International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:2, No:10, 2008

Integral Operators Related to Problems of Interface Dynamics

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***Abstract*—**This research work is concerned with the eigenvalue problem for the integral operators which are obtained by linearization of a nonlocal evolution equation. The purpose of section II.A is to describe the nature of the problem and the objective of the project. The problem is related to the "stable solution" of the evolution equation which is the so-called "instanton" that describe the interface between two stable phases. The analysis of the instanton and its asymptotic behavior are described in section II.C by imposing the Green function and making use of a probability kernel. As a result , a classical Theorem which is important for an instanton is proved. Section III devoted to a study of the integral operators related to interface dynamics which concern the analysis of the Cauchy problem for the evolution equation with initial data close to different phases and different regions of space.

***Keywords*—** Evolution, Green function, instanton, integral operators.

I. Introduction

THE eigenvalue problem for an integral operator A on the space C^{Sym} (3) of symmetric, bounded function on the real line 3 with sup norm, has attracted a great deal of attention in the field of engineering science since such an operator A is related to problems of interface dynamics. Suppose that the kernel of A has the form $A(x,y)= p(x)J(x,y)$,with $p(x)$ a symmetric, strictly, regular bounded function and $J(x,y)$a regular, nonnegative function of the variable y, x with compact support in the interval [-1,1] and integral equal to 1. Under certain conditions on P and J, motivated by applications to interface dynamics, its aims to estimate the bound for the isolated positive maximal eigenvalue λ with positive eigenvector of the integral operator A on C^{Sym} (3). For stability questions it is important to determine the part of the spectrum outside the unit ball. This is not simple in this present context because $p(x)$ occurs in two possibilities: one possibility is greater than one and the other possibility is less than one. It is possible to investigate the problem only when the eigenvalue λ is >1, while the rest of the spectrum is strictly inside the unit ball. This kind of problem has been studied by some other authors such as [1]-[2].The results will hopefully provide for many physical applications.

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II. Evolution Equation

A. Nature of the Problem

This paper is concerned with the problem which arises from the analysis of the evolution equation in $C^{sym}(\mathbb{3};[-1,1])$:

$$\frac{\partial m_t}{\partial t} = -m_t + \tanh(\beta J * m_t + \beta h) \tag{1}$$

where $\beta > 1, h \geq 0, m_t$ is a real-valued function on 3 for each $t \in \mathbb{3}$, $*$ is the convolution product defined by:

$$(J * m)(x) = \int_{\mathbb{3}} J(x,y)m(y)dy \tag{2}$$

for each $m \in C^{sym}(\mathbb{3};[-1,1])$. The operators $A: C^{sym}(\mathbb{3};[-1,1]) \to C^{sym}(\mathbb{3};[-1,1])$ can be considered. Which are related to the linearization of the right hand side of (1), namely given $m \in C^{sym}(\mathbb{3};[-1,1])$,

$$p_m(x) = \frac{\beta}{\cosh^2\{\beta J * m(x)\}} \tag{3}$$

is set and defined A_m, as the function of m with kernel $J(x, y)$ by

$$A_m(x, y) = p_m(x)\, J(x, y). \tag{4}$$

Then (when $h = 0$), the operator $L_m = A_{m-1}$ is the linearization around m of the right hand side of (1). The operators A_m for each $m \in C^{sym}(3; [-1, 1])$ can be studied. Let C_b (3) be the space of all bounded real-valued functions on 3. Solutions m of (1) in $C_b(X)$ with sup norm ?? m ??$_\infty \leq 1$ can be observed and that are differentiable with respect to time t. The Cauchy problem in this setup is well-posed with a unique global solution because the right hand side of (1) is uniformly Lipschitz because the set $\{m \in C_b(3)$: ?? m ??$_\infty \leq 1\}$is Left invariant, since $\tanh z < 1$ for all z. Equation (1) arises in the study of spin systems with Glauber dynamic and Kac interactions where it is derived in a continuum limit, [3]; m is then interpreted as a magnetization density and β^{-1} as the product of the absolute temperature and the Boltzmann constant. The analysis of Gibbs measures with Kac interactions, that started in the late sixties [4]- [5] is by now a well established theory. It proves the validity, in equilibrium Statistical Mechanics setting, of the Van der Waals theory by showing that its typical phase diagram is exhibited by systems

