Designing FIR Filters with Polynomial Approach

Sunil Bhooshan and Vinay Kumar

Abstract—This paper discusses a method for designing the Finite Impulse Response (FIR) filters based on polynomial approach.

Index Terms—FIR filter, Polynomial.

I. INTRODUCTION

The design of FIR filters is a field where an immense amount of work had been carried out since a long period of time, and numerous distinctive approaches had been taken to carry out designs of different types of FIR filters [12]. FIR filters have the distinction of introducing no phase distortion and hence are important.

In this paper we will present an approach to realize FIR filters using polynomials. The polynomial approach had been discussed in several papers [1-11], but here, along with the polynomial approach, we will be using a special type of transformation coupled with a specialized procedure.

II. PROCEDURE

The procedure to realize an low-pass FIR filter with linear phase is as follows.

Step 1. Choose a polynomial $f(x)$ which has the following properties:
a. The chosen polynomial must have real roots,
b. The polynomial should have zeros near the origin; i.e., $0 \le |x| \le 1$, and let the zeros be at $x = x_i, i = 1,..n$,
c. The polynomial should increase sharply for real values of x, far away from $x = 1$; i.e., $x > 1$.

Step 2. We will use the following transformation for mapping x to frequency ω

$$x = x_m cos(\omega/2) \qquad (1)$$

Where,
x_m represents the maximum value of x, which we will be using in our approach. At this value the polynomial has the value b, (see Figure 1) How b and x_m are to be calculated will be clear shortly.

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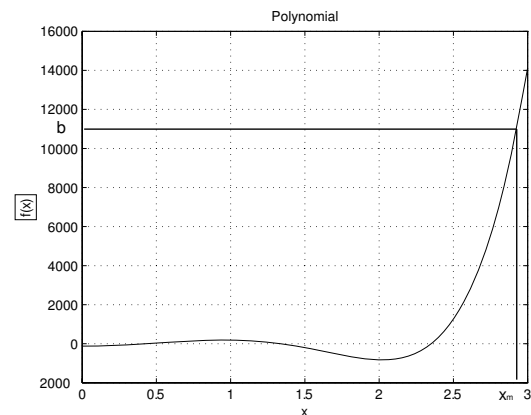
Fig. 1. Example Polynomial

Step 3. By using the above transformation, the zeros of the transfer function of the FIR filter are calculated using Equation (1). Let these zeros be represented by $z_i = \exp(j\omega_i), i = 1,..n$, where the ω_is are calculated using the transformation equation:

$$x_i = x_m \cos \omega_i/2 \qquad (2)$$

Note that the x_is are the zeros of the original polynomial, $f(x)$. The FIR filter transfer function is then given by:

$$H(z) = (z - z_1)(z - z_2)...(z - z_n) \qquad (3)$$

The values of x_m and b are related to the bandwidth of the filter or the ripple in the stop-band as will be clear through actual examples.

III. EXAMPLES

A. Example 1

Let us consider a polynomial where the values of x_i's are all equal to 0, in this case the polynomial is

$$f(x) = x^n \qquad (4)$$

This polynomial satisfies all the conditions specified in *Step 1* . The value of the function where we want our stopband to start can be taken to be $x = 1$. This value is arbitrary, but $x = 1$ gives good results. Therefore, at the start of the stopband:

$$x^n = 1 \qquad (5)$$

Looking at the transformation $x = x_m \cos(\omega/2)$ we observe that $x = 0$ tansforms to $\omega = \pi$, $x = 1$ tansforms to

$\omega = 2cos^{-1}(1/x_m)$ which is where the stop-band starts. x_m transforms to $\omega = 0$. Using these ideas, we want that at $x = x_m$ the function $f(x_m)$ is b times its value than that at the stop band $f(x_s)$. So

$$x_m^n = b \qquad (6)$$

$$x_m = b^{(1/n)} \qquad (7)$$

Now, we will use the transform, defined in *Step 2* , to calculate the stop band frequency, ω_s. The value of x will be 1, as discussed above, and $x_m = b^{1/n}$

