi_S}(x) = \begin{cases} 0 & \text{if } x \in S; \\ 1 & \text{otherwise.} \end{cases}$$

χ_S is called the I-vague characteristic set of S in I. Thus

$$V_{\chi_S}(x) = \begin{cases} [1, 1] & \text{if } x \in S; \\ [0, 0] & \text{otherwise.} \end{cases}$$

Definition 2.15: [19] Let $A = (t_A, f_A)$ and $B = (t_B, f_B)$ be I-vague sets of a set G.

(i) Their union $A \cup B$ is defined as $A \cup B = (t_{A \cup B}, f_{A \cup B})$ where $t_{A \cup B}(x) = t_A(x) \vee t_B(x)$ and

$f_{A \cup B}(x) = f_A(x) \wedge f_B(x)$ for each $x \in G$.

(ii) Their intersection $A \cap B$ is defined as $A \cap B = (t_{A \cap B}, f_{A \cap B})$ where $t_{A \cap B}(x) = t_A(x) \wedge t_B(x)$ and $f_{A \cap B}(x) = f_A(x) \vee f_B(x)$ for each $x \in G$.

Definition 2.16: [19] Let $B_1 = [a_1, b_1]$ and $B_2 = [a_2, b_2]$ be I-vague values. Then

(i) $\text{isup}\{B_1, B_2\} = [\text{sup}\{a_1, a_2\}, \text{sup}\{b_1, b_2\}]$.

(ii) $\text{iinf}\{B_1, B_2\} = [\text{inf}\{a_1, a_2\}, \text{inf}\{b_1, b_2\}]$.

Lemma 2.17: [19] Let A and B be I-vague sets of a set G. Then $A \cup B$ and $A \cap B$ are also I-vague sets of G.

Let $x \in G$. From the definition of $A \cup B$ and $A \cap B$ we have

(i) $V_{A \cup B}(x) = \text{isup}\{V_A(x), V_B(x)\}$;

(ii) $V_{A \cap B}(x) = \text{iinf}\{V_A(x), V_B(x)\}$.

Definition 2.18: [19] Let I be complete and $\{A_i : i \in \Delta\}$ be a non empty family of I-vague sets of G where $A_i = (t_{A_i}, f_{A_i})$. Then

(i) $\bigcap_{i \in \Delta} A_i = (\bigwedge_{i \in \Delta} t_{A_i}, \bigvee_{i \in \Delta} f_{A_i})$

(ii) $\bigcup_{i \in \Delta} A_i = (\bigvee_{i \in \Delta} t_{A_i}, \bigwedge_{i \in \Delta} f_{A_i})$

Lemma 2.19: [19] Let I be complete. If $\{A_i : i \in \Delta\}$ is a non empty family of I-vague sets of G, then $\bigcap_{i \in \Delta} A_i$ and

$\bigcup_{i \in \Delta} A_i$ are I-vague sets of G.

Definition 2.20: [19] Let I be complete and $\{A_i = (t_{A_i}, f_{A_i}) : i \in \Delta\}$ be a non empty family of I vague sets of G. Then for each $x \in G$,

(i) $\text{isup}\{V_{A_i}(x) : i \in \Delta\} = [\bigvee_{i \in \Delta} t_{A_i}(x), \bigvee_{i \in \Delta} (1 - f_{A_i})(x)]$.

(ii) $\text{iinf}\{V_{A_i}(x) : i \in \Delta\} = [\bigwedge_{i \in \Delta} t_{A_i}(x), \bigwedge_{i \in \Delta} (1 - f_{A_i})(x)]$.

III. I-VAGUE GROUPS

Definition 3.1: Let G be a group. An I-vague set A of a group G is called an I-vague group of G if

(i) $V_A(xy) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for all $x, y \in G$ and

(ii) $V_A(x^{-1}) \geq V_A(x)$ for all $x \in G$.

Lemma 3.2: If A is an I-vague group of a group G, then $V_A(x) = V_A(x^{-1})$ for all $x \in G$.

Proof: Since A is an I-vague group of G, $V_A(x^{-1}) \geq V_A(x)$ for all $x \in G$. $V_A(x) = V_A((x^{-1})^{-1}) \geq V_A(x^{-1})$. Hence the lemma follows.

Lemma 3.3: If A is an I-vague group of a group G, then $V_A(e) \geq V_A(x)$ for all $x \in G$.

