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A Study of Under Actuator Dynamic System by Comparing between Minimum Energy and Minimum Jerk Problems

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***Abstract*—**This paper deals with under actuator dynamic systems such as spring-mass-damper system when the number of control variable is less than the number of state variable. In order to apply optimal control, the controllability must be checked. There are many objective functions to be selected as the goal of the optimal control such as minimum energy, maximum energy and minimum jerk. As the objective function is the first priority, if one like to have the second goal to be applied; however, it could not fit in the objective function format and also avoiding the vector cost for the objective, this paper will illustrate the problem of under actuator dynamic systems with the easiest to deal with comparing between minimum energy and minimum jerk.

***Keywords*—**Under actuator, Dynamic optimal control, Minimum jerk, Minimum energy.

I. Introduction

The Most of the robots and advanced mobile machines nowadays are designed so that they are either optimized on their energy consumption or on their greatest smoothness of motion, [3]. Consequently, the trajectory planning and designs of these robots are done exclusively through many approaches such as the minimum energy and minimum jerk, [4]. Nevertheless, in some applications, the robot is needed to work very smoothly in order to avoid damaging the specimen that the robot is handling while consuming least amount of energy at the same time. In other words, we may want to minimize the jerk of the movement of the robot as to give it the smoothest motion as well as optimize that robot in the energy consumption issue.

The general format of the dynamic problems is consisting of the equation of motion, the initial conditions, and the boundary conditions. The area of interest in this paper will involve the problems with two-point-boundary-value conditions. Each of the problems may contain many possible solutions depending on the objective of application. Obviously, the robot that aims to run at lowest cost of energy will be designed to have the lowest actuator inputs during the motion. This is basically the optimization problem of the dynamic systems. Research shows that many of the researchers pay a lot of their attention on the minimization of energy while many tend to seek for the smoothness of the system. According to the second law of Newton's laws, there is a relationship between acceleration and summation of all forces including the control inputs of any linear dynamic system. By taking derivative with respect to time, there is a relationship between derivative of the acceleration called Jerk and derivative of all forces including the derivative of the control inputs of the dynamic system. In this paper, the derivative of the control inputs with respect to time are called indirect jerks.

The problem of under actuator dynamic system is quite challenge in the way of applying minimum energy or minimum jerk in order to have the most benefit. One very important condition for the under actuator dynamic system is the state interval of the link that has control input applied on. This state interval becomes the major consideration in this paper.

Therefore, this research paper aims to compare the state interval of the link that has control input applied on between the minimum energy and minimum jerk by using the optimization method so that conclusion can be applied.

II. Problem Statement

Dynamic systems can be described as the first order derivative function of state as

$$\dot{x}_i = f_i(x_1,...,x_n,u_1,...u_m,t); \quad i = 1,...,n \,, \tag{1}$$

where $x \in R^n$, $u \in R^m$ and t are state, control input, and time respectively, [5]. The problem of interest is to find the states $x(t)$ and control inputs $u(t)$ that make our system operates according to the desired objective of minimum energy or minimum jerk. Note that this paper is focusing on the system with fixed end time and fixed end points. Therefore, states and control inputs that serve the necessary condition must also be able to bring the system from initial conditions $x(t_0)$ at initial time t_0 to the end point $x(t_f)$ at time t_f.

The optimization problem of minimum energy will take the form of

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$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} u_i^2 dt , \tag{2}$$

where u_i is the control input, which can be force or torque applied to the system, and $i = 1,...,m$. J is the cost function of the energy consumed by the system from initial time t_0 to end time t_f.

The same kind of concept is used to the minimum jerk problem. It is well known that jerk is the change of input force with respect to time. It is, thus, the third derivative with respect to time of x, or first order derivative of control input u. Therefore,

$$Jerk = \dddot{x} \propto \dot{u} \ . \tag{3}$$

Defining

$$\dot{u} = \tilde{u} , \tag{4}$$

so that (1) becomes

$$\dot{x}_i = f_i(x_1,...,x_{n+m},\tilde{u}_1,...\tilde{u}_m,t); \quad i = 1,...,n+m \tag{5}$$

From now on, $\tilde{u}$ is treated as a variable and as the control input of our dynamic system. Consequently, (2) can be rewritten for the objective function of the minimum indirect jerk problem as

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} \tilde{u}_i^2 dt . \tag{6}$$

Similarly, (2) also can be rewritten for the objective function of the minimum direct jerk problem as

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{n} \dddot{x}_i^2 dt . \tag{7}$$

This time, J is the cost function of the jerks.

