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Anti-periodic solutions for Cohen-Grossberg shunting inhibitory neural networks with delays

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Abstract—By using the method of coincidence degree theory and constructing suitable Lyapunov functional, several sufficient conditions are established for the existence and global exponential stability of anti-periodic solutions for Cohen-Grossberg shunting inhibitory neural networks with delays. An example is given to illustrate our feasible results.

Keywords—Anti-periodic solution; Coincidence degree; Global exponential stability; Cohen-Grossberg shunting inhibitory cellular neural networks.

I. Introduction

In recent years, the Cohen-Grossberg neural networks [1] have been extensively studied because of their immense potential of application perspective in different areas such as pattern recognition, optimization, signal and image processing. Hence, they have been the object of intensive analysis by numerous authors and some good results on the existence and exponential stability of periodic solutions for Cohen-Grossberg neural networks with delays or with delays impulses have been obtained [2-11].

On the other hand, shunting inhibitory cellular neural networks (SICNNs) have many applications in psychophysics, speech, perception, robotics, adaptive pattern recognition, vision, and image processing. Since all these applications closely relate to their dynamics, the dynamical behaviors of SICNNs with delays have been widely investigated (see e.g. [12-15]). Many important results on the dynamics behaviors of SICNNs have been established and successfully applied to signal processing, pattern recognition, associative memories, and so on.

In this paper, we are concerned with the following Cohen-Grossberg shunting inhibitory cellular neural networks with delays:

$$\begin{aligned} u'_{ij}(t) &= -a_{ij}(u_{ij}(t))\Big\{b_{ij}(t,u_{ij}(t)) \\ &\quad + \sum_{c^{kl}\in N_r(i,j)} C^{kl}_{ij}(t) f_{ij}(t,u_{kl}(t-\tau_{kl}(t))) \\ &\quad \times u_{ij}(t) - I_{ij}(t)\Big\}, \end{aligned} \tag{1}$$

where $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, C_{ij} denotes the cell at the (i,j) position of the lattice, the r-neighborhood $N_r(i,j)$ of C_{ij} is given by

$$\begin{aligned} N_r(i,j) = \Big\{&C_{ij}:\max(|k-i|,|l-j|)\le r, \\ &1\le k\le m, 1\le l\le n\Big\}, \end{aligned}$$

u_{ij} acts as the activity of the cell C_{ij}, $L_{ij}(t)$ is the external input to C_{ij}, $a_{ij}(u_{ij}(t))$ and $b_{ij}(t,u_{ij}(t))$ represent an amplification function at time t and an appropriately behaved function at time t, respectively, $C^{kl}_{ij}(t)>0$ is the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}, and the activity function $f_{ij}(t,x_{kl})$ is a continuous function representing the output or firing rate of the cell C_{kl}, C_{kl} are periodic functions with period $\frac{\omega}{2}$, τ_{kl} are periodic functions with period $\frac{\omega}{2}$, ω is a positive constant.

Let $u=(u_{11},u_{12},\ldots,u_{1m},\ldots,u_{n1},\ldots,u_{nm})^T$ be a column vector. The initial conditions of (1) is of the form

$$u_{ij}(s)=\phi_{ij}(s),\ s\in[-\tau,0],\ \tau=\max_{(i,j)}\{\sup_{t\in[0,\omega]}|\tau_{ij}(t)|\},$$

where $\phi_{ij}(s), i=1,2,\ldots,n,\ j=1,2,\ldots,m$, are continuous $\frac{\omega}{2}$-anti-periodic solutions.

Arising from problems in applied sciences, the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations (see [16-20]). Since SICNNs can be analog voltage transmission which is often an anti-periodic process, it is worth continuing the investigation of the existence and stability of anti-periodic solutions of SICNNs. To the best of the authors' knowledge, this is the first paper to study the existence and global exponential stability of the anti-periodic solution of system (1).

The rest of this paper is organized as follows. In Section 2, we obtain system (2) which is equivalent to system (1). After that, we shall introduce some notations and definitions and state some preliminary results needed in later sections. In Section 3, by using the method of coincidence degree, we obtain the existence of the anti-periodic solutions of system (2). In Section 4, we give the criteria of global exponential stability of the anti-periodic solutions of system (2). In Section 5, an example is also provided to illustrate the effectiveness of the main results in Sections 3 and 4. The conclusions are drawn in Section 6.