Proof: Let $x \in G$.

$V_A(e) = V_A(xx^{-1}) \geq \text{iinf}\{V_A(x), V_A(x^{-1})\} = V_A(x)$ for all $x \in G$. Therefore $V_A(e) \geq V_A(x)$ for all $x \in G$.

Lemma 3.4: Let $m \in \mathbb{Z}$. If A is an I-vague group of a group G, then $V_A(x^m) \geq V_A(x)$ for all $x \in G$.

Proof: Let $m \in \mathbb{Z}$. We prove that $V_A(x^m) \geq V_A(x)$ for all $x \in G$. Since $V_A(e) \geq V_A(x)$ for all $x \in G$ by lemma 3.3, the statement is true for $m = 0$.

First we prove that the lemma is true for positive integers by induction.

Since $V_A(x) \geq V_A(x)$, it is true for $m = 1$.

Assume it is true for m.

$V_A(x^{m+1}) = V_A(x^m x) \geq \text{iinf}\{V_A(x^m), V_A(x)\} = V_A(x)$.

Thus $V_A(x^{m+1}) \geq V_A(x)$. Hence the statement is true for non-negative integers.

Suppose that m is a negative integer.

$V_A(x^m) = V_A((x^{-1})^{-m}) \geq V_A(x^{-1}) = V_A(x)$. We have $V_A(x^m) \geq V_A(x)$.

Consequently, $V_A(x^m) \geq V_A(x)$ for all $x \in G$ and for every integer m. Hence the lemma follows.

Lemma 3.5: A necessary and sufficient condition for an I-vague set A of a group G to be an I-vague group of G is that $V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for all $x, y \in G$.

Proof: Let A be an I-vague set of G.

Suppose that $V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for all $x, y \in G$. Let $x \in G$.

Then $V_A(e) = V_A(xx^{-1}) \geq \text{iinf}\{V_A(x), V_A(x)\} = V_A(x)$.

Thus $V_A(e) \geq V_A(x)$ for all $x \in G$.

$V_A(x^{-1}) = V_A(ex^{-1}) \geq \text{iinf}\{V_A(e), V_A(x)\} = V_A(x)$.

Thus $V_A(x^{-1}) \geq V_A(x)$ for each $x \in G$.

Let $x, y \in G$. Then

$V_A(xy) = V_A(x(y^{-1})^{-1}) \geq \text{iinf}\{V_A(x), V_A(y^{-1})\}$

$\geq \text{iinf}\{V_A(x), V_A(y)\}$. Hence

$V_A(xy) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for each $x, y \in G$, so A is an I-vague group of G.

Conversely, suppose that A is an I-vague group of G. Let $x, y \in G$. Then

$V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y^{-1})\} = \text{iinf}\{V_A(x), V_A(y)\}$.

Therefore $V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for all $x, y \in G$. Hence the theorem follows.

Lemma 3.6: Let H be a subgroup of G and $[\gamma, \delta] \leq [\alpha, \beta]$ with $\alpha, \beta, \gamma, \delta \in I$ where $\alpha \leq \beta$ and $\gamma \leq \delta$. Then the I-vague set A of G defined by

$$V_A(x) = \begin{cases} [\alpha, \beta] & \text{if } x \in H; \\ [\gamma, \delta] & \text{otherwise} \end{cases}$$

is an I-vague group of G.

Proof: Let H be a subgroup of G. We prove that the I-vague set A defined as above is an I-vague group of G.

Let $x, y \in G$. If $xy^{-1} \in H$, then $V_A(xy^{-1}) = [\alpha, \beta]$. Hence $V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\}$.

If $xy^{-1} \notin H$, then either $x \notin H$ or $y \notin H$.

Thus, $\text{iinf}\{V_A(x), V_A(y)\} = [\gamma, \delta]$. It follows that

$V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\}$. Hence

$V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for every $x, y \in G$.

Therefore A is an I-vague group of G.

Example: Consider the group $(Z, +)$. Let I be the unit interval $[0, 1]$ of real numbers. Let $a \oplus b = \min\{1, a + b\}$. With the usual ordering $(I, \oplus, \leq, -)$ is an involutory DRL-semigroup.