with Kac interactions, in a suitable scaling limit. The critical temperature corresponds to $\beta = 1$, according to our normalization condition on the interaction, so that $\{\beta > 1\}$ is the phase transition region. For each value of $\beta > 1$ there are two pure thermodynamic phases with magnetization respectively equal to $\pm m_\beta : m_\beta$ being the positive standard of the equation

$$m_\beta = \tanh(\beta m_\beta) \tag{5}$$

B. Objective and Motivation

The main objective of this paper is to investigate spectral property of integral operators in problem of interface dynamics. Interface dynamics concerns the analysis of the Cauchy problem for (1) with initial data close to different phases in different regions of space. This problem has been extensively studied in the last ten years with special attention to the multi-dimensional case where it has been proved that on a suitable space time scaling limit the evolution is ruled by a motion by mean curvature, [6]. In one dimension, when $h > 0$ there are traveling fronts describing the growth of the stable phase at the expense of the metastable one, [7]. When $h = 0$ there are stationary solutions with two coexisting phases: they are all identical, modulo translations and reflection, [5], to "the instanton" $\bar{m}(x)$, which is a C^∞, strictly increasing, antisymmetric function which identically verifies

$$\bar{m}(x) = \tanh(\beta J * \bar{m}(x)) \tag{6}$$

where $\bar{m}(x)$ is the stationary pattern that connects the minus and plus phases, as

$$\lim_{x \to \pm\infty} \bar{m}(x) = \pm m_\beta \tag{7}$$

and it has therefore the interpretation of a "diffuse interface". However, since the derivative $\bar{m}'(x)$ of $\bar{m}(x)$ vanishes exponentially fast as ? x ?$\to \infty$, [3], then loosely speaking, the fraction of space not occupied by pure phase is vanishingly small. In this sense, that can be made precisely by introducing scaling, the interface is sharp and the transition from one phase to the other one is "instantaneous". That is why ($\bar{m}$ or, more properly $\bar{m}'$) is called *the instanton*.

As proved in [1], the interface described by the instanton is "stable" and any initial datum "close to an instanton" is attracted and eventually converges to some translate of the instanton. In the present context, (1) in a neighborhoods of functions m of the form

$$m_\xi(x) = \bar{m}(\xi - |x|), \quad \xi > 0 \tag{8}$$

is studied. By linearization leads to the operator A_m with either $m = m_\xi$ or m close to it (and ξ large).This is a first motivation for studying A_m. In the analysis of the operator A_m, with $m = m_\xi + h$ is interested or closed to such a function. The existence of these attractive one dimensional unstable manifolds reflects the presence of an isolated, simple, maximal eigenvalue $\lambda_m > 1$ for A_m and of a spectral gap, with the rest of the spectrum strictly in the unit ball. In this paper, these can be established and other properties of the operators A_m which are the building ingredients for the applications to interface dynamics.

C. Asymptotic Behavior of the Instanton

In this section, the asymptotic behavior of the instanton $\bar{m}(x)$ as $x \to \infty$ is studied. The instanton $\bar{m}(x)$ is an antisymmetric, continuous increasing function of $x \in 3$ that solves (6). In Proposition 2.2 of [3] it is proved that there exists $c > 0$ and $\eta > 0$ such that for all $x > 0$,

$$\left|\bar{m}(x) - m_\beta\right| \le c e^{-\eta x} \tag{9}$$

and in proposition 2.1 of [6] it is also proved that $\bar{m}' > 0$ and $\bar{m}'$ is an eigenvector of $A_{\bar{m}}$ with eigenvalue 1. Namely