$$1 = b^{1/n}cos(\omega_s/2) \qquad (8)$$

$$\omega_s = 2cos^{-1}(1/b^{(1/n)}) \qquad (9)$$

The passband of filter characteristic will occur when the function is $b/\sqrt{2}$ times its value than that at the stopband. So,

$$x_p^n = b/\sqrt{2} \qquad (10)$$

$$x_p = \frac{b^{1/n}}{2^{1/2n}} \qquad (11)$$

Using the same transform, discussed in *Step 2* , we can calculate the value of passband, ω_p, as follows

$$\frac{b^{1/n}}{2^{1/2n}} = b^{1/n}cos(\omega_p/2) \qquad (12)$$

$$\omega_p = 2cos^{-1}(1/2^{1/2n}) \qquad (13)$$

Zeros of $f(x)$ lie at $x = 0$, so there is no need of calculation involved in *Step 3* in this case.

Calculation of b:

As discussed above the function is b times its value than that at the stopband. So, if we want our stopband to be, lets say p dB down than that of maximum value of passband, then the value of b will be absolute value of p dB;i.e.,

$$b = 10^{p/20} \qquad (14)$$

Suppose in the polynomial discussed above, the value of n is 6; i.e., our filter is of the order of 6, and we want our stopband to be 40 dB down; i.e., $b = 10^{40/20} = 100$ from Equation 14 and the values of ω_s and ω_p are 2.17622 and 0.6733 as calculated from the equations above. The transfer function, as discussed in *Step 3* , for this FIR filter, at $\omega_i = \pi$ at $i = 1, 2, ...6$, is

$$H(z) = (z - e^{j\pi})^6 \qquad (15)$$

The polynomial, frequency response in dB, and phase response for this FIR filter is shown in Figures (2), (3), and (4) respectively.
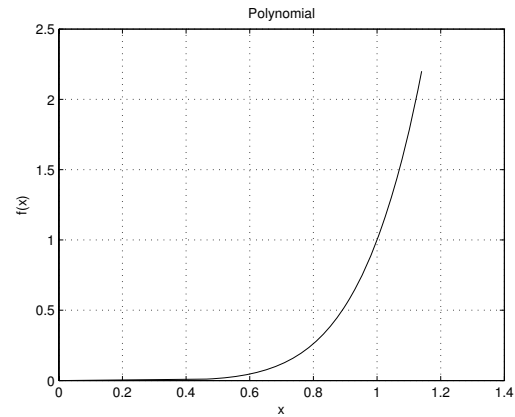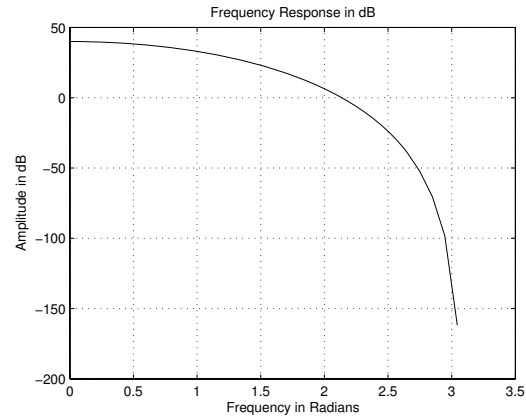

Fig. 2.   Polynomial x^6

Fig. 3.   Frequency Response in dB of $H(z)$ of Equation (15)

B. Example 2

Let us consider another polynomial, whose zeros all do not lie at $x = 0$. The following polynomial follows the characteristics discussed in *Step 1* as well as it has zeros at locations other than $x = 0$ except one.

$$f(x) = (x-0)(x-0.1)(x-0.2)(x-0.3)(x-0.4)(x-0.5) \qquad (16)$$

In this case the value of n is 6 and suppose we want our stopband to be 40 dB down; i.e., b=100. Now we will Follow the *Step 2* for the transformation and *Step 3* to calculate the position of zeros for this polynomial.