Define the I-vague set A of G as follows:

$$V_A(x) = \begin{cases} [a_1, b_1] & \text{if } x \in 4Z; \\ [a_2, b_2] & \text{if } x \in 2Z - 4Z; \\ [a_3, b_3] & \text{otherwise} \end{cases}$$

where $[a_3, b_3] \leq [a_2, b_2] \leq [a_1, b_1]$ and $a_i, b_i \in [0, 1]$ for $i = 1, 2, 3$. Then A is an I-vague group of G.

We prove that $V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for all $x, y \in G$.

(i) If $xy^{-1} \in 4Z$, then $V_A(xy^{-1}) = [a_1, b_1] \geq \text{iinf}\{V_A(x), V_A(y)\}$.

(ii) If $xy^{-1} \in 2Z - 4Z$, then there exist $x, y \in Z$ such that $x \notin 4Z$ or $y \notin 4Z$. This implies $\text{iinf}\{V_A(x), V_A(y)\} \leq [a_2, b_2] = V_A(xy^{-1})$.

(iii) If $xy^{-1} = x - y$ is odd, then one of them must be odd. Hence $\text{iinf}\{V_A(x), V_A(y)\} = [a_3, b_3] \leq V_A(xy^{-1})$.

Therefore A is an I-vague group of G.

Lemma 3.7: Let $H \neq \emptyset$ and $H \subseteq G$. The I-vague characteristic set of H, χ_H is an I-vague group of G iff H is a subgroup of G.

Proof: Suppose that H is a subgroup of G. By Lemma 3.6, χ_H is an I-vague group of G.

Conversely, suppose that χ_H is an I-vague group of G.

We show that H is a subgroup of G. Let $x, y \in H$. Then

$V_{\chi_H}(xy^{-1}) \geq \text{iinf}\{V_{\chi_H}(x), V_{\chi_H}(y)\} = [1, 1]$. Hence

$V_{\chi_H}(xy^{-1}) = [1, 1]$, so $xy^{-1} \in H$. Therefore H is a subgroup of G. Hence the lemma follows.

Lemma 3.8: If A and B are I-vague groups of a group G, then $A \cap B$ is also an I-vague group of G.

Proof: Let A and B are I-vague groups of G. Then $A \cap B$ is an I-vague set of G by lemma 2.17. Now we show that

$V_{A \cap B}(xy^{-1}) \geq \text{iinf}\{V_{A \cap B}(x), V_{A \cap B}(y)\}$ for each $x, y \in G$.

Let $x, y \in G$. Then

$$\begin{aligned} V_{A \cap B}(xy^{-1}) &= \text{iinf}\{V_A(xy^{-1}), V_B(xy^{-1})\} \\ &\geq \text{iinf}\{\text{iinf}\{V_A(x), V_A(y)\}, \text{iinf}\{V_B(x), V_B(y)\}\} \\ &= \text{iinf}\{\text{iinf}\{V_A(x), V_B(x)\}, \text{iinf}\{V_A(y), V_B(y)\}\} \\ &= \text{iinf}\{V_{A \cap B}(x), V_{A \cap B}(y)\}. \end{aligned}$$

Thus $V_{A \cap B}(xy^{-1}) \geq \text{iinf}\{V_{A \cap B}(x), V_{A \cap B}(y)\}$ for every $x, y \in G$. Therefore $A \cap B$ is an I-vague group of G.

Lemma 3.9: Let I be complete. If $\{A_i: i \in \Delta\}$ is a non empty family of I-vague groups of G, then $\bigcap_{i \in \Delta} A_i$ is an I-vague group of G.

Proof: Let $A = \bigcap_{i \in \Delta} A_i$. Then A is an I-vague set of G by lemma 2.19.

Now we prove that $V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for every $x, y \in G$. Let $x, y \in G$. Then

$$\begin{aligned} V_A(xy^{-1}) &= V \bigcap_{i \in \Delta} A_i(xy^{-1}) \\ &= \text{iinf}\{V_{A_i}(xy^{-1}) : i \in \Delta\} \\ &\geq \text{iinf}\{\text{iinf}\{V_{A_i}(x), V_{A_i}(y)\} : i \in \Delta\} \\ &= \text{iinf}\{\text{iinf}\{V_{A_i}(x) : i \in \Delta\}, \text{iinf}\{V_{A_i}(y) : i \in \Delta\}\} \\ &= \text{iinf}\{V_A(x), V_A(y)\}. \end{aligned}$$

Hence $V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for every $x, y \in G$. Therefore $\bigcap_{i \in \Delta} A_i$ is an I-vague group of G.