$$\bar{m}' = pJ * \bar{m}' \tag{10}$$

where the shorthand notation

$$p(x) \equiv p_{\bar{m}}(x) = \beta\,[1 - \bar{m}(x)^2] \tag{11}$$

is used. Equation (10) is obtained by differentiating with respect to x the instanton equation (6). After integration by parts the convolution on the right hand side of (10) is deduced that $\bar{m}'$ is a bounded continuous function and by further differentiations that all the derivatives of $\bar{m}$ share such a property. Since

$$\lim_{|x| \to \infty} p(x) = p_\infty \text{ (say)} = \beta\,[1 - \bar{m}_\beta^2] < 1 \tag{12}$$

the obvious conjecture is that the asymptotic behavior of $\bar{m}'(x)$ as $x \to +\infty$ is ruled by the equation

$$v = p_\infty J * v. \tag{13}$$

Looking for a solution of (13) of the form

$$v(x) = e^{-\alpha x} \tag{14}$$

α must be solved

$$p_\infty \int J(0, y) e^{-\alpha y} dy = 1 \tag{15}$$

When the domain of an integral is not specified then it will be taken the whole 3. Now, basic results for instanton can be given as follows:

***Proposition* 1 .**

There is a strictly positive, decreasing C^1 function α_p, $p \in (0,1)$, such that $\pm \alpha_p$ are the only solutions of the equation

$$p \int J(0, y) e^{-\lambda y} dy = 1, \quad \lambda \in 3 \ . \tag{16}$$

$\alpha = \alpha_{p_\infty}, p_\infty = \beta(1 - m_\beta^2)$ is written and introduced the following kernel $K(x, y) = p_\infty(x, y) e^{-\alpha(y-x)}$. This kernel can be considered as *the asymptotic expression for x and y* large of the transition probability $P(x, y)$ given by

$$P(x,y) = A_{\bar{m}}(x,y)\frac{\bar{m}'(y)}{\bar{m}'(x)}, \qquad x,y \in 3\ . \tag{17}$$

The following is an important result for the instanton.

***Theorem* 1.**
There are M > 0 and $\delta \in (0,\alpha)$ positive such that

$$\lim_{x\to+\infty} e^{\alpha x}\bar{m}'(x) = M \quad , \lim_{x\to+\infty} e^{\delta x}(e^{\alpha x}\bar{m}'(x) - M) = 0\ . \tag{18}$$

An analogous statement holds for $x \to -\infty$, as $\bar{m}(x)$ is antisymmetric (and $\bar{m}'(x)$ symmetric). Without loss of generality, it is restricted to x $\geq$ 0.To prove Theorem 1 some useful facts have been developed which will be needed in the proof of the Theorem 1. So the theorem will be proved later.

D. The Green Function $G_s(x, y)$

First, the following convention is introduced. Given a positive integer s the following equation in $C(3)$ is considered.

$$\begin{cases} v(x) = p(x)J * v(x) & \text{for } x \geq s \\ \\ v(x) = \bar{m}'(x) & \text{for } x < s. \end{cases} \tag{19}$$

It can be seen that $\bar{m}'$ solves (18) and it is the only solution. It can also be proved that for s large enough there is *a Green function* $G_S(x, y)$ [9] for (19). An expression for $\bar{m}'(x)$, $x > s$, in terms of $G_S(\cdot,\cdot)$ can be obtained and of $\bar{m}'(y)$, $s-1 \leq y < s$, and that will eventually lead to the proof of Theorem 1. An identity is satisfied by $\bar{m}'(\cdot)$.

***Proposition* 2.**
For any $x \geq s$, the following identity holds:

$$\bar{m}'(x) = \int_{s-1}^{s} G_S(x,y)\bar{m}'(y)dy \tag{20}$$

Where

$$G_s(x,y) = \sum_{n=1}^{\infty} R_s^{(n)}(x,y) \tag{21}$$

and, setting $x = y_0$ and $y = y_n$, $n>1$

$$R_s^{(n)}(y_0,y_n) = \int_s^{\infty} dy_1 \ldots \int_s^{\infty} \prod_{i=1}^{n} p(y_{i-1})J(y_{i-1},y_i)dy_{n-1}$$

***Proposition* 3.**
If $x \geq s$ and $y \in [s-1, s)$ then

$$e^{-\alpha(x-y)}g_s(x,y) \leq G_s(x,y) \leq e^{-\alpha(x-y)}[1+\varepsilon(s)]g_s(x,y) \tag{22}$$

where

$$\varepsilon(s) = ce^{-\eta s} \tag{23}$$

with c a suitable, positive constant.

