

A neighborhood condition for fractional k -deleted graphs

Sizhong Zhou, Hongxia Liu

Abstract—Let $k \geq 3$ be an integer, and let G be a graph of order n with $n \geq 9k + 3 - 4\sqrt{2(k-1)^2 + 2}$. Then a spanning subgraph F of G is called a k -factor if $d_F(x) = k$ for each $x \in V(G)$. A fractional k -factor is a way of assigning weights to the edges of a graph G (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is k . A graph G is a fractional k -deleted graph if there exists a fractional k -factor after deleting any edge of G . In this paper, it is proved that G is a fractional k -deleted graph if G satisfies $\delta(G) \geq k + 1$ and $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$ for each pair of nonadjacent vertices x, y of G .

Keywords—graph, minimum degree, neighborhood union, fractional k -factor, fractional k -deleted graph.

I. INTRODUCTION

IN this paper, we consider only finite undirected graphs without loops or multiple edges. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set of vertices adjacent to x in G , and $N_G[x]$ for $N_G(x) \cup \{x\}$. For any $S \subseteq V(G)$, $N_G(S) = \bigcup_{x \in S} N_G(x)$ and we denote by $G[S]$ the subgraph of G induced by S , and $G - S = G[V(G) \setminus S]$. We say that S is independent if $N_G(S) \cap S = \emptyset$. Let S and T be disjoint subsets of $V(G)$. We use $e_G(S, T)$ to denote the number of edges joining S and T in G . The minimum vertex degree of G is denoted by $\delta(G)$.

Let k be a positive integer. Then a spanning subgraph F of G is called a k -factor if $d_F(x) = k$ for each $x \in V(G)$. If $k = 1$, then a k -factor is simply called a 1-factor. A fractional k -factor is a way of assigning weights to the edges of a graph G (with all weights between 0 and 1) such that for each vertex the sum of the weights of the edges incident with that vertex is k . If $k = 1$, then a fractional k -factor is a fractional 1-factor. A graph G is a fractional k -deleted graph if there exists a fractional k -factor after deleting any edge of G . If $k = 1$, then a fractional k -deleted graph is a fractional 1-deleted graph. Some other terminologies and notations can be found in [1,2].

Many authors have studied graph factors [3-8]. Many authors have investigated fractional k -factors [9-12] and fractional k -deleted graphs [13,14]. The following results on k -factors, fractional k -factors and fractional k -deleted graphs are known.

Sizhong Zhou is with the School of Mathematics and Physics, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang, Jiangsu 212003, People's Republic of China, e-mail: zsz_cumt@163.com.

Hongxia Liu is with the School of Mathematics and Informational Science, Yantai University, Yantai, Shandong 264005, People's Republic of China, e-mail: mqy7174@sina.com

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Theorem 1^[15] Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, kn is even, and the minimum degree is at least k . If G satisfies $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a k -factor.

Theorem 2^[11] Let k be an integer such that $k \geq 2$, and let G be a connected graph of order n such that $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2}$, and the minimum degree $\delta(G) \geq k$. If $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G has a fractional k -factor.

Theorem 3^[16] Let $k \geq 2$ be an integer. Let G be a connected graph of order n with $n \geq 13k + 1 - 4\sqrt{2(k-1)^2 + 2}$, $\delta(G) \geq k + 2$. If $|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$ for each pair of nonadjacent vertices x, y of G , then G is a fractional k -deleted graph.

The purpose of this paper is to weaken the conditions on the order, minimum degree and connectivity of G in Theorem 3. The main result is the following theorem.

Theorem 4 Let $k \geq 3$ be an integer. Let G be a graph of order n with $n \geq 9k + 3 - 4\sqrt{2(k-1)^2 + 2}$, $\delta(G) \geq k + 1$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$$

for each pair of nonadjacent vertices x, y of G , then G is a fractional k -deleted graph.

II. THE PROOF OF THEOREM 4

The following result is essential to the proof of our main theorem.

Lemma 2.1^[17] A graph G is a fractional k -deleted graph if and only if for any $S \subseteq V(G)$ and $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$

$$\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq \varepsilon(S, T),$$

where $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$ and $\varepsilon(S, T)$ is defined as follows,

$$\varepsilon(S, T) = \begin{cases} 2, & \text{if } T \text{ is not independent,} \\ 1, & \text{if } T \text{ is independent, and} \\ & e_G(T, V(G) \setminus (S \cup T)) \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Theorem 4. Let G be a graph satisfying the hypothesis of Theorem 4, we prove the theorem by contradiction.