Example: Let I = The positive divisors of 30 = $\{1, 2, 3, 5, 6, 10, 15, 30\}$ in which

$x \vee y$ = The least common multiple of x and y .

$x \wedge y$ = The greatest common divisor of x and y .

$x' = \frac{30}{x}$. Then $I = (I, \vee, \wedge, ', 1, 30)$ is a Boolean algebra.

Hence it is an involutory DRL-semigroup.

Consider the group $G = (Z, +)$. Then $H = (2Z, +)$ and

$K = (3Z, +)$ are subgroups of G. Define the I-vague groups A and B of G as follows:

$$V_A(x) = \begin{cases} [15, 30] & \text{if } x \in H; \\ [5, 10] & \text{otherwise} \end{cases}$$

and

$$V_B(x) = \begin{cases} [15, 30] & \text{if } x \in K; \\ [5, 10] & \text{otherwise.} \end{cases}$$

Let $x = 2$ and $y = 3$. $xy = x + y = 5$.

$V_{A \cup B}(xy) = V_{A \cup B}(5) = \text{isup}\{V_A(5), V_B(5)\} = [5, 10]$.

$V_{A \cup B}(x) = V_{A \cup B}(2) = \text{isup}\{V_A(2), V_B(2)\} = [15, 30]$.

$V_{A \cup B}(y) = V_{A \cup B}(3) = \text{isup}\{V_A(3), V_B(3)\} = [15, 30]$.

$\text{iinf}\{V_{A \cup B}(x), V_{A \cup B}(y)\} = [15, 30]$.

But $V_{A \cup B}(xy) = [5, 10] < [15, 30] =$

$\text{iinf}\{V_{A \cup B}(x), V_{A \cup B}(y)\}$. Therefore $A \cup B$ is not an I-vague group of G.

The above example shows that the union of two I-vague groups of G is not an I-vague group of G.

However we have the following.

Lemma 3.10: Let A be an I-vague group of G and B be a constant I-vague group of G. Then $A \cup B$ is an I-vague group of G.

Proof: Let A be an I-vague group of G and B be a constant I-vague group of G. Then $A \cup B$ is an I-vague set of G by lemma 2.17.

We prove that $A \cup B$ is an I-vague group of G.

Since B is a constant I-vague group of G, $V_B(x) = V_B(y)$ for all $x, y \in G$. Let $x, y \in G$. Then

$$\begin{aligned} V_{A \cup B}(xy^{-1}) &= \text{isup}\{V_A(xy^{-1}), V_B(xy^{-1})\} \\ &\geq \text{isup}\{\text{iinf}\{V_A(x), V_A(y)\}, V_B(x)\} \\ &= \text{iinf}\{\text{isup}\{V_A(x), V_B(x)\}, \text{isup}\{V_A(y), V_B(y)\}\} \\ &= \text{iinf}\{\text{isup}\{V_A(x), V_B(x)\}, \text{isup}\{V_A(y), V_B(y)\}\} \\ &= \text{iinf}\{V_{A \cup B}(x), V_{A \cup B}(y)\}. \end{aligned}$$

Thus $V_{A \cup B}(xy^{-1}) \geq \text{iinf}\{V_{A \cup B}(x), V_{A \cup B}(y)\}$ for all $x, y \in G$. Hence $A \cup B$ is an I-vague group of G.

Theorem 3.11: An I-vague set A of a group G is an I-vague group of G if and only if for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the I-vague cut $A_{(\alpha, \beta)}$ is a subgroup of G whenever it is non empty.

Proof: Let A be an I-vague set of G.

Suppose that A is an I-vague group of G. We prove that $A_{(\alpha, \beta)}$ is a subgroup of G whenever it is non empty.

Let $x, y \in A_{(\alpha, \beta)}$. Then $V_A(x) \geq [\alpha, \beta]$ and

$V_A(y) \geq [\alpha, \beta]$. Since A is an I-vague group of G, $V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\} \geq [\alpha, \beta]$. Hence $xy^{-1} \in A_{(\alpha, \beta)}$, so $A_{(\alpha, \beta)}$ is a subgroup of G. Conversely, suppose that for all $\alpha, \beta \in I$ with $\alpha \leq \beta$, the non empty set $A_{(\alpha, \beta)}$ is a subgroup of G. We prove that A is an I-vague group of G.