III. Necessary Conditions

In this paper, we use the calculus of variations in solving for the extremal solutions of the dynamic system, [1]. Representing the control input with u, the principle of calculus of variations helps us solve the optimization problem by finding the time history of the control input that would minimize the cost function of the form

$$J = \phi(t,x_1,...,x_n)_{t_f} + \int_{t_0}^{t_f} L(t,x_1,...,x_n,u_1,...,u_m)\, dt \tag{8}$$

where

$$\phi(t,x_1,...,x_n)_{t_f} , \tag{9}$$

is the cost based on the final time and the final states of the system, and

$$\int_{t_i}^{t_f} L(t,x_1,...,x_n,u_1,...,u_m)\, dt , \tag{10}$$

is an integral cost dependent on the time history of the state and control variables. Since the cost of the final states would be equal in all feasible time histories of the control input; therefore, the first term of (8) is omitted.

To find the extremum of the function, the dynamic equations are augmented via Lagrange Multipliers to the cost functional as follow:

$$J'(x_1,...,x_n,u_1,...,u_m) = \int_{t_i}^{t_f} L'(t,x_1,...x_n,u_1,...u_m)dt. \tag{11}$$

Where

$$L'(t,x_1,...,x_n,u_1,...,u_m) = L + \sum_{i=1}^{n} \lambda_i(f_i) , \tag{12}$$

and $\lambda_i(t)$ are Lagrange multipliers. Consequently, (11) becomes:

$$J'(x_1,...,x_n,u_1,...,u_m) = \int_{t_i}^{t_f} [L(t,x_1,...x_n,u_1,...u_m) + \sum_{i=1}^{n} \lambda_i(t)[\dot{x}_i - f_i(t,x_1,...,x_n,u_1,...,u_m)]]dt \tag{13}$$

Since the problem with fixed end time and end points are considered, initial time t_0, end time t_f, initial state $x(t_0)$, and final state $x(t_f)$ must be set prior to solving the problem. The differentiable functions are dependent on the boundary condition of $x(t_0)= x_0$, $x(t_f)= x_f$, $u(t_0)= u_0$ and $u(t_f)= u_f$ where time used falls in the interval $t_i \le t \le t_f$.

Let function $L(t,x_1,...,x_n,u_1,...,u_m,\dot{x}_1,...,\dot{x}_n)$ be represented as a functional

$$J[x_1,...,x_n,u_1,...,u_m] = \int_{t_0}^{t_f} L(t,x_1,...,x_n,u_1,...,u_m,\dot{x}_1,...,\dot{x}_n)dt \tag{14}$$

Let $x(t_0)$ be incremented by $h_{xj}(t_0)$, $u(t_0)$ be incremented by $h_{uk}(t_0)$, and still satisfy the boundary conditions, then $h_{xj}(t_0)$ $= h_{xj}(t_f)$ $= h_{uk}(t_0)$ $= h_{uk}(t_f)$ $= 0$. So, the change in functional ΔJ will be

$$\Delta J = J[x_1 + h_{x1},...,x_n + h_{xj},u_1 + h_{u1},...,u_m + h_{uk}] - J[x_1,...,x_n,u_1,...,u_m]$$

$$= \int_{t_i}^{t_f} \left[L\begin{pmatrix} t,x_1 + h_{x1},...,x_n + h_{xj},\dot{x}_1 + \dot{h}_{x1},...,\dot{x}_n + \dot{h}_{xj}, \\ u_1 + h_{u1},...,u_m + h_{uk},\dot{u}_1 + \dot{h}_{u1},...,\dot{u}_m + \dot{h}_{uk} \end{pmatrix} - L(t,x_1,...,x_n,\dot{x}_1,...,\dot{x}_n,u_1,...,u_m,\dot{u}_1,...,\dot{u}_m) \right] dt \tag{15}$$

Applying Taylor's Series to (15), disregard the higher order terms, and apply it to the problem results in

$$\delta J' = \int_{t_i}^{t_f} \sum_{j=1}^{n} \left(\frac{\partial L'}{\partial x_j} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{x}_j}\right) h_{x_j}\, dt$$
$$+ \int_{t_i}^{t_f} \sum_{k=1}^{m} \left(\frac{\partial L'}{\partial u_k} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{u}_k}\right) h_{u_k}\, dt$$
$$+ \sum_{k=1}^{m} \left(\frac{\partial L'}{\partial \dot{u}_k} h_{u_k}|_{t_f} - \frac{\partial L'}{\partial \dot{u}_k} h_{u_k}|_{t_i}\right)$$
$$+ \sum_{j=1}^{m} \left(\frac{\partial L'}{\partial \dot{x}_j} h_{x_j}|_{t_f} - \frac{\partial L'}{\partial \dot{x}_j} h_{x_j}|_{t_i}\right) \tag{16}$$