Throughout this paper, we assume that

(H_1) $C^{kl}_{ij}(t+\frac{\omega}{2})=C^{kl}_{ij}(t)$, $\tau_{ij}(t+\frac{\omega}{2})=\tau_{ij}(t)$, $I_{ij}(t+\frac{\omega}{2})=-I_{ij}(t)$, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$;

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(H_2) $a_{ij}(u)$ are even continuous functions, i.e. $a_{ij}(u) = a_{ij}(-u)$, and there exist positive constants $\overline{a_{ij}}$ and $\underline{a_{ij}}$ such that $0 < \underline{a_{ij}} \leq a_{ij}(t) \leq \overline{a_{ij}}$, $u \in R$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$;

(H_3) $b_{ij}(t, u) \in C(R^2, R)$, $b_{ij}(t + \frac{\omega}{2}, -u) = -b_{ij}(t, u)$. There is a positive constant μ_{ij} such that $\frac{\partial b_{ij}(t,u)}{\partial u} \geq \mu_{ij}$, $u \in R$, $b_{ij}(t, 0) = 0$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$;

(H_4) $f_{ij}(t, u) \in C(R^2, R)$, $f_{ij}(t + \frac{\omega}{2}, -u) = f_{ij}(t, u)$ and there are ω-periodic functions $\gamma_{ij}(t)$ such that $\gamma_{ij}(t) = \sup_{u \in R} |f_{ij}(t, u)|$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$;

(H_5) there are positive ω-periodic solutions $\beta_{ij}(t)$ such that $|f_{ij}(t, u) - f_{ij}(t, v)| \leq \beta_{ij}(t)|u - v|$ for all $u, v \in R, i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

II. Preliminaries

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

From (H_2), the antiderivative $1/a_{ij}(u_{ij})$ exists. We choose an antiderivative $h_{ij}(u_{ij})$ of $1/a_{ij}(u_{ij})$ that satisfies $h_{ij}(0) = 0$. Obviously, $(\mathrm{d}/\mathrm{d}u_{ij})h_{ij}(u_{ij}) = 1/a_{ij}(u_{ij})$. By $a_{ij}(u_{ij}) > 0$, we obtain that $h_{ij}(u_{ij})$ is strictly monotone increasing about u_{ij}. In view of derivative theorem for inverse function, the inverse function $h_{ij}^{-1}(u_{ij})$ of $h_{ij}(u_{ij})$ is differential and $(\mathrm{d}/\mathrm{d}u_{ij})h_{ij}^{-1}(u_{ij}) = a_{ij}(h_{ij}^{-1}(u_{ij}))$. By (H_3), composition function $b_{ij}(t, h_{ij}^{-1}(z))$ is differentiable. Denote $x_{ij}(t) = h_{ij}(u_{ij}(t))$. It is easy to see that $x'_{ij}(t) = u'_{ij}(t)/a_{ij}(u_{ij}(t))$ and $u_{ij}(t) = h_{ij}^{-1}(x_{ij}(t))$. Substituting these equalities into system (1), we get

$$
\begin{aligned}
x'_{ij}(t) &= -b_{ij}(t, h_{ij}^{-1}(x_{ij}(t))) \\
&\quad - \sum_{c^{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f_{ij}(t, h_{kl}^{-1}(x_{kl}(t - \tau_{kl}(t)))) \\
&\quad \times h_{ij}^{-1}(x_{ij}(t)) + I_{ij}(t), \\
&\quad i = 1, 2, \ldots, n, j = 1, 2, \ldots, m,
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
x'_{ij}(t) &= -d_{ij}(t, x_{ij}(t)) x_{ij}(t) \\
&\quad - \sum_{c^{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f_{ij}(t, h_{kl}^{-1}(x_{kl}(t - \tau_{kl}(t)))) \\
&\quad \times h_{ij}^{-1}(x_{ij}(t)) + I_{ij}(t), \qquad (2)
\end{aligned}
$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $d_{ij}(t, x_{ij}(t)) \triangleq \partial b_{ij}(t, h_{ij}^{-1}(z))/\partial z|_{z=\xi_{ij}}$, $\partial b_{ij}(t, h_{ij}^{-1}(z))/\partial z|_{z=\xi_{ij}}$ denotes the partial derivative of $b_{ij}(t, h_{ij}^{-1}(z))$ at point $z = \xi_{ij}$, $z \in R$, ξ_{ij} is between 0 and $x_{ij}(t)$.

By (H_2), (H_3) and the definition of $h_{ij}(x_{ij})$, we obtain $b_{ij}(t, h_{ij}^{-1}(t))$ is strictly monotone in creasing about x_{ij}. Hence, $d_{ij}(t, x_{ij}(t))$ is unique for any $x_{ij}(t)$ and continuous about $x_{ij}(t)$. Moreover, $d_{ij}(t, x_{ij}(t)) \geq \underline{a}_{ij}\mu_{ij}$.

From the definition of $h_{ij}^{-1}(u)$, use the Lagrange mean-value theorem, one gets

$$
|h_{ij}^{-1}(x) - h_{ij}^{-1}(y)| = |(h_{ij}^{-1})'(\xi)(x - y)| \leq \overline{a}_{ij}|x - y|
$$

for all $x, y \in R$, where ξ is between x and y, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

Let $x(t) = (x_{11}(t), \ldots, x_{1m}(t), \ldots, x_{n1}(t), \ldots, x_{nm}(t))^T \in C(\mathbb{R}, \mathbb{R}^{nm})$. The initial conditions associated with system (2) are of the form

$$
x_{ij}(s) = h_{ij}(\phi_{ij}(s)) = \varphi_{ij}(s), \quad s \in [-\tau, 0],
$$

where $\varphi_{ij}(t), i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ are continuous functions on $[0, \omega]$.