E. Estimates on $g_s(x, y)$

In this section, the function $g_s(x, y)$ can be estimated.

***Proposition* 4.**

For x $\geq s$ and $y < s$,

$$g_s(x,y) = g_0(x-s, y-s), \tag{24}$$

and for all $x \geq 0$,

$$\int_{-1}^{0} g_0(x,y)dy = 1. \tag{25}$$

***Proposition* 5.**
There are $\delta_1 > 0$ and a probability density $\rho(y)$, for $y \in [-1, 0]$, such that

$$\lim_{N\to\infty} g_0(x,y) = \rho(y), \lim_{N\to\infty} e^{\delta_1 x}\int_{-1}^{0} |g_0(x,y) - \rho(y)|dy = 0 \tag{26}$$

F. Properties of Instanton

Now, Theorem 1 can be restated and proved.

***Theorem* 1.**
There are $M > 0$ and $\delta \in (0, \alpha)$ such that

$$\lim_{x\to\infty} e^{\alpha x}\bar{m}'(x) = M \quad , \lim_{x\to\infty} e^{\delta x}\left(e^{\alpha x}\bar{m}'(x)\right) - M = 0$$

Proof**:** Let $x \geq s$ and

$$I_s(x) = \int_{-1}^{0} g_0(x-s,y)e^{\alpha(s+y)}\bar{m}'(s+y)dy. \tag{27}$$

If $s + y = y$ is put then it follows from the translation invariance property that

$$I_s(x) = \int_{s-1}^{s} g_0(x-s,y-s)e^{\alpha y}\bar{m}'(y)dy \tag{28}$$

$$= \int_{s-1}^{s} g_s(x,y)e^{\alpha x}\bar{m}'(y)dy$$

But from (22), it is known that

$$g_s(x,y) \leq e^{\alpha(x-y)}G_s(x,y) \tag{29}$$

and

$$\frac{e^{\alpha(x-y)}}{1+\varepsilon(s)}G_s(x,y) \leq g_s(x,y)\ . \tag{30}$$

From (28), (29) and (20),

$$I_s(x) \leq \int_{s-1}^{s} e^{\alpha x}G_s(x,y)\bar{m}'(y)dy = e^{\alpha x}\bar{m}'(x)\ .$$

From (28), (30) and (20),

$$I_s(x) \geq \int_{s-1}^{s} \frac{e^{\alpha x}}{1+\varepsilon(s)}G_s(x,y)\bar{m}'(y)dy = \frac{e^{\alpha x}}{1+\varepsilon(s)}\bar{m}'(x)$$

is obtained. Thus,

$$I_s(x) \leq e^{\alpha x}\bar{m}'(x) \leq I_s(x)(1+\varepsilon(s)). \tag{31}$$

Let

$$I_s^* = \int_{-1}^{0} \rho(y)\left[e^{\alpha(s+y)}\bar{m}'(s+y)\right]dy\ . \tag{32}$$

Then from (27) and (32),

$$\left|I_s(x)-I_s^*\right|$$
$$\le \int_{-1}^{0}\left[g_0(x-s,y)-\rho(y)\right]\left[e^{\alpha(s+y)}\bar{m}'(s+y)\right]dy$$
$$\le \sup_{|z|\le s} e^{\alpha z}\bar{m}'(z)\int_{-1}^{0}\left[g_0(x-s,y)-\rho(y)\right]dy .$$

Hence by (26), there exists $c > 0$ and $\delta_1 > 0$ such that

$$\left|I_s(x)-I_s^*\right| \le ce^{-\delta_1(x-s)}\sup_{|z|\le s} e^{\alpha z}\bar{m}'(z). \quad (33)$$