Suppose that G is not a fractional k -deleted graph. Then by Lemma 2.1, there exists a subset S of $V(G)$ such that

$$\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \leq \varepsilon(S, T) - 1, \quad (1)$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k\}$. Firstly, we prove the following claims.

Claim 1. $S \neq \emptyset$.

Proof. Note that $\varepsilon(S, T) \leq |T|$. If $S = \emptyset$, then by (1) we have

$$\begin{aligned} \varepsilon(S, T) - 1 &\geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \\ &= d_G(T) - k|T| \geq (\delta(G) - k)|T| \\ &\geq |T| \geq \varepsilon(S, T). \end{aligned}$$

It is a contradiction. This completes the proof of Claim 1.

Claim 2. $|T| \geq k + 1$.

Proof. Assume that $|T| \leq k$. Then from (1) and $|S| + d_{G-S}(x) - k \geq d_G(x) - k \geq \delta(G) - k \geq 1$, we get

$$\begin{aligned} \varepsilon(S, T) - 1 &\geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \\ &\geq |T||S| + d_{G-S}(T) - k|T| \\ &= \sum_{x \in T} (|S| + d_{G-S}(x) - k) \\ &\geq |T| \geq \varepsilon(S, T). \end{aligned}$$

That is a contradiction. This completes the proof of Claim 2.

Claim 3. $|T| \geq |S| + 1$.

Proof. Let $|T| \leq |S|$. Then by (1), we obtain

$$\varepsilon(S, T) - 1 \geq k|S| + d_{G-S}(T) - k|T| \geq d_{G-S}(T). \quad (2)$$

On the other hand, according to the definition of $\varepsilon(S, T)$, we have

$$d_{G-S}(T) \geq \varepsilon(S, T),$$

which contradicts (2). The proof of Claim 3 is complete.

Claim 4. $|S| \leq \frac{n-1}{2}$.

Proof. In terms of Claim 3 and $|S| + |T| \leq n$, we have

$$n \geq |S| + |T| \geq 2|S| + 1,$$

that is,

$$|S| \leq \frac{n-1}{2}.$$

The proof of Claim 4 is complete.

In terms of Claim 2, $T \neq \emptyset$. Now we define

$$h_1 = \min\{d_{G-S}(x) : x \in T\}$$

and choose $x_1 \in T$ such that $d_{G-S}(x_1) = h_1$. Clearly, we have $0 \leq h_1 \leq k$. In the following, we consider two cases.

Case 1. $T = N_T[x_1]$.

Using Claim 2, $T = N_T[x_1]$ and $0 \leq h_1 \leq k$, we obtain

$$k \geq h_1 = d_{G-S}(x_1) \geq |T| - 1 \geq k,$$

which implies

$$h_1 = k. \quad (3)$$

In terms of (3) and Claim 1, we get

$$\begin{aligned} \delta_G(S, T) &= k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| + h_1|T| - k|T| = k|S| \\ &\geq k > 2 \geq \varepsilon(S, T). \end{aligned}$$

That contradicts (1).

Case 2. $T \setminus N_T[x_1] \neq \emptyset$.

Define

$$h_2 = \min\{d_{G-S}(x) : x \in T \setminus N_T[x_1]\}.$$

We choose $x_2 \in T \setminus N_T[x_1]$ such that $d_{G-S}(x_2) = h_2$. Obviously, $0 \leq h_1 \leq h_2 \leq k$ and $x_1 x_2 \notin E(G)$. According to the hypothesis of Theorem 4, we have

$$\begin{aligned} \frac{n+k-2}{2} &\leq |N_G(x_1) \cup N_G(x_2)| \\ &\leq d_{G-S}(x_1) + d_{G-S}(x_2) + |S| \\ &= h_1 + h_2 + |S|, \end{aligned}$$

which implies

$$|S| \geq \frac{n+k-2}{2} - h_1 - h_2. \quad (4)$$

By (4) and Claim 4, we obtain

$$\frac{n-1}{2} \geq \frac{n+k-2}{2} - h_1 - h_2,$$

that is,

$$h_1 + h_2 \geq \frac{k-1}{2}. \quad (5)$$

In terms of (5), $k \geq 3$, $0 \leq h_1 \leq h_2 \leq k$ and the integrity of h_2 , we get

$$h_2 \geq 1. \quad (6)$$

Claim 5. $0 \leq h_1 \leq k-1$.

Proof. If $h_1 = k$, then by (1) and Claim 1 we get

$$\begin{aligned} \varepsilon(S, T) - 1 &\geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| + h_1|T| - k|T| = k|S| \geq k \\ &> 2 \geq \varepsilon(S, T), \end{aligned}$$

which is a contradiction. This completes the proof of Claim 5.