The zeros for this polynomial will lie at:

$$\omega_1 = 2cos^{-1}\frac{0}{x_m} = \pi, \omega_2 = 2cos^{-1}\frac{0.1}{x_m} = 3.0586,$$

$$\omega_3 = 2cos^{-1}\frac{0.2}{x_m} = 2.9755, \omega_4 = 2cos^{-1}\frac{0.3}{x_m} = 2.8921,$$

$$\omega_5 = 2cos^{-1}\frac{0.4}{x_m} = 2.8083, \omega_6 = 2cos^{-1}\frac{0.5}{x_m} = 2.7238$$
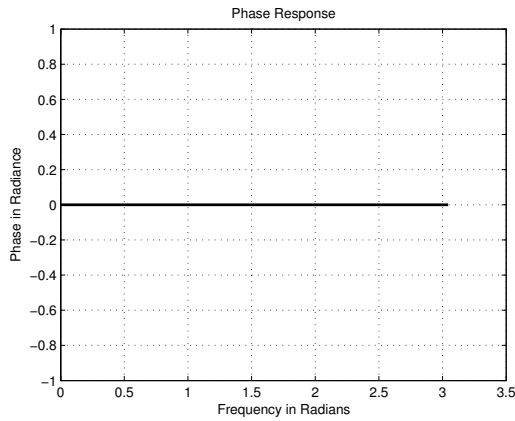
$$(17)$$

Fig. 4.   Phase Response of Equation (15)

The value of stopband for the above polynomial will be calculated by putting $f(x) = 1$; i.e.,

$$(x - 0)(x - 0.1)(x - 0.2)(x - 0.3)(x - 0.4)(x - 0.5) = 1 \quad (18)$$

It will give us six different values of x, here we will consider only the positive value amongst them. The positive value we get will correspond to x_s and it is $x_s = 1.2646$.

Now to calculate the value of x_m we will equate $f(x)$ to $b = 100$; i.e.,

$$(x - 0)(x - 0.1)(x - 0.2)(x - 0.3)(x - 0.4)(x - 0.5) = 100 \quad (19)$$

It will also give us six different values of x, here we will consider only the positive value amongst them. The positive value we get is $x_m = 2.4112$.

The value of stopband, ω_s, can be calculated by using the transform discussed in *Step 2* ; i.e.,

$$1.2646 = 2.41121 cos(\omega_s/2) \quad (20)$$

$$\omega_s = 2cos^{-1}(1.2646/2.41121) = 2.0374 \quad (21)$$

Similarly the value of passband can be calculated by equating the polynomial to $b/\sqrt{2}$; i.e.,

$$(x-0)(x-0.1)(x-0.2)(x-0.3)(x-0.4)(x-0.5) = b/\sqrt{2} = 100/\sqrt{2} \quad (22)$$

It will give us six values, but we will consider only positive value, $x_p = 2.29069$ and then by using the same transformation as discussed in *Step 2* we will get passband frequency; i.e., ω_p
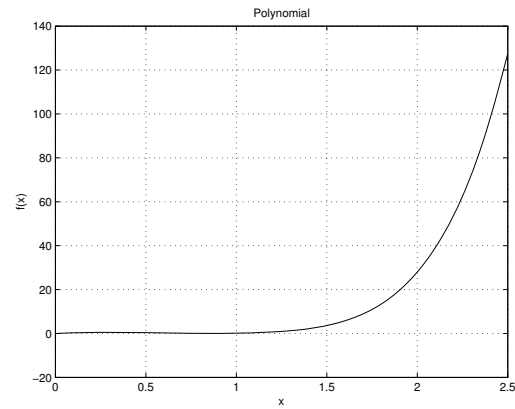
$$2.29069 = 2.41121 cos(\omega_p/2) \quad (23)$$

$$\omega_p = 2cos^{-1}(2.29069/2.41121) = 0.6350 \quad (24)$$

So, ω_s and ω_p comes out to be 2.0374 and 0.6350 respectively.

The transfer function for this FIR filter will be, using *Step 2*

$$H(z) = (z - e^{j\omega_1})(z - e^{j\omega_2})(z - e^{j\omega_3})(z - e^{j\omega_4})(z - e^{j\omega_5})(z - e^{j\omega_6}) \quad (25)$$

where, $\omega_1, \omega_2, ...$ are calculated above from Equations (17). Figures (5), (6) and (7) shows the polynomial, frequency response in dB, and phase response for this FIR filter.