Let $x, y \in G$. Suppose that $V_A(x) = [\alpha, \beta]$ and $V_A(y) = [\gamma, \delta]$. Then $x \in A_{(\alpha, \beta)}$ and $y \in A_{(\gamma, \delta)}$. Let $\text{iinf}\{V_A(x), V_A(y)\} = [\alpha \wedge \gamma, \beta \wedge \delta] = [\eta, \zeta]$. It follows that $x, y \in A_{(\eta, \zeta)}$. Since $A_{(\eta, \zeta)}$ is a subgroup of G, $xy^{-1} \in A_{(\eta, \zeta)}$. Thus $V_A(xy^{-1}) \geq [\eta, \zeta]$. As a result we have $V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\}$. Therefore A is an I-vague group of G. Hence the theorem follows.

Theorem 3.12: Let A be an I-vague group of a group G. If $V_A(xy^{-1}) = V_A(e)$ for $x, y \in G$, then $V_A(x) = V_A(y)$.

Proof: Suppose that $V_A(xy^{-1}) = V_A(e)$ for $x, y \in G$. $V_A(x) = V_A(xe) = V_A(xy^{-1}y) \geq \text{iinf}\{V_A(xy^{-1}), V_A(y)\} = \text{iinf}\{V_A(e), V_A(y)\} = V_A(y)$. Thus $V_A(x) \geq V_A(y)$. Since $V_A(xy^{-1}) = V_A(yx^{-1})$, we have $V_A(y) \geq V_A(x)$. Therefore $V_A(x) = V_A(y)$. Hence the theorem follows.

The following example shows that the converse of the preceding theorem is not true.

Example: Let I be the unit interval [0, 1] of real numbers. Define $a \oplus b = \min\{1, a + b\}$. With the usual ordering $(I, \oplus, \leq, -)$ is an involutory DRL-semigroup. Consider $G = (Z, +)$ and $H = (3Z, +)$. Let A be the I-vague group of G defined by

$$V_A(x) = \begin{cases} [\frac{1}{2}, 1] & \text{if } x \in H ; \\ [0, \frac{3}{4}] & \text{otherwise.} \end{cases}$$

Let $x = 2$ and $y = 1$. $V_A(x) = V_A(y) = [0, \frac{3}{4}]$ but $V_A(xy^{-1}) = V_A(2 - 1) = V_A(1) = [0, \frac{3}{4}] \neq V_A(0)$.

Theorem 3.13: Let A be an I-vague group of a group G and $x \in G$. Then $V_A(yx) = V_A(xy) = V_A(y)$ for all $y \in G$ iff $V_A(x) = V_A(e)$.

Proof: Let A be an I-vague group of a group G and $x \in G$. Suppose that $V_A(yx) = V_A(xy) = V_A(y)$ for all $y \in G$. Take $y = e$. It follows that $V_A(x) = V_A(e)$.

Conversely, suppose that $V_A(x) = V_A(e)$. We prove that $V_A(yx) = V_A(xy) = V_A(y)$ for all $y \in G$.

For any $y \in G$, $V_A(y) \leq V_A(e) = V_A(x)$.

$V_A(xy) \geq \text{iinf}\{V_A(x), V_A(y)\} = V_A(y)$.

Hence $V_A(xy) \geq V_A(y)$.

$$\begin{aligned} V_A(y) = V_A(ey) &= V_A(x^{-1}xy) \\ &\geq \text{iinf}\{V_A(x^{-1}), V_A(xy)\} \\ &= \text{iinf}\{V_A(x), V_A(xy)\} \\ &= \text{iinf}\{V_A(e), V_A(xy)\} \\ &= V_A(xy). \end{aligned}$$

Thus $V_A(y) \geq V_A(xy)$. Hence we have $V_A(xy) = V_A(y)$. Similarly, $V_A(yx) = V_A(y)$. Therefore $V_A(yx) = V_A(xy) = V_A(y)$. Hence the theorem follows.

Lemma 3.14: Let A be an I-vague group of a group G. Then $GV_A = \{x \in G : V_A(x) = V_A(e)\}$ is a subgroup of G.