Definition 1. *Let x^* be an $\frac{\omega}{2}$-anti-periodic solution of (2) with initial value φ^*. If there exist constants $\alpha > 0, P \geq 1$ such that, for every solution $x(t)$ of (2) with initial value φ, the following inequalities hold for $i = 1, \ldots, n, j = 1, \ldots, m$,*

$$
|x_{ij}(t) - x_{ij}^*(t)| \leq P\|\varphi - \varphi^*\|e^{-\alpha t}, \quad t > 0,
$$

where $\|\varphi - \varphi^*\| = \max_{(i,j)} \sup_{-\tau \leq s \leq 0}\{|\varphi_{ij}(s) - \varphi_{ij}^*(s)|\}$. Then $x^*(t)$ is said to be global exponentially stable.

The following fixed point theorem of coincidence degree is crucial in the arguments of our main results.

Lemma 1. [21] *Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces, $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L) \cap \bar{\Omega} \neq \emptyset$ and $N : \bar{\Omega} \to \mathbb{Y}$ is L-compact. Further, we also assume that*

(H) *$Lx - Nx \neq \lambda(-Lx - N(-x))$ for all $x \in D(L) \cap \partial\Omega$, $\lambda \in (0, 1]$.*

Then equation $Lx = Nx$ has at least one solution on $D(L) \cap \bar{\Omega}$.

For the sake of convenience, we introduce some notations

$$
\overline{I}_{ij} = \max_{t \in R} |I_{ij}(t)|, \quad \overline{C}_{ij}^{kl} = \max_{t \in R} |C_{ij}^{kl}(t)|,
$$
$$
\overline{\gamma}_{ij} = \max_{t \in R} |\gamma_{ij}(t)|, \quad \overline{\beta}_{ij} = \max_{t \in R} |\beta_{ij}(t)|,
$$

where $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$.

III. Existence of anti-periodic solutions

In this section, by Lemma 1, we will study the existence of at least one anti-periodic solution of (1).

Theorem 1. *Assume that (H_1)-(H_5) hold. Suppose further that*

(H_6) $1 - A \sum_{C_{kl} \in N_r(i,j)} \overline{C}_{ij}^{kl} \overline{\gamma}_{ij} \overline{a_{ij}} \omega > 0$, *where*

$$
A = \frac{e^{\underline{a}_{ij}\mu_{ij}\omega}}{e^{\underline{a}_{ij}\mu_{ij}\omega} - 1}.
$$

Then system (1) has at least one $\frac{\omega}{2}$-anti-periodic solution.

Proof: We first prove that system (2) has at least one $\frac{\omega}{2}$-anti-periodic solution. Take

$$
\mathbb{X} = \mathbb{Y} = \{x \in C(R, R^{nm}) : x(t + \frac{\omega}{2}) = -x(t), t \in [0, \frac{\omega}{2}]\}
$$

be two Banach spaces equipped with the norms

$$
\|x\|_{\mathbb{X}} = \|y\|_{\mathbb{Y}} = \sum_{i=1}^{n} \sum_{j=1}^{m} |x_{ij}|_0 \quad \text{for all } x \in \mathbb{X},
$$

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where $|x_{ij}|_0 = \max_{t\in[0,\omega]}|x_{ij}(t)|$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$.

In system (2), let

$$\begin{aligned}&F_{ij}(t,x)\\ &= -\sum_{c^{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f_{ij}(t,h_{kl}^{-1}(x_{kl}(t-\tau_{kl}(t))))\\ &\quad\times h_{ij}^{-1}(x_{ij}(t))+I_{ij}(t),\end{aligned}\tag{3}$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$. Then we have

$$\left(x_{ij}(t)e^{\int_0^t d_{ij}(s,x_{ij}(s))\mathrm{d}s}\right)' = e^{\int_0^t d_{ij}(s,x_{ij}(s))\mathrm{d}s}F_{ij}(t,x),\tag{4}$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$. Integrating (4) from t to $t+\omega$ leads to

$$\begin{aligned}&x_{ij}(t+\omega)e^{\int_0^t d_{ij}(s,x_{ij}(s))\mathrm{d}s} - x_{ij}(t)e^{\int_0^t d_{ij}(s,x_{ij}(s))\mathrm{d}s}\\ &= \int_t^{t+\omega} e^{\int_0^s d_{ij}(r,x_{ij}(r))\mathrm{d}r}F_{ij}(s,x)\mathrm{d}s,\end{aligned}$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$. So