Fig. 5.   Polynomial $(x - 0)(x - 0.1)(x - 0.2)(x - 0.3)(x - 0.4)(x - 0.5)$
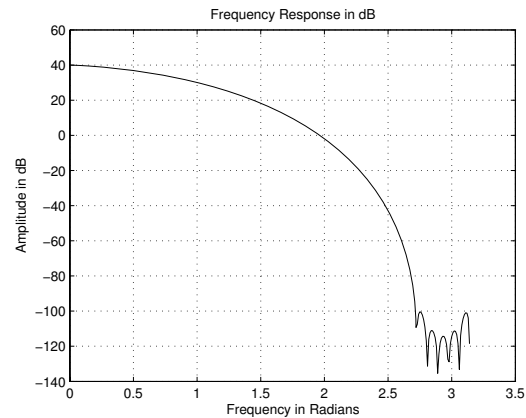
Fig. 6.   Frequency Response in dB of $H(z)$ of Equation (25)

C. *Example 3*

Figures (8), (9) and (10) are polynomial, frequency response in dB and phase response for a polynomial having zero at $x = 1$; i.e., at the point where we want our stopband, such a polynomial is

$$f(x) = x^5(x - 0.7071) \quad (26)$$

The plots shown have stopband at 40 dB down and the values of $x_m, x_s, x_p, \omega_s, \omega_p$ are 2.2911, 1.1677, 2.1715,
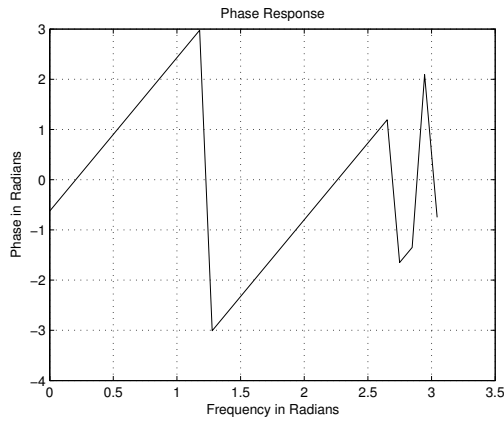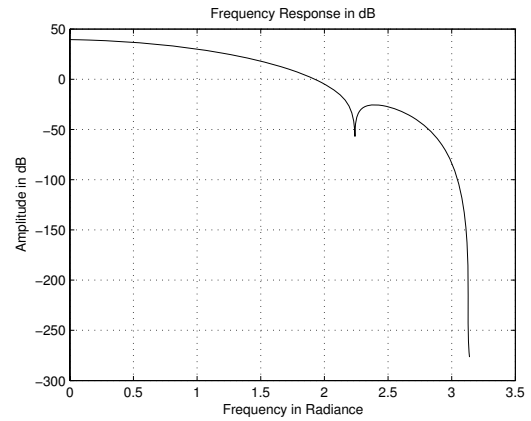
Fig. 7. Phase Response of Equation (25)

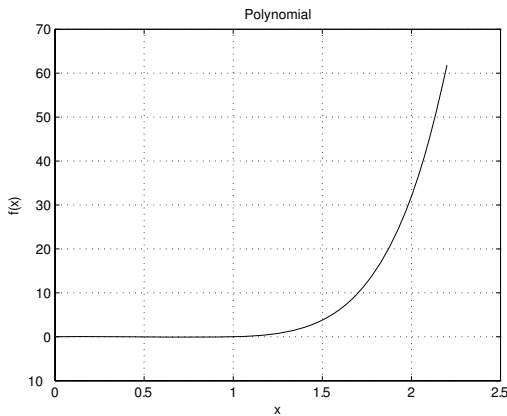Fig. 9. Frequency Response in dB of $H(z)$ of Equation (28)

2.0720, 0.6492 respectively using *Step 2* and by *Step 3* we get the transfer function as

The values of zeros in frequency domain is

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega_5 = 2cos^{-1}\frac{0}{x_m} = \pi,$$