This shows that

$$I_s^* = \lim_{x\to+\infty} I_s(x). \quad (34)$$

Let

$$\underline{M} = \liminf_{x\to+\infty} e^{\alpha x}\bar{m}'(x) \quad , \overline{M} = \limsup_{x\to+\infty} e^{\delta x}\bar{m}'(x). \quad (35)$$

Thus from (31), (34) and (35),

$$I_s^* \le \underline{M} \le \overline{M} \le I_s^*(1+\varepsilon(s)) \quad (36)$$

is obtained. This yield,

$$\overline{M} - \underline{M} \le I_s^*(1+\varepsilon(s)) - I_s^* = \varepsilon(s) I_s^* . \quad (37)$$

But, by (31) and (32), for $s > s_0$ large enough,

$$I_s^* \le \int_{-1}^{0}\rho(y)\,(1+\varepsilon(s_0))\,I_{s_0}(s+y)\,dy \quad (38)$$

and hence (37) become

$$\overline{M} - \underline{M} \le \varepsilon(s)\int_{-1}^{0}\rho(y)\,(1+\varepsilon(s_0))\,I_{s_0}(s+y)\,dy \quad (39)$$

for $s > s_0$ and s_0 large enough

$$\varepsilon(s) = ce^{-\eta s} \quad (40)$$

for $c > 0$, $\eta > 0$,
and, by (31) and (36),

$$\limsup_{s\to+\infty} I_{s_0}(s+y) \le \limsup_{s\to+\infty} e^{\alpha(s+y)}\bar{m}'(s+y)$$
$$\le I_{s_0}^*(1+\varepsilon(s_0)) \quad (41)$$

is obtained. Using (40) and (41),it is seen that the right hand side of (39) vanishes when $s \to +\infty$ and hence $\overline{M} - \underline{M}$.

This shows that $\limsup_{s\to+\infty} e^{\alpha x}\bar{m}'(x)$ exists.

Therefore there exists $M > 0$ so that

$$\lim_{s\to+\infty} e^{\alpha x}\bar{m}'(x) = M.$$

Thus the first limit in the theorem (as well as (18)) is proved.
The proof of the second one is similar. Indeed both M and $e^{\alpha x}\bar{m}'(x)$ are in the interval with extremes min $\{I_s(x), I_s^*\}$ and $(1 + \varepsilon(s))$ max $\{I_s(x), I_s^*\}$.

Then by (33) there is $c > 0$ such that all s large enough and all $x > s$,

$$\left|M - e^{\alpha x}\bar{m}'(x)\right| \le \left|I_s(x)-I_s^*\right| + \varepsilon(s)\left[I_s(x)+I_s^*\right]$$
$$\le c(e^{-\delta(x-s)} + e^{-\eta s}) .$$

If s is chosen such that $(x - s)d_1 = \eta s$,

$$\left|M - e^{\alpha x}\bar{m}'(x)\right| \le 2ce^{-\eta s} .$$

If $\delta = \dfrac{\eta\delta_1}{\eta+\delta_1}$, then

$$\left(M - e^{\alpha x}\bar{m}'(x)\right) \le 2ce^{-\delta x} .$$

Thus, Theorem 1 is proved.
The following Theorems are Corollaries of Theorem 1.

***Theorem* 2.**
Let $a = M\alpha^{-1}$ and $0 < \delta 9 < \delta$ with δ as in Theorem 1. Then

$$\lim_{x\to+\infty} e^{(\alpha+\delta')x}[\bar{m}(x) - (m_\beta - ae^{-\alpha x})] = 0 . \quad (42)$$

Proof: Since, from (7) $\bar{m}(x) \to m_\beta$ as $x \to +\infty$, it follows that

$$m_\beta - \bar{m}(x) = \int_x^\infty \bar{m}'(y)dy .$$

Then, recalling that $M = \alpha a$,

$$\left|e^{(\alpha+\delta')x}[\bar{m}(x) - (m_\beta - ae^{-\alpha x})]\right|$$
$$= \left|e^{(\alpha+\delta')x}\int_x^\infty [a\alpha e^{-\alpha y} - \bar{m}'(y)dy]\right| \quad (43)$$
$$\le \int_x^\infty e^{(\alpha+\delta')y}\left|Me^{-\alpha y} - \bar{m}'(y)\right|dy$$

$$= \int_x^\infty e^{(\delta-\delta')y}e^{\delta y}\left|M - e^{\alpha y}\bar{m}'(y)\right|dy .$$