$$\begin{aligned}&x_{ij}(t)\left(e^{\int_0^{t+\omega} d_{ij}(s,x_{ij}(s))\mathrm{d}s} - e^{\int_0^t d_{ij}(s,x_{ij}(s))\mathrm{d}s}\right)\\ &= \int_t^{t+\omega} e^{\int_0^s d_{ij}(r,x_{ij}(r))\mathrm{d}r}F_{ij}(s,x)\mathrm{d}s,\end{aligned}$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$. Hence

$$\begin{aligned}&x_{ij}(t)e^{\int_0^t d_{ij}(s,x_{ij}(s))\mathrm{d}s}\left(e^{\int_t^{t+\omega} d_{ij}(s,x_{ij}(s))\mathrm{d}s} - 1\right)\\ &= \int_t^{t+\omega} e^{\int_0^s d_{ij}(r,x_{ij}(r))\mathrm{d}r}F_{ij}(s,x)\mathrm{d}s,\end{aligned}$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$. Thus

$$\begin{aligned}x_{ij}(t) &= \frac{e^{\int_t^s d_{ij}(r,x_{ij}(r))\mathrm{d}r}}{e^{\int_t^{t+\omega} d_{ij}(s,x_{ij}(s))\mathrm{d}s}-1}\int_t^{t+\omega}F_{ij}(s,x)\mathrm{d}s\\ &= \int_t^{t+\omega}G_{ij}(t,s)F_{ij}(s,x)\mathrm{d}s,\end{aligned}\tag{5}$$

where

$$\begin{aligned}G_{ij}(t,s) &= \frac{e^{\int_t^s d_{ij}(r,x_{ij}(r))\mathrm{d}r}}{e^{\int_0^\omega d_{ij}(s,x_{ij}(s))\mathrm{d}s}-1}\\ &\le \frac{e^{\int_0^\omega d_{ij}(s,x_{ij}(s))\mathrm{d}s}}{e^{\int_0^\omega d_{ij}(s,x_{ij}(s))\mathrm{d}s}-1}\\ &\le \frac{e^{\underline{a}_{ij}\mu_{ij}\omega}}{e^{\underline{a}_{ij}\mu_{ij}\omega}-1} \triangleq A,\end{aligned}\tag{6}$$

in which $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

Set

$$L:\mathrm{Dom}\,L\cap\mathbb{X}\to\mathbb{Y},\quad x\to x', Lx_{ij}=x'_{ij}(t),$$

where $i=1,2,\ldots,n$, $j=1,2,\ldots,m$,

$$\mathrm{Dom}\,L=\{x\in C^1(R,R^{nm}):x(t+\frac{\omega}{2})=-x(t),t\in[0,\frac{\omega}{2}]\},$$

and $N:\mathbb{X}\to\mathbb{Y}$

$$Nx_{ij} \;=\; -d_{ij}(t,x_{ij}(t))x_{ij}(t)+F_{ij}(t,x_{ij}(t)),$$

where $F_{ij}(t,x_{ij}(t))$ are defined as (3), $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

It is easy to see that

$$\mathrm{Ker}\,L=\{0\}\quad\text{and}\quad\mathrm{Im}\,L=\{y\in\mathbb{Y}:\int_0^\omega y(s)\,\mathrm{d}s=0\}\equiv\mathbb{Y}.$$

Thus dim Ker $L=0=$ codim Im L, and L is a linear Fredholm operator of index zero.

Define the continuous projector $P:\mathbb{X}\to\mathrm{Ker}L$ and the averaging projector $Q:\mathbb{Y}\to\mathbb{Y}$ by

$$Px=\int_0^\omega x(s)\,\mathrm{d}s=0\quad\text{and}\quad Qy=\int_0^\omega y(s)\,\mathrm{d}s=0.$$

Hence Im $P=$ Ker L and Ker $Q=$ Im L=Im $(I-Q)$. Denoting by $L_P^{-1}:\mathrm{Im}\,L\to\mathrm{Dom}(L)\cap\mathrm{Ker}\,P$ the inverse of $L|_{D(L)\cap\mathrm{Ker}P}$, we have

$$L_P^{-1}y=\int_0^t y(s)\,\mathrm{d}s-\frac{1}{2}\int_0^{\frac{\omega}{2}}y(s)\,\mathrm{d}s.$$

It is not difficult to show that $QN(\bar\Omega)$, $L_P^{-1}(I-Q)N(\bar\Omega)$ are relatively compact for any open bounded set $\Omega\subset\mathbb{X}$. Therefore, N is L-compact on $\bar\Omega$ for any open bounded set $\Omega\subset\mathbb{X}$.

In order to apply Lemma 1, we need to find an appropriate open bounded subset Ω in $\mathbb{X}$. Corresponding to the operator equation $Lx-Nx=\lambda(-Lx-N(-x)),\lambda\in(0,1]$, we have

$$\begin{aligned}x'_{ij}(t) &= \frac{1}{1+\lambda}[-d_{ij}(t,x_{ij}(t))x_{ij}(t)+F_{ij}(t,x_{ij}(t))]\\ &\quad-\frac{\lambda}{1+\lambda}[d_{ij}(t,-x_{ij}(t))x_{ij}(t)+F_{ij}(t,-x_{ij}(t))]\\ &= -d_{ij}(t,x_{ij}(t))x_{ij}(t)+\frac{1}{1+\lambda}F_{ij}(t,x_{ij}(t))\\ &\quad-\frac{\lambda}{1+\lambda}F_{ij}(t,-x_{ij}(t)),\end{aligned}$$

where $i=1,2,\ldots,m$, $j=1,2,\ldots,n$.