Proof: Let A be an I-vague group of G. Since $e \in GV_A$, $GV_A \neq \emptyset$ and $GV_A \subseteq G$. Let $x, y \in GV_A$. We prove that $xy^{-1} \in GV_A$.

$V_A(xy^{-1}) \geq \text{iinf}\{V_A(x), V_A(y)\} = V_A(e)$. Since $V_A(e) \geq$

$V_A(xy^{-1})$ for all $x, y \in GV_A$, $V_A(xy^{-1}) = V_A(e)$. Thus $xy^{-1} \in GV_A$. Therefore GV_A is a subgroup of G.

Lemma 3.15: Let A be an I-vague group of a group G. If $\langle x \rangle \subseteq \langle y \rangle$ then $V_A(y) \leq V_A(x)$.

Proof: Suppose that $\langle x \rangle \subseteq \langle y \rangle$. Then $x \in \langle y \rangle$. It follows that $x = y^m$ for some $m \in Z$.

$V_A(x) = V_A(y^m) \geq V_A(y)$. Therefore $V_A(x) \geq V_A(y)$.

The following example shows that the converse of lemma 3.15 is not true.

Example: Let I be the unit interval [0, 1] of real numbers. Let $a \oplus b = \min\{1, a + b\}$. With the usual ordering $(I, \oplus, \leq, -)$ is an involutory DRL-semigroup. Let $G =$ The klein-4-group $= \{e, a, b, c\}$.

Define the I-vague set A of G by

$$V_A(x) = \begin{cases} [\frac{1}{2}, 1] & \text{if } x \in \langle a \rangle ; \\ [0, \frac{3}{4}] & \text{otherwise.} \end{cases}$$

Then $V_A(c) = [0, \frac{3}{4}] \leq [\frac{1}{2}, 1] = V_A(a)$ but $\langle a \rangle$ is not a subset of $\langle c \rangle$.

Definition 3.16: Let A be an I-vague group of a group G. Image of A is defined as $ImA = \{V_A(x) : x \in G\}$.

Since $V_A(e) \geq V_A(x)$ for all $x \in G$, $V_A(e)$ is the greatest element of ImA .

Theorem 3.17: Let A be an I-vague group of a group G.

(i) If G is cyclic then ImA has a least element.

(ii) If $V_A(x) \leq V_A(y)$ then $\langle x \rangle \supseteq \langle y \rangle$ and ImA has a least element then G is cyclic.

Proof: Let A be an I-vague group of G.

(i) Suppose that G is cyclic. Then $G = \langle x \rangle$ for some $x \in G$. We prove that $V_A(x)$ is the least element of ImA .

Let $y \in G$. Then $y = x^m$ for some $m \in Z$. $V_A(y) = V_A(x^m) \geq V_A(x)$. We have $V_A(x) \leq V_A(y)$ for every $y \in G$. Thus $V_A(x)$ is the least element of image of A. Hence ImA has a least element.

(ii) Suppose that ImA has a least element say $V_A(x)$ for some $x \in G$. Let $y \in G$. Thus $V_A(y) \geq V_A(x)$ for all $y \in G$. By our condition we have $\langle y \rangle \subseteq \langle x \rangle$. Since $y \in \langle y \rangle$, $y \in \langle x \rangle$. Hence $G \subseteq \langle x \rangle$. Consequently, we have $G = \langle x \rangle$. Therefore G is cyclic.

Lemma 3.18: Let A be an I-vague group of G. Let $x, y \in G$. The two conditions

i) $V_A(x) = V_A(y) \Rightarrow \langle x \rangle = \langle y \rangle$

ii) $V_A(x) > V_A(y) \Rightarrow \langle x \rangle \subseteq \langle y \rangle$ are equivalent to the condition $V_A(x) \geq V_A(y) \Rightarrow \langle x \rangle \subseteq \langle y \rangle$.

Proof: Assume that the two conditions are given.

We prove that $V_A(x) \geq V_A(y) \Rightarrow \langle x \rangle \subseteq \langle y \rangle$.

If $V_A(x) > V_A(y)$, then $\langle x \rangle \subseteq \langle y \rangle$ by (ii).

If $V_A(x) = V_A(y)$, then $\langle x \rangle = \langle y \rangle$ by (i).

We have $\langle x \rangle \subseteq \langle y \rangle$.