Now, if $x \to +\infty$, then $y \to +\infty$ so that

$$e^{\delta y}\left|M - e^{\alpha y}\bar{m}'(y)\right| \to 0 .$$

By Theorem 1. Hence the left hand side of (43) vanishes as $x \to +\infty$. Thus the Theorem is proved.

***Theorem* 3.**
Let $\delta > 0$ be as in Theorem 1. Then

$$\lim_{x\to+\infty} e^{\delta y}\left|e^{\alpha x} - \bar{m}''(x) + \alpha M\right| = 0 . \quad (44)$$

Proof: By differentiating (10) which is stated as $\bar{m}' = p(x)J * \bar{m}'(x)$ where $p(x) = \beta(1 - \bar{m}(x)^2]$ (see (11))

$$\bar{m}''(x) = -2\beta\bar{m}(x)\bar{m}'(x)J * \bar{m}'(x) + p(x)J' * \bar{m}'(x) \quad (45)$$

where $J9(x, y) = \dfrac{\partial}{\partial x}J(x, y)$.

By Theorem 1,

$$\lim_{x\to+\infty} e^{(\alpha+\delta)x}[-2\beta\bar{m}(x)\bar{m}'(x)J * \bar{m}'(x)] = 0 . \quad (46)$$

is obtained. So that

$$\lim_{x\to+\infty} e^{\delta x}\left|e^{\alpha x}\bar{m}''(x) - p(x)\int J'(x,y)e^{-\alpha(y-x)}\left[e^{\alpha y}\bar{m}'(y)dy\right]\right|$$
$$= 0 \quad (47)$$

Thus,

$$\left|e^{\delta x}p(x)\int J'(x,y)e^{-\alpha(y-x)}\left[e^{\alpha y}\bar{m}'(y)-M\right]dy\right|$$
$$\le p(x)\int_{x-1}^{x+1}\left|J'(x,y)\right|e^{-\alpha(y-x)}e^{\delta}\left|e^{\delta y}(e^{\alpha y}\bar{m}'(y)-M)\right|dy$$

so by Theorem 2, $e^{\delta y}[e^{\alpha y}\bar{m}'(y)-M] = 0$ as $x \to +\infty$. This yield

$$\lim_{x\to+\infty} e^{\delta x}p(x)\int J'(x,y)e^{-\alpha(y-x)}e^{\alpha y}\bar{m}'(y)dy$$
$$= \lim_{x\to+\infty} e^{\delta x}p(x)\int J'(x,y)e^{-\alpha(y-x)}Mdy\,.$$

Thus from (47)

$$\lim_{x\to+\infty} e^{\delta x}\left|e^{\alpha x}\bar{m}''(x)-p(x)\int J'(x,y)e^{-\alpha(y-x)}Mdy\right|=0\,. \quad (48)$$

Again from (15),

$$\int J'(x,y)e^{-\alpha(y-x)}dy = -\alpha\int J(x,y)e^{-\alpha(y-x)}dy \quad (49)$$
$$= \frac{-\alpha}{p_\infty}$$

(recall that $\alpha = \alpha p_\infty$) is obtained. Then (49) becomes

$$\lim_{x\to+\infty} e^{\delta x}\left|e^{\alpha x}\bar{m}''(x)+\frac{p(x)}{p_\infty}\alpha M\right|=0 \quad (50)$$

Thus,

$$\frac{p(x)}{p_\infty}-1=\frac{1}{1-m_\beta^2}\left[m_\beta+\bar{m}(x)\right]\left[m_\beta-\bar{m}(x)\right]$$

is obtained. Since $0 < \delta < \alpha$, it follows from Theorem 2 that $\lim_{x\to+\infty} e^{\delta x}\left[m_\beta+\bar{m}(x)\right]=0$.