Set

$$\begin{aligned}H_{ij}(t,x_{ij}(t)) &= \frac{1}{1+\lambda}F_{ij}(t,x_{ij}(t))\\ &\quad-\frac{\lambda}{1+\lambda}F_{ij}(t,-x_{ij}(t)),\end{aligned}$$

where $i=1,2,\ldots,m$, $j=1,2,\ldots,n$. So we have

$$\begin{aligned}x'_{ij}(t) &= -d_{ij}(t,x_{ij}(t))x_{ij}(t)+H_{ij}(t,x_{ij}(t)),\\ &\quad i=1,2,\ldots,m,j=1,2,\ldots,n.\end{aligned}$$

As (5), we have

$$x_{ij}(t)=\int_t^{t+\omega}G_{ij}(t,s)H_{ij}(s,x_{ij}(t))\,\mathrm{d}s,$$

where $G_{ij}(t,s)$ are defined as (6), $i=1,2,\ldots,n$, $j=1,2,\ldots,m$.

So we get for $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$,

$$\begin{aligned}
|x_{ij}(t)| &\le \int_t^{t+\omega} |G_{ij}(t,s)||H_{ij}(s, x_{ij}(s))|\mathrm{d}s \\
&\le A\int_t^{t+\omega} |H_{ij}(s, x_{ij}(t))|\mathrm{d}s \\
&\le A\int_t^{t+\omega} \frac{1}{1+\lambda}|F_{ij}(s, x_{ij}(s))|\mathrm{d}s \\
&\quad + A\int_t^{t+\omega} \frac{\lambda}{1+\lambda}|F_{ij}(s, -x_{ij}(s))|\mathrm{d}s \\
&\le \frac{A}{1+\lambda}\int_t^{t+\omega}\Big\{\Big|\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s) \\
&\quad \times f_{ij}(s, h_{kl}^{-1}(x_{kl}(s-\tau_{kl}(t))))h_{ij}^{-1}(x_{ij}(s))\Big|\Big\}\mathrm{d}s \\
&\quad + \frac{\lambda A}{1+\lambda}\int_t^{t+\omega}\Big\{\Big|\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s) \\
&\quad \times f_{ij}(s, h_{kl}^{-1}(x_{kl}(s-\tau_{kl}(s))))h_{ij}^{-1}(x_{ij}(s))\Big|\Big\}\mathrm{d}s \\
&\quad + A\int_t^{t+\omega} |I_{ij}(s)|\mathrm{d}s \\
&\le \frac{A}{1+\lambda}\int_t^{t+\omega} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\overline{\gamma}_{ij}\overline{a_{ij}}|x_{ij}(s)|\mathrm{d}s \\
&\quad + \frac{\lambda A}{1+\lambda}\int_t^{t+\omega} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\overline{\gamma}_{ij}\overline{a_{ij}}|x_{ij}(s)|\mathrm{d}s \\
&\quad + A\int_t^{t+\omega} |I_{ij}(s)|\mathrm{d}s \\
&\le A\sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\overline{\gamma}_{ij}\overline{a_{ij}}\omega|x_{ij}|_0 \\
&\quad + A\int_t^{t+\omega} |I_{ij}(s)|\mathrm{d}s.
\end{aligned}$$

Set $C = \max_{s\in[0,\omega]} A\int_0^\omega |I_{ij}(s)|\mathrm{d}s$. So we have

$$|x_{ij}|_0 \le A\sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\overline{\gamma}_{ij}\overline{a_{ij}}\omega|x_{ij}|_0 + C,$$

where $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$.

From (H_5), we get

$$|x_{ij}|_0 \le \frac{C}{1 - A\sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\overline{\gamma}_{ij}\overline{a_{ij}}\omega} = M_{ij},$$

where $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$. Let

$$M = \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}.$$

where $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$.

Clearly, M is independent of λ. Take

$$\Omega = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} < M + 1\}.$$

It is clear that Ω satisfies all the requirement in Lemma 1 and the condition (H) is satisfied. In view of all the discussions above, we conclude from Lemma 1 that system (2) has at least one $\frac{\omega}{2}$-anti-periodic solution. That is, system (1) has at least one $\frac{\omega}{2}$-anti-periodic solution. This completes the proof. ■

IV. Global exponential stability of anti-periodic solutions

Suppose that $x^*(t) = (x_{11}^*(t), \ldots, x_{1m}^*(t), \ldots, x_{n1}^*(t), \ldots, x_{nm}^*(t))^T$ is an $\frac{\omega}{2}$-anti-periodic solution of system (1). In this section, we will construct some suitable Lyapunov functions to study the global exponential stability of this anti-periodic solution.