Since $h_{x_j}|_{t_f} = h_{x_j}|_{t_i} = 0$ and $\frac{\partial L'}{\partial \dot{u}_k} = 0$, the last two terms of (16) become zero. In order that the cost functional of jerk in (13) can be solved for minimal solution, the condition that make $\delta J' = 0$ at arbitrary variation of h_{x_j} and h_{u_k} are needed. From (16), obviously the mentioned conditions are as follow:

$$\frac{\partial L'}{\partial x_j} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{x}_j} = 0, \tag{17}$$

and

$$\frac{\partial L'}{\partial u_k} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{u}_k} = 0, \tag{18}$$

for $j = 1, \ldots, n$ and $k = 1, \ldots, m$.

Equations (17) and (18) are the necessary conditions that will lead to solve for Lagrange multipliers $\lambda_j(t)$, and control inputs $u_k(t)$. Alternatively, we can use the derived relationship below to solve for the unknowns necessary conditions:

For

$$\dot{x}_i = f_i(x_1, \ldots, x_n, u_1, \ldots u_m, t), \quad i = 1, \ldots, n \tag{19}$$

Necessary conditions are (19) and

$$\dot{\lambda}_j = -\frac{\partial L}{\partial x_j} - \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial x_j}, \quad j = 1, \ldots, n, \tag{20}$$

$$\frac{\partial L}{\partial u_k} + \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial u_k} = 0, \quad k = 1, \ldots, m. \tag{21}$$

As of above the necessary conditions are in the form of differential and algebraic equations which are known as two-point boundary valued problem, [2].

IV. EXAMPLES

The procedure outlined in this paper for dynamic optimization is illustrated with the following example of a two degree-of-freedom spring-mass-damper system sketched in equation as

$$A\dot{x} = Bu \tag{22}$$

The matrices A and B for this system is as follows:

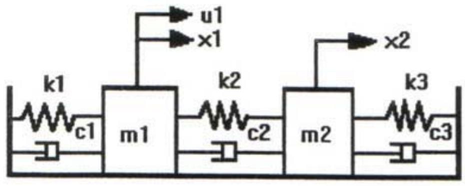

Fig. 1 Two degree of-freedom of spring mass and damper system

$$A = \begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I_2 & 0 \end{bmatrix} \tag{23}$$

$$B = \begin{bmatrix} \frac{1}{m_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{24}$$

where the matrices M, C and K are:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \tag{25}$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \tag{26}$$

The equation (22) can also be rewritten in the second order differential equation according to the second law of Newton. The parameters used in the model in MKS units are: $m_1 = m_2 = 1.0$, $c_1 = c_3 = 1.0$, $c_2 = 2.0$, $k_1 = k_2 = k_3 = 3.0$. The boundary conditions are $x(t_0) = (0\ 0\ 10\ 20)^T$ and $x(t_f) = (0\ 0\ 0\ 0)^T$, where $t_0 = 0$ and $t_f = 1.0$

A. Minimum Energy

The cost function of minimum energy is defined as

$$J = \int_0^1 u_1^2\, dt. \tag{27}$$

In order for the cost function in (27) to be minimized, the Calculus of Variations as stated in previous section has been used.

B. Minimum Jerk

The cost function of minimum jerk is also defined as

$$J = \int_0^1 \tilde{u}_1^2 \, dt \,. \tag{28}$$

Similarly for (28) to be minimized, the Calculus of Variations must be applied here.

C. Numerical Results

The minimum jerk problem has the exact same format as the minimum energy problem in (2). However, since the time derivative of control inputs are considered, the (22) must be rewritten as to include the consideration of jerk into the system:

$$\begin{aligned} &\dddot{x}_1 + 3\ddot{x}_1 - 2\ddot{x}_2 + 6\dot{x}_1 - 3\dot{x}_2 = \frac{du_1}{dt} = \tilde{u}_1 \\ &\dddot{x}_2 - 2\ddot{x}_1 + 3\ddot{x}_2 - 3\dot{x}_1 + 6\dot{x}_2 = 0. \end{aligned} \tag{29}$$

Therefore, the extra boundary conditions can be applied at both ends are assigned to be free. These conditions can be applied in the numerical scheme through the original dynamic equations as follow:

$$\begin{aligned} &\ddot{x}_1 + 3\dot{x}_1 - 2\dot{x}_2 + 6x_1 - 3x_2 = u_1 \\ &\ddot{x}_2 - 2\dot{x}_1 + 3\dot{x}_2 - 3x_1 + 6x_2 = 0. \end{aligned} \tag{30}$$

By using software developed by Tawiwat Veeraklaew, [6], the problems of minimum direct and indirect jerks can be solved to obtain the optimal solutions. The idea behind this software is to transform the necessary conditions of the dynamic optimization to static optimization. Then one kind of the well known methods called nonlinear programming or linear programming has been used to solve for all parameters that are parameterized through collocation technique. The comparison for each variable such as state and control variables of the dynamic systems in this example are shown in figure below as Fig. 2 to Fig. 5.