Then, by (50),

$$\lim_{x\to+\infty} e^{\delta x}\left|e^{\alpha x}\bar{m}''(x)+\alpha M\right|=0$$

is got. This completes the proof of the Theorem.From Theorem 1, 2 and 3, the following Theorem is obtained.

***Theorem* 4.**

If $m=\bar{m}$ is an instanton. Then there are α and a positive $\alpha_0 > \alpha$ and $c > 0$ such that for all $x \ge 0$,

$$\left|\bar{m}(x)-\left(m_\beta-ae^{-\alpha x}\right)\right|+\left|\bar{m}'(x)-a\alpha e^{-\alpha x}\right|$$
$$+\left|\bar{m}''(x)+a\alpha^2e^{-\alpha x}\right|\le ce^{-\alpha_0 x} \quad (51)$$

III. Operators on Bounded Domains

In this section, the problem in bounded domains with Neumann conditions can be considered. The operator A_m on $C^{sym}(3)$ can be isomorphically regarded as an operator on $C(3_+)$, by setting

$$A_m^+ f = A_m f^* \quad (52)$$

where $f^* \in C^{sym}(3)$ is defined for each $f \in C(3_+)$ by

$$f^*(x) = f(?\,x\,?). \quad (53)$$

A_m^+ is still an integral operator and its kernel is

$$A_m^+(x,y) = A_m(x,y) + A_m(x,-y)\,,\ x,y \in 3_+. \quad (54)$$

This can be interpreted as a reflecting boundary condition called the Neumann condition at 0 and the original problem for A_m on $C^{sym}(3)$ is actually the problem on the half line with Neumann conditions at 0. By adding another reflecting at $\ell > 1$, a new operator $A_{m,\ell}$ on $C([0,\ell])$ can be defined by setting the reflection rule

$$R_\ell(x) = \begin{cases} |x| & \text{for } -1 \le x \le \ell \\ \ell-(x-\ell) & \text{for } \ell \le x \le \ell+1 \end{cases}. \quad (55)$$

Now, for x and y in $[0, \ell]$,

$$A_{m,\ell}(x,y) = \sum_{z:R_\ell(z)=y} A_m(x,z) \quad (56)$$

is defined. Then $A_{m,\ell}$ (in fact $A_{(\cdot),\ell}$) is the operator on $C[0,\ell]$ with kernel $A_{m,\ell}(x,y)$. The case $\ell = +\infty$ is included by setting $R_{+\infty}(x) = ?\,x\,?$, then $A_{m,+\infty} = A_m^+$. It can be worked in finite volume and by proving estimates uniform in ℓ, the original case in the limit $\ell \to +\infty$. This is not only a technical device, but in fact the analysis in the bounded domains has its own interest. When m is taken close to a double instanton, the analysis in $C([0,\ell])$ with Neumann conditions corresponds to two double instantons, one across 0 and the other one across ℓ. The spectral properties in this case reflect the interaction between these two structures. When ℓ is finite, the classical Perron-Frobenius theorem is obtained. It can be stated the Perron-Frobenius Theorem with several other properties of the maximal eigenvalue $\lambda_{m,\ell}$ and the corresponding left and right eigenvectors $u_{m,\ell}$ and $v_{m,\ell}$. It is not yet supposing that m is close to a double instanton, which statements refer to general $m \in C^{sym}([0,\ell])$, but the results are not uniform in ℓ.Now, the following Perron - Frobenius Theorem can be stated.