Theorem 2. *Assume that (H_1)-(H_6) hold. Suppose further that*

(H_7) *For $i = 1, 2, \ldots, n, j = 1, 2, \ldots, m$,*

$$\sum_{c^{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl}\bar{a}_{ij}(\bar{\gamma}_{ij} + M\bar{\beta}_{ij}\bar{a}_{kl}) < q_{ij}.$$

Then the $\frac{\omega}{2}$-anti-periodic solution of system (1) is globally exponentially stable.

Proof: According to Theorem 1, we know that system (1) has an $\frac{\omega}{2}$-anti-periodic solution $x^*(t) = (x_{11}^*(t), \ldots, x_{1m}^*(t), \ldots, x_{n1}^*(t), \ldots, x_{nm}^*(t))^T$ with initial value $\varphi^*(t) = (\varphi_{11}^*(t), \ldots, \varphi_{1m}^*(t), \ldots, \varphi_{n1}^*(t), \ldots, \varphi_{nm}^*(t))^T$ and $\|x^*\| \le M$. Suppose that $x(t) = (x_{11}(t), \ldots, x_{1m}(t), \ldots, x_{n1}(t), \ldots, x_{nm}(t))^T$ is an arbitrary solution of system (1) with initial value $\varphi(t) = (\varphi_{11}(t), \ldots, \varphi_{1m}(t), \ldots, \varphi_{n1}(t), \ldots, \varphi_{nm}(t))^T$. Set $z(t) = (z_{11}(t), \ldots, z_{1m}(t), \ldots, z_{n1}(t), \ldots, z_{nm}(t))^T = x(t) - x^*(t)$. Then it follows that

$$\begin{aligned}
z_{ij}'(t) &= \big(x_{ij}(t) - x_{ij}^*(t)\big)' \\
&= -[b_{ij}(t, h_{ij}^{-1}(x_{ij}(t))) - b_{ij}(t, h_{ij}^{-1}(x_{ij}(t)))] \\
&\quad - \sum_{c^{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\big[f_{ij}(t, h_{kl}^{-1}(x_{kl}(t-\tau_{kl}(t)))) \\
&\quad \times h_{ij}^{-1}(x_{ij}(t)) \\
&\quad - f_{ij}(t, h_{kl}^{-1}(x_{kl}^*(t-\tau_{kl}(t))))h_{ij}^{-1}(x_{ij}^*(t))\big],
\end{aligned} \tag{7}$$

where $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$. By (H_7), we have

$$-q_{ij} + \sum_{c^{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl}\bar{a}_{ij}(\bar{\gamma}_{ij} + M\bar{\beta}_{ij}\bar{a}_{kl}) < 0, \tag{8}$$

where $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$. Set

$$h_{ij}(\lambda) = \lambda - q_{ij} + \sum_{c^{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl}\bar{a}_{ij}(\bar{\gamma}_{ij} + e^{\lambda\tau}M\bar{\beta}_{ij}\bar{a}_{kl}).$$

Clearly, $h_{ij}(\lambda)$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$ are continuous functions on R. Since $h_{ij}(0) < 0$,

$$\frac{\mathrm{d}h_{ij}(\lambda)}{\mathrm{d}\lambda} = 1 + \lambda e^{\lambda\tau}\sum_{c^{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl}\bar{a}_{ij}M\bar{\beta}_{ij}\bar{a}_{kl} > 0$$

and $h_{ij}(+\infty) = +\infty$, hence $h_{ij}(\lambda)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$ are strictly monotone increasing functions. Therefore, for any $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$, there is a unique $\lambda_{ij} > 0$ such that

$$\lambda_{ij} - q_{ij} + \sum_{c^{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl}\bar{a}_{ij}(\bar{\gamma}_{ij} + e^{\lambda_{ij}\tau}M\bar{\beta}_{ij}\bar{a}_{kl}) = 0.$$

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Let $\alpha = \min\{\lambda_{11}, \lambda_{12}, \ldots, \lambda_{1m}, \ldots, \lambda_{n1}, \ldots, \lambda_{nm}\}$. Obviously, we have from (8) that

$$h_{ij}(\alpha) = \alpha - q_{ij} + \sum_{c^{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl}\bar{a}_{ij}(\bar{\gamma}_{ij} + e^{\alpha\tau}M\bar{\beta}_{ij}\bar{a}_{kl}) \leq 0, \tag{9}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

It is obvious that

$$|z_{ij}(t)| \leq \|\varphi - \varphi^*\| \leq \|\varphi - \varphi^*\|e^{-\alpha t} \quad \text{for } t \in [-\tau, 0],$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, $\|\varphi - \varphi^*\|$ is defined as that in Definition 1.