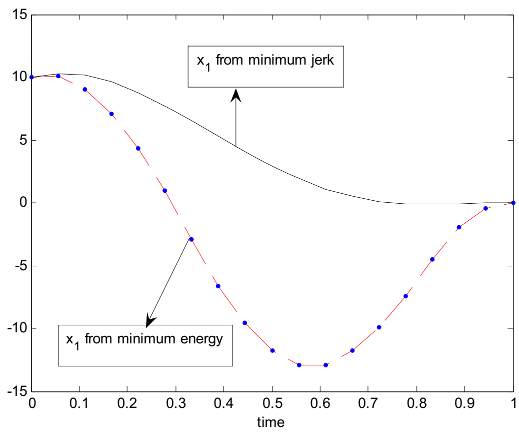

Fig. 2 Solutions of the first state variables from minimum jerk and energy

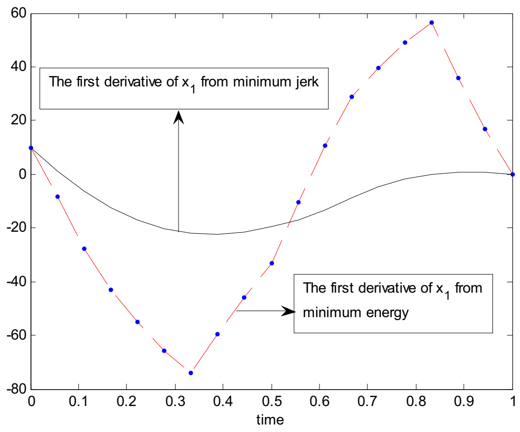

Fig. 3 Solutions of the first derivative of the first state variables from minimum jerk and energy

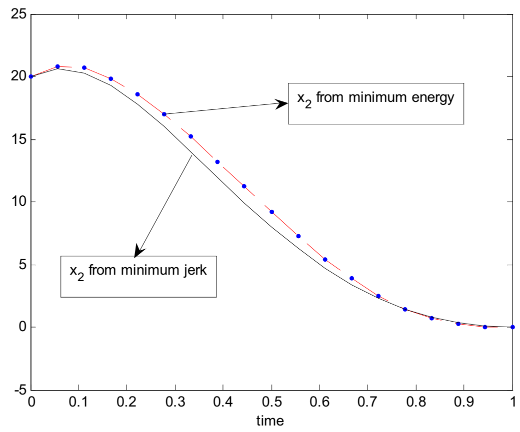

Fig. 4 Solutions of the second state variables from minimum jerk and energy

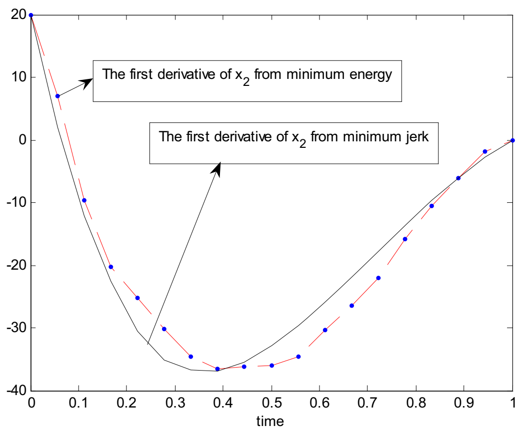

Fig. 5 Solutions of the first derivative of the second state variables from minimum jerk and energy

V. Conclusion

From the solutions in the previous section, $x_1(t)$, $\dot{x}_1(t)$, $x_2(t)$ and $\dot{x}_2(t)$ from both minimum jerk and energy have quite different especially the solution of $x_1(t)$. The state interval (gap between minimum and maximum values) of the link that has control input applied on it are 10 units and 22.922 units while the cost values are 5.7×10^4 and 6×10^5 for the minimum jerk and minimum energy problems, respectively.

The above results can be concluded that applying minimum jerk has much less gap for the control. This result makes more flexible in order to design some dynamic system that has similar situation as under actuator dynamic system.

The results in this paper show that the minimum jerk can be used instead of minimum energy strongly if the energy is not the first priority issue. Moreover, the result can be concluded for the linear dynamic systems. However, the under actuator nonlinear dynamic problems could be used to compare for the future work which very high expectation that both problems will have the similar results.

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