Conversely, assume that $V_A(x) \geq V_A(y) \Rightarrow \langle x \rangle \subseteq \langle y \rangle$.

(i) Suppose that $V_A(x) = V_A(y)$.

$V_A(x) = V_A(y) \Rightarrow V_A(x) \geq V_A(y)$ and $V_A(y) \geq V_A(x)$.

$\Rightarrow \langle x \rangle \subseteq \langle y \rangle$ and $\langle y \rangle \subseteq \langle x \rangle$.

$\Rightarrow \langle x \rangle = \langle y \rangle$.

(ii) $V_A(x) > V_A(y) \Rightarrow V_A(x) \geq V_A(y)$

$\Rightarrow \langle x \rangle \subseteq \langle y \rangle$.

Thus $V_A(x) > V_A(y) \Rightarrow \langle x \rangle \subseteq \langle y \rangle$. Therefore

(i) and (ii) are equivalent to $V_A(x) \geq V_A(y) \Leftrightarrow \langle x \rangle \subseteq \langle y \rangle$.

Theorem 3.19: Let A be an I-vague group of a group G such that the image set of A is given by $\text{Im}A = \{I_0 > I_1 > \dots > I_n\}$ and such that

(i) $V_A(x) = V_A(y) \Rightarrow \langle x \rangle = \langle y \rangle$;

(ii) $V_A(x) < V_A(y) \Rightarrow \langle x \rangle \supseteq \langle y \rangle$.

Then G is a cyclic group of prime power order.

Proof: Let A be an I-vague group of a group G. Since $\text{Im}A = \{I_0 > I_1 > \dots > I_n\}$, Im A has a least element. By theorem 3.17, G is cyclic. It follows that $G \cong Z$ or $G \cong Z_m$ for some $m \in N$. Suppose that $G \cong Z$. Consider $V_A(2)$ and $V_A(3)$.

If $V_A(2) = V_A(3)$, then $\langle 2 \rangle = \langle 3 \rangle$ by (i). But this is not true since $2 \notin \langle 3 \rangle$. So either $V_A(2) > V_A(3)$ or $V_A(3) > V_A(2)$.

If $V_A(2) > V_A(3)$, then $\langle 2 \rangle \supseteq \langle 3 \rangle$ by (ii). But this is not true since $2 \notin \langle 3 \rangle$.

If $V_A(3) > V_A(2)$, then $\langle 3 \rangle \supseteq \langle 2 \rangle$ by (ii). But this is not true since $3 \notin \langle 2 \rangle$. Therefore G is not isomorphic to Z. Thus $G \cong Z_m$ for some $m \in N$.

Suppose that m is not a prime power. Then there exist prime numbers p and q such that $p \neq q$ which are factors of m. Consider $V_A(p)$ and $V_A(q)$.

Since $\text{Im}A = \{I_0 > I_1 > \dots > I_n\}$, either $V_A(p) \geq V_A(q)$ or $V_A(p) < V_A(q)$. It follows that $\langle p \rangle \supseteq \langle q \rangle$ or $\langle q \rangle \supseteq \langle p \rangle$, a contradiction.

Thus our supposition is false. Therefore m is prime power. Hence the theorem follows.

Theorem 3.20: Let G be a cyclic group of prime power order then there is an I and an I-vague group A of G such that for all $x, y \in G$

(i) $V_A(x) = V_A(y) \Rightarrow \langle x \rangle = \langle y \rangle$;

(ii) $V_A(x) > V_A(y) \Rightarrow \langle x \rangle \supseteq \langle y \rangle$.

Proof: Suppose that G is a cyclic group of order p^n where p is prime and $n \in N \cup \{0\}$. We find an I and an I-vague group A of G satisfying (i) and (ii).

Step(1) We construct an I and an I-vague set of G.

Let I be the unit interval [0, 1] of real numbers. Define

$$a \oplus b = \min \{1, a + b\}$$

With the usual ordering $(I, \oplus, \leq, -)$ is an involutory DRL-semigroup.

Now we construct our I-vague set of G.

Let $z \in G$. Then $O(z) = p^i$ where $i = 0, 1, 2, \dots, n$.