Define a Lyapunov functional $V = (V_{11}, V_{12}, \ldots, V_{1m}, \ldots, V_{n1} \ldots, V_{nm})^T$ by $V_{ij}(t) = e^{\alpha t}|z_{ij}(t)|$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. In view of (7), we get

$$\begin{aligned}
&\frac{\mathrm{d}^+V_{ij}(t)}{\mathrm{d}t}\\
&= e^{\alpha t}\mathrm{sgn} z_{ij}\Big\{ - [b_{ij}(t, h_{ij}^{-1}(x_{ij}(t)))\\
&\quad -b_{ij}(t, h_{ij}^{-1}(x_{ij}(t)))]\\
&\quad - \sum_{c^{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\big[f_{ij}(t, h_{kl}^{-1}(x_{kl}(t - \tau_{kl}(t))))\\
&\quad \times h_{ij}^{-1}(x_{ij}(t))\\
&\quad -f_{ij}(t, h_{kl}^{-1}(x_{kl}^*(t - \tau_{kl}(t))))h_{ij}^{-1}(x_{ij}^*(t))\big]\Big\}\\
&\quad +\alpha e^{\alpha t}|z_{ij}(t)|\\
&\leq e^{\alpha t}\Big\{(\alpha - q_{ij})|z_{ij}(t)|\\
&\quad + \sum_{c^{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\big[\big|f_{ij}(t, h_{kl}^{-1}(x_{kl}(t - \tau_{kl}(t))))\\
&\quad \times h_{ij}^{-1}(x_{ij}(t))\\
&\quad -f_{ij}(t, h_{kl}^{-1}(x_{kl}(t - \tau_{kl}(t))))h_{ij}^{-1}(x_{ij}^*(t))\big|\\
&\quad +\big|f_{ij}(t, h_{kl}^{-1}(x_{kl}(t - \tau_{kl}(t))))h_{ij}^{-1}(x_{ij}^*(t))\\
&\quad -f_{ij}(t, h_{kl}^{-1}(x_{kl}^*(t - \tau_{kl}(t))))h_{ij}^{-1}(x_{ij}^*(t))\big|\big]\Big\}\\
&\leq e^{\alpha t}\Big\{\Big(\alpha - q_{ij} + \sum_{c^{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl}\bar{a}_{ij}\bar{\gamma}_{ij}\Big)|z_{ij}(t)|\\
&\quad + \sum_{c^{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl}\bar{a}_{ij}M\bar{\beta}_{ij}\bar{a}_{kl}|z_{kl}(t - \tau_{kl}(t))|\Big\}\\
&\leq \Big(\alpha - q_{ij} + \sum_{c^{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl}\bar{a}_{ij}\bar{\gamma}_{ij}\Big)V_{ij}(t)\\
&\quad + \sum_{c^{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl}\bar{a}_{ij}M\bar{\beta}_{ij}\bar{a}_{kl}e^{\alpha\tau}V_{kl}(t - \tau_{kl}(t)).
\end{aligned}\tag{10}$$

We claim that

$$V_{ij}(t) = |z_{ij}(t)|e^{\alpha t} \leq \|\varphi - \varphi^*\|,\ t > 0, \tag{11}$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Contrarily, there must exists $i_0 \in \{1, 2, \ldots, n\}$, $j_0 \in \{1, 2, \ldots, m\}$ such that

$$V_{i_0j_0}(\tilde{t}) = \|\varphi - \varphi^*\|, \frac{\mathrm{d}^+V_{i_0j_0}(\tilde{t})}{\mathrm{d}t} > 0, V_{ij}(t) \leq \|\varphi - \varphi^*\|, \tag{12}$$

$t \in [-\tau, \tilde{t}]$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Together with (10) and (12), we obtain

$$\begin{aligned}
0 &< \frac{\mathrm{d}^+V_{i_0j_0}(\tilde{t})}{\mathrm{d}t}\\
&\leq \Big(\alpha - q_{i_0j_0} + \sum_{c^{kl}\in N_r(i_0,j_0)} \bar{C}_{i_0j_0}^{kl}\bar{a}_{i_0j_0}\bar{\gamma}_{i_0j_0}\Big)V_{i_0j_0}(\tilde{t})\\
&\quad + \sum_{c^{kl}\in N_r(i_0,j_0)} \bar{C}_{i_0j_0}^{kl}\bar{a}_{i_0j_0}M\bar{\beta}_{i_0j_0}\bar{a}_{kl}e^{\alpha\tau}V_{kl}(\tilde{t} - \tau_{kl}(\tilde{t}))\\
&\leq \|\varphi - \varphi^*\|\Big\{\alpha - q_{i_0j_0} + \sum_{c^{kl}\in N_r(i_0,j_0)} \bar{C}_{i_0j_0}^{kl}\bar{a}_{i_0j_0}\bar{\gamma}_{i_0j_0}\\
&\quad + \sum_{c^{kl}\in N_r(i_0,j_0)} \bar{C}_{i_0j_0}^{kl}\bar{a}_{i_0j_0}M\bar{\beta}_{i_0j_0}\bar{a}_{kl}e^{\alpha\tau}\Big\}.
\end{aligned}$$