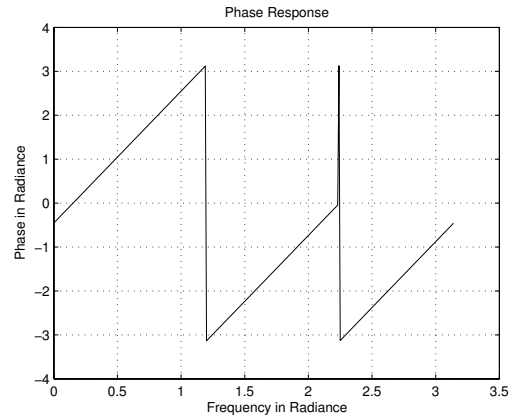$$\omega_6 = 2cos^{-1}\frac{1}{x_m} = 2.2383 \qquad (27)$$

$$H(z) = (z - e^{j\omega_1})(z - e^{j\omega_2})(z - e^{j\omega_3})(z - e^{j\omega_4})(z - e^{j\omega_5})(z - e^{j\omega_6}) \qquad (28)$$

where $\omega_1, \omega_2, ..., \omega_6$ are calculated above.

Fig. 10. phase Response of $H(z)$ of Equation (28)

Fig. 8. Polynomial $x^5(x - 0.7071)$

described in *Step 1* .

$$f(x) = x^4(x^2 + 0.5) \qquad (29)$$

It is a 6th order polynomial and has 40 dB down stopband; i.e., b=100. Following the steps described for previous polynomials, we will get $x_m = 2.1168$, $x_s = 0.9263$, $x_p = 1.9937$, $\omega_s = 2.2357$, $\omega_p = 0.6854$ and zeros at

$$\omega_1 = \omega_2 = \omega_3 = \omega_4 = 2cos^{-1}\frac{0}{x_m} = \pi,$$

$$\omega_5 = 2cos^{-1}\frac{0.707i}{x_m} = \pi - 0.6562i,$$

$$\omega_6 = 2cos^{-1}\frac{-0.707i}{x_m} = \pi + 0.6562i \qquad (30)$$

Following the *Step 3* we can write the transfer function for this filter

$$H(z) = (z - e^{j\omega_1})(z - e^{j\omega_2})(z - e^{j\omega_3})(z - e^{j\omega_4})(z - e^{j\omega_5})(z - e^{j\omega_6}) \qquad (31)$$

where the values of $\omega_1, \omega_2, ..., \omega_6$ are calculated above.

D. Example 4

As a last example let us consider a polynomial *not* having zeros which are real, but having a complex conjugate pair of zeros. The following polynomial has such a pair of zeros but follows the characteristics

Figures (11), (12), and (13) shows the polynomial, frequency response in dB and phase response for our polynomial

It is clear from the Figure (13) that the phase response of such a system is not completely linear. This leads us to the conclusion that zeros which are not real or complex in nature may lead to a non-linear phase FIR filter, realized using the tranformation discussed in *Step 2* . But the Figure(12) shows that the bandwidth is increased when compared with previous 6th order polynomial (Example 3), which has 5 zeros at zero and 6th at the same point as in this case. So, we can say that if we can compromise with the phase and need flat frequency response than we could use the polynomial with non real zeros as well.
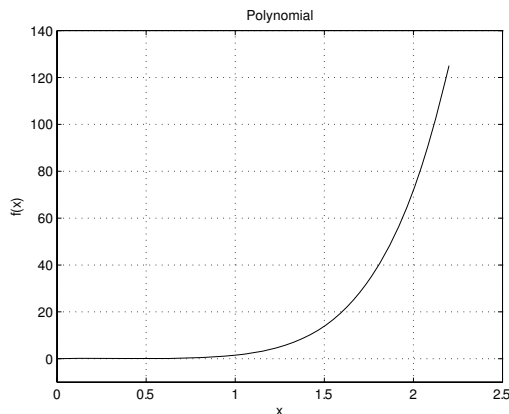