***Theorem* 5.**

Let $l > 1$ and $m \in C([0,l],[-1,1])$. Then there are $\lambda_{m,\ell} > 0$, $u_{m,\ell}$ and $v_{m,\ell}$ in $C[0,l]$, $u_{m,\ell}$ and $v_{m,\ell}$ strictly positive, such that

$$A_{m,\ell} * v_{m,\ell} = \lambda_{m,\ell} v_{m,\ell} \quad , \quad u_{m,\ell} * A_{m,\ell} = \lambda_{m,\ell} u_{m,\ell} \quad (57)$$

($u_{m,l}$ and $v_{m,l}$ are left and right eignevectors with eigenvalue $\lambda_{m,l}$) and for any $x \in [0,\ell]$.

$$v_{m,l}(x) = p_m(x)\,u_{m,l}(x). \quad (58)$$

Any other point of the spectrum is strictly inside the ball of radius $\lambda_{m,l}$. Recall that

$$p_m(x) = \frac{\beta}{\cosh^2\{\beta J * m(x)\}} \quad .$$

Since the theorem is too general it cannot say much about the localization of the spectrum and the dependence on ℓ of $\lambda_{m,\ell}$, $v_{m,\ell}$ and $u_{m,\ell}$, for that more assumptions on m is needed. To complete, the proof of Theorem is needed that a Markov chain whose transition probability is conjugated to $A_{m,\ell}(x, y)$.

A. Auxiliary Markov Chains

In Theorem 5, it is mentioned that $\lambda_{m,\ell}$ and $v_{m,\ell}$ are strictly positive. Hence the function

$$Q_{m,\ell}(x, y) = A_{m,\ell}(x, y)\ \frac{v_{m,\ell}(y)}{\lambda_{m,\ell} v_{m,\ell}(x)},\ x, y \in [0,\ \ell\} \qquad (59)$$

is well-posed and its defines a transition probability on $[0, \ell]$ conjugated to the operator $A_{m,\ell}$: the spectrum of $A_{m,\ell}$ is obtained from that of $Q_{m,\ell}$ after multiplication by $\lambda_{m,\ell}$. In particular the spectral gap in Theorem 5 is related to the mixing properties of the Markov chain with transition probability $Q_{m,\ell}$ .If m is an instanton, $m = \bar{m}$, then $\lambda_{\bar{m}} = 1$ and $v_{\bar{m}} = \bar{m}'$, i.e. $A_{\bar{m}} = \bar{m}'$, obtained by differentiating the instanton equation (6).The analogue of (28) defines the basic transition probability:

$$P(x,y) = A_{\bar{m}}(x, y) = \frac{\bar{m}'(y)}{\bar{m}'(x)},\ x,y \text{ in } 3. \qquad (60)$$

In the problem with a (reflected) instanton at ξ, i.e. $(\xi - x)$, and Neumann conditions, i.e. reflection at 0 and ℓ, $\ell > 2\xi$, $\xi > 1$, an important role will be played by the transition probability:

$$Q_{\xi,\ell}(x, y) = \sum_{R_\ell(z)=y} P(\xi - x, \xi - z),\ x \text{ and } y \text{ in } [0,\ \ell\ ]\ . \qquad (61)$$

The above three Markov chains can be seen as describing similar, discrete time, jump processes of a particle on the line 3.

B. Bounds for Eigen-vectors

The following proposition provides the local bounds for eigenvectors for the operator $A_{m,\ell}$.

Proposition6*:* There is $b > 1$ so that for any $|x - y| \leq 1$

$$b^{-1} \leq \frac{v_{m,\ell}(x)}{v_{m,\ell}(y)} \leq b. \qquad (62)$$

IV. Conclusion

The evolution equation in the *space* of symmetric bounded function is given by

$$\frac{\partial m_t}{\partial t} = -m_t + \tanh(\beta J * m_t + \beta h)$$

with $\beta > 1$, $h \geq 0$ and

$$(J * m)(x) = \int_3 J(x, y) m(y) dy$$

This project is devoted to a study of the eigenvalue problem for the integral operator $A_{\ m}$ related to the given evolution equation. It is noted that spectral properties of the operator $A_{\ m}$ are obtained by showing the existence of a simple positively of the corresponding eigenvector. The analysis of the instanton and of its asymptotic behavior has been discussed.

Acknowledgment

The author is grateful to His Excellency U Thaung, Minister, Ministry of Science and Technology for allowing me conduct this research. She wish to my gratitude to Dr.Khaing Khaing Aye,Associate Professor and Head, Department of Engineering Mathematics, Mandalay Technological University for her support of this research.

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