Hence,

$$0 < \alpha - q_{i_0j_0} + \sum_{c^{kl}\in N_r(i_0,j_0)} \bar{C}_{i_0j_0}^{kl}\bar{a}_{i_0j_0}(\bar{\gamma}_{i_0j_0} + e^{\alpha\tau}M\bar{\beta}_{i_0j_0}\bar{a}_{kl}),$$

which contradicts (9). Hence, (11) holds. It follows that

$$|z_{ij}(t)| \leq \|\varphi - \varphi^*\|e^{-\alpha t},$$

where $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. In view of Definition 1, the $\frac{\omega}{2}$-anti-periodic solution x^* of system (1) is globally exponentially stable. This completes the proof. ■

V. An example

In this section, we give an example to illustrate that our results are feasible.

Example 1. *Consider the following Cohen-Grossberg shunting inhibitory neural networks with delays:*

$$\begin{aligned}
u_{ij}'(t) &= -a_{ij}(u_{ij}(t))\Big\{b_{ij}(t, u_{ij}(t))\\
&\quad + \sum_{c^{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f_{ij}(t, u_{kl}(t - \tau_{kl}(t)))\\
&\quad \times u_{ij}(t) - I_{ij}(t)\Big\},
\end{aligned}\tag{13}$$

where $i, j = 1, 2$ $a_{11}(u) = 1 - \frac{1}{2}|\sin u|$, $a_{12}(u) = 1 - \frac{1}{2}|\cos u|$, $a_{21}(u) = 1 - \frac{1}{2}|\sin\sqrt{|u|}|$, $a_{22}(u) = 1 - \frac{1}{2}|\cos\sqrt{3|u|}|$, $b_{ij}(t, u) = u$, $C_{11}^{kl}(t) = C_{22}^{kl}(t) = 1 - \frac{1}{4}|\cos t|$, $C_{12}^{kl}(t) = C_{21}^{kl}(t) = 0$, $\tau_{kl}(t) = 1$, $f_{11}(t, u) = f_{22}(t, u) = \frac{e^{\pi}-1}{4\pi e^{2\pi}+1}\sin u^2$, $f_{12}(t, u) = f_{21}(t, u) = \frac{e^{\pi}-1}{4\pi e^{2\pi}+1}\cos u$, $I_{ij}(t) = \frac{e^{\pi}-1}{4\pi e^{4\pi}-e^{2\pi}}\cos t$, $\omega = 2\pi$, *system* (13) *has at least one globally exponentially stable* π*-anti-periodic solution.*

Proof: By calculation, we have

$$\sum_{c^{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} = 2,\ \bar{\gamma}_{ij} = \frac{e^{\pi} - 1}{4\pi e^{2\pi} + 1},$$

$$\bar{\beta}_{ij} = \frac{e^{\pi} - 1}{4\pi e^{2\pi} + 1},\ \bar{a}_{ij} = 1,\ \underline{a}_{ij} = \frac{1}{2},$$

$$d_{ij}(t, x) = \frac{\partial b_{ij}(t, u)}{\partial u} \cdot \frac{\partial h_{ij}^{-1}(z)}{\partial z}|_{z=\xi} = a_{ij}(\xi) \leq \bar{a}_{ij}(\xi) = 1,$$

$$d_{ij}(t,x) \geq \underline{a}_{ij}(\xi) = \frac{1}{2},\ A \leq \frac{e^{\int_0^{2\pi} \overline{a}_{ij}(\xi)\mathrm{d}s}}{e^{\int_0^{2\pi} \underline{a}_{ij}(\xi)\mathrm{d}s} - 1} \leq \frac{e^{2\pi}}{e^{\pi}-1},$$

$$q_{ij} = \frac{1}{2}.$$

It is obvious that (H_1)-(H_5) are satisfied. Furthermore, we can easily calculate that

$$A \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} \bar{\gamma}_{ij} \bar{a}_{ij} \omega \leq \frac{4\pi e^{2\pi}}{e^{\pi}-1} \cdot \frac{e^{\pi}-1}{4\pi e^{2\pi}+1} < 1,$$
$$\sum_{c^{kl}\in N_r(i,j)} \bar{C}_{ij}^{kl} \bar{a}_{ij} (\bar{\gamma}_{ij} + M\bar{\beta}_{ij} \bar{a}_{kl}) < q_{ij}.$$

So (H_6)-(H_7) hold. By Theorem 1 and Theorem 2, system (13) has at least one globally exponentially stable π-anti-periodic solution. This completes the proof. ■

VI. Conclusion

In this letter, Cohen-Grossberg shunting inhibitory cellular neural networks with delays have been studied. Some sufficient conditions for the existence and global exponential stability of the anti-periodic solutions have been established by using the method of coincidence degree theory and constructing suitable Lyapunov functional. Moreover, an example is given to illustrate the effectiveness of our results.

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