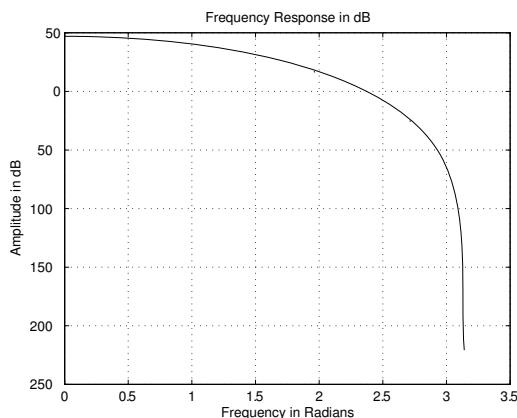Fig. 11.   Polynomial $x^4(x^2 + 0.5)$

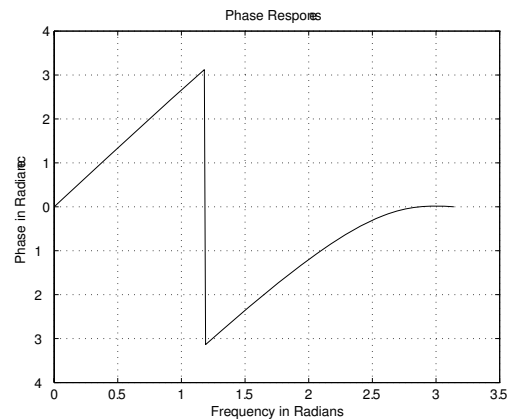Fig. 12.   Frequency Response in dB of $H(z)$ of Equation (31)

Fig. 13.   Phase Response of $H(z)$ of Equation (31)

REFERENCES

[1] Boite R. and H. Leich, "A New Procedure for the Design of High Order Minimum Phase FIR Digital or CCD FIlters", *Signal Processing,* vol. 3, pages 101-108, 1981.
[2] Herman 0. and H.W. Schussler, "Design of Non Recursive Digital Filters With Minimum Phase", *Electronics Letters,* vol. 6, pages 329-330, 1970.
[3] Stathaki, T.; Constantinides, A.; Stathakis, G.,"Minimum phase FIR filter design from linear phase systems using root moments",*International Conference on Acoustics, Speech, and Signal Processing, 1998. ICASSP '98. Proceedings of the 1998 IEEE Volume 3,* Page(s):1317 - 1320, 12-15 May 1998
[4] Mian G.A. and A.P. Naider,"A Fast Procedure to Design Equiripple Minimum Phase FIR FIlters" , *IEEE Trans. on Circuits and Systems,* vol. 29, no. 5, pages 327-331, 1982.
[5] T.Stathaki, "Root moments: a digital signal-processing perspective", *IEE Proc.- Vis. Image Signal Process.,* Vol. 145, No. 3, pages 293-302, August 1998
[6] Tanskanen, J.M.A.; Dimitrov, V.S., "Round-off error free fixed-point design of polynomial FIR predictors",*Conference on Signals, Systems, and Computers, 1999. Conference Record of the Thirty-Third Asilomar Volume 2,* 1999 Page(s):1317 - 1321 vol.2, 24-27 Oct.
[7] Fotinopoulos, I.; Stathaki, T., "Equiripple minimum phase FIR filter design from linear phase systems using a novel technique for polynomial factorisation", *Global Telecommunications Conference, 2000. GLOBECOM '00. IEEE Volume 3,* 27 Nov.-1 Dec. 2000 Page(s):1659 - 1663 vol.3
[8] Kamp Y. and Wellekens C.J., "Optimal design of minimum phase FIR filters ", *IEEE Trans. on ASSP, Vol. 31,* pages 922-926, 1983.
[9] A. G. Constantinides, "Spectral transformation for digital filters ", *Proc. Inst. Elect. Eng.,* pp. 1585-1590, 1970.
[10] Swamy, M.N.S.; Thyagarajan, K.S., "Frequency transformations for digital filters ", *Proceedings of the IEEE Volume 65,* Page(s):165 - 166, Issue 1, Jan. 1977
[11] Moorer, J., "General spectral transformations for digital filters ", *Acoustics, Speech, and Signal Processing, IEEE Transactions on Volume 29,* Page(s):1092 - 1094, Issue 5, Oct 1981
[12] Proakis J.G., "Digital Signal Processing: Principles, Algorithms and Applications", *Prentice Hall of India,* 3rd Edition.

IV. CONCLUSION

In this paper we have discussed the connection of poynomials and FIR filters. Polynomials could be used to design user specific pass band characteristics. We are working in that field.

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