Note that

$$|N_T[x_1]| \leq d_{G-S}(x_1) + 1 = h_1 + 1. \quad (7)$$

From (4), (7), $0 \leq h_1 \leq h_2 \leq k$ and $|S| + |T| \leq n$, we have

$$\begin{aligned} \delta_G(S, T) &= k|S| + d_{G-S}(T) - k|T| \\ &\geq k|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) \\ &\quad - k|T| \\ &= k|S| - (h_2 - h_1)|N_T[x_1]| - (k - h_2)|T| \\ &\geq k|S| - (h_2 - h_1)(h_1 + 1) \\ &\quad - (k - h_2)(n - |S|) \\ &= (2k - h_2)|S| - (h_2 - h_1)(h_1 + 1) \\ &\quad - (k - h_2)n \\ &\geq (2k - h_2)\left(\frac{n+k-2}{2} - h_1 - h_2\right) \\ &\quad - (h_2 - h_1)(h_1 + 1) - (k - h_2)n, \end{aligned}$$

that is,

$$\delta_G(S, T) \geq (2k - h_2)\left(\frac{n + k - 2}{2} - h_1 - h_2\right) - (h_2 - h_1)(h_1 + 1) - (k - h_2)n. \quad (8)$$

Let $F(h_1, h_2) = (2k - h_2)\left(\frac{n + k - 2}{2} - h_1 - h_2\right) - (h_2 - h_1)(h_1 + 1) - (k - h_2)n$. Then by Claim 5, we have

$$\begin{aligned} F'_{h_1}(h_1, h_2) &= -(2k - h_2) + (h_1 + 1) - (h_2 - h_1) \\ &= 2h_1 - 2k + 1 \leq 2(k - 1) - 2k + 1 \\ &= -1 < 0. \end{aligned}$$

Combining this with $h_1 \leq h_2$, we obtain

$$F(h_1, h_2) \geq F(h_2, h_2). \quad (9)$$

Using (8) and (9), we get

$$\delta_G(S, T) \geq (2k - h_2)\left(\frac{n + k - 2}{2} - 2h_2\right) - (k - h_2)n. \quad (10)$$

According to (1), (10) and $\varepsilon(S, T) \leq 2$, we get

$$\begin{aligned} 1 &\geq \varepsilon(S, T) - 1 \geq \delta_G(S, T) \\ &\geq (2k - h_2)\left(\frac{n + k - 2}{2} - 2h_2\right) - (k - h_2)n \\ &= \frac{1}{2}(4h_2^2 + (n - 9k + 2)h_2 + 2k^2 - 4k), \end{aligned}$$

which implies

$$4h_2^2 + (n - 9k + 2)h_2 + 2k^2 - 4k - 2 \leq 0. \quad (11)$$

Claim 6. For $k \geq 3$, we have $\sqrt{\frac{(k-1)^2+1}{2}} - 1 > \frac{1}{2}$.

Proof. Since $k \geq 3$, we have

$$\frac{(k-1)^2+1}{2} \geq \frac{5}{2} > \frac{9}{4},$$

that is,

$$\sqrt{\frac{(k-1)^2+1}{2}} > \frac{3}{2}.$$

Thus, we obtain

$$\sqrt{\frac{(k-1)^2+1}{2}} - 1 > \frac{1}{2}.$$

The proof of Claim 6 is complete.

According to (6), (11), $n \geq 9k + 3 - 4\sqrt{2(k-1)^2+2}$, $k \geq 3$ and Claim 6, we obtain

$$\begin{aligned} 0 &\geq 4h_2^2 + (n - 9k + 2)h_2 + 2k^2 - 4k - 2 \\ &\geq 4h_2^2 + (-4\sqrt{2(k-1)^2+2} + 5)h_2 + 2k^2 - 4k - 2 \\ &\geq 4h_2^2 - 8\sqrt{\frac{(k-1)^2+1}{2}}h_2 + 2(k-1)^2 + 2 + 5h_2 - 6 \\ &= 4\left(\sqrt{\frac{(k-1)^2+1}{2}} - h_2\right)^2 + 5h_2 - 6 \\ &\geq 4\left(\sqrt{\frac{(k-1)^2+1}{2}} - 1\right)^2 - 1 \\ &> 4\left(\frac{1}{2}\right)^2 - 1 \geq 0, \end{aligned}$$

which is a contradiction.

From all the cases above, we deduce the contradictions. Hence, G is a fractional k -deleted graph. This completes the proof of Theorem 4.

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