Define $A = (t_A, f_A)$ where $t_A : G \rightarrow I$ and $f_A : G \rightarrow I$ such that $t_A(z) = a_i$, $f_A(z) = b_i$ where $a_i, b_i \in [0, 1]$ satisfying $a_i \leq 1 - b_i$ for $i = 0, 1, 2, \dots, n$. Choose the intervals I_0, I_1, \dots, I_n in such a way that $I_0 > I_1 > \dots > I_n$ where $I_i = [a_i, 1 - b_i]$. Then $V_A(z) = I_i$. Hence A is an I-vague set of G. We have $V_A(e) = I_0$.

Step(2) We show that A is an I-vague group of G.

Let $x \in G$. $O(x) = O(x^{-1})$ implies $V_A(x) = V_A(x^{-1})$.

To show A is an I-vague group of G it remains to prove that $V_A(xy) \geq \text{iinf}\{V_A(x), V_A(y)\}$ for every $x, y \in G$.

Let $x, y \in G$. Since G is a cyclic group of order p^n and the order of the subgroup divides the order of the group, $O(\langle x \rangle) = p^j$, $O(\langle y \rangle) = p^k$ and $O(\langle xy \rangle) = p^m$ for some $j, k, m \in \{0, 1, \dots, n\}$ say.

Therefore $V_A(x) = I_j$, $V_A(y) = I_k$ and $V_A(xy) = I_m$. Moreover, since G is a cyclic group of prime power order, $\langle x \rangle \supseteq \langle y \rangle$ or $\langle y \rangle \supseteq \langle x \rangle$.

If $\langle x \rangle \supseteq \langle y \rangle$ then $x, y \in \langle y \rangle$. Hence $\langle xy \rangle \supseteq \langle y \rangle$.

If $\langle y \rangle \supseteq \langle x \rangle$ then $x, y \in \langle x \rangle$. Hence $\langle xy \rangle \supseteq \langle x \rangle$.

Therefore $\langle xy \rangle \supseteq \langle y \rangle$ or $\langle xy \rangle \supseteq \langle x \rangle$.

Assume that $\langle xy \rangle \supseteq \langle x \rangle$. It follows that $O(\langle xy \rangle) \leq O(\langle x \rangle)$ or $O(\langle xy \rangle) = O(\langle x \rangle)$.

If $O(\langle xy \rangle) < O(\langle x \rangle)$ then $m < j$. It follows that $I_m > I_j$.

$$\text{Hence } V_A(xy) = I_m \geq \text{iinf}\{I_j, I_k\} = \text{iinf}\{V_A(x), V_A(y)\}.$$

$$\text{Thus } V_A(xy) \geq \text{iinf}\{V_A(x), V_A(y)\}.$$

If $O(\langle xy \rangle) = O(\langle x \rangle)$ then $m = j$. Hence $I_m = I_j$.

$$V_A(xy) = I_m \geq \text{iinf}\{I_m, I_k\} = \text{iinf}\{V_A(x), V_A(y)\}.$$

$$\text{Thus } V_A(xy) \geq \text{iinf}\{V_A(x), V_A(y)\}.$$

In both cases $V_A(xy) \geq \text{iinf}\{V_A(x), V_A(y)\}$ and

$$V_A(x) \geq V_A(x^{-1}) \text{ for all } x, y \in G.$$

Thus A is an I-vague group of G.

Step(3) We show that A satisfies the conditions (i) and (ii) of the theorem.

(a) Suppose that $V_A(x) = V_A(y)$ for $x, y \in G$.

By the definition of A we have $O(\langle x \rangle) = O(\langle y \rangle)$.

Since G is a cyclic group of prime power order, $O(\langle x \rangle) = O(\langle y \rangle)$ implies $\langle x \rangle = \langle y \rangle$.

$$\text{Hence } V_A(x) = V_A(y) \Rightarrow \langle x \rangle = \langle y \rangle.$$

(b) Suppose that $V_A(x) > V_A(y)$ for $x, y \in G$. Then $I_j > I_k$.

It follows that $j < k$.

$$\text{Hence } p^j < p^k, \text{ so } O(\langle x \rangle) < O(\langle y \rangle).$$

Since G is a cyclic group of order p^n and

$$O(\langle x \rangle) < O(\langle y \rangle), \langle x \rangle \supseteq \langle y \rangle.$$

$$\text{Thus } V_A(x) > V_A(y) \Rightarrow \langle x \rangle \supseteq \langle y \rangle.$$

Therefore A satisfies (i) and (ii).

Hence the theorem follows.

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