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Physical conserved quantities for the axisymmetric liquid, free and wall jets

Rehana Naz, D P Mason and Fazal Mahomed

***Abstract*—A systematic way to derive the conserved quantities for the axisymmetric liquid jet, free jet and wall jet using conservation laws is presented. The flow in axisymmetric jets is governed by Prandtl's momentum boundary layer equation and the continuity equation. The multiplier approach is used to construct a basis of conserved vectors for the system of two partial differential equations for the two velocity components. The basis consists of two conserved vectors. By integrating the corresponding conservation laws across the jet and imposing the boundary conditions, conserved quantities are derived for the axisymmetric liquid and free jet. The multiplier approach applied to the third-order partial differential equation for the stream function yields two local conserved vectors one of which is a non-local conserved vector for the system. One of the conserved vectors gives the conserved quantity for the axisymmetric free jet but the conserved quantity for the wall jet is not obtained from the second conserved vector. The conserved quantity for the axisymmetric wall jet is derived from a non-local conserved vector of the third-order partial differential equation for the stream function. This non-local conserved vector for the third-order partial differential equation for the stream function is obtained by using the stream function as multiplier.

***Keywords*—Axisymmetric jet,liquid jet, free jet, wall jet, conservation laws, conserved quantity

I. Introduction

Conserved quantities are required when solving boundary value problems for jets. The conserved quantities have been established by integrating the momentum boundary layer equation across the jet. The boundary conditions and continuity equation are used in the derivation of the conserved quantity. Recently, a new method of deriving conserved quantities was introduced by Naz, Mason and Mahomed [1] by which conserved quantities are constructed with the help of conservation laws. Goldstein [2] established the conserved quantity for an axisymmetric free jet by integrating the momentum equation across the jet. Later, Duck and Bodoyni [3] established the conserved quantity for the axisymmetric wall jet by the same procedure. The conserved quantity for the axisymmetric liquid jets was obtained from a physical argument based on conservation of volume flux in an incompressible fluid.

The conserved quantity not only determines the unknown exponent in the similarity solution for the homogeneous boundary value problems of jets but also it is a measure of the strength of the jet. The conserved quantity for an axisymmetric liquid jet is the volume flux. Goldstein [2] showed that the conserved quantity for the axisymmetric free jet is the total momentum flux in the downstream direction. This flux is constant along the jet. For the wall jet on an axisymmetric body, the conservation integral involves a balance between the viscous and inertial forces of the jet (see [3]). There is no physical interpretation of this conservation integral.

In this paper, we construct the conserved quantities for the axisymmetric liquid, free and wall jets using the approach introduced in [1]. The conserved quantities for the axisymmetric liquid, free and wall jets are derived using conservation laws. Mason and Ruscic [4] discussed the elementary conservation law for the third-order partial differential equation for the stream function for an axisymmetric free jet. These authors wrote that conservation law from inspection of the momentum equation. To the best of our knowledge the conservation laws for the system of equations for the velocity components and the third-order partial differential equation for the stream function have not been computed. We will construct the conservation laws for the system of two partial differential equations for the velocity components as well as for the third-order partial differential equation for the stream function for the axisymmetric jets using the multiplier approach (see for example [5], [6], [7], [8]).

II. Conservation laws for system of equations for axisymmetric jet flows

We consider the axisymmetric liquid jet, free jet and wall jet. An axisymmetric free jet is formed when fluid emerges from a small circular orifice in a wall into the surrounding fluid which is the same fluid as the jet ([2]). An axisymmetric wall jet and liquid jet on an axisymmetric body is formed when an axisymmetric jet is incident on the surface of the body along the axis of symmetry and spreads out over the surface. For the wall jet the surrounding fluid is the same as the jet ([3]) but for the liquid jet the surrounding fluid is a gas.

The fluid of the jet is viscous and incompressible. There is no slip or suction or blowing at the solid boundary for the axisymmetric liquid and wall jets. Cylindrical polar coordinates (z, r, θ) are used. The $z-$axis is along the axis of the jet and all the fluid variables are independent of θ. For the free jet the origin of the coordinate system is at the orifice. For the wall and liquid jet on a circular cylinder the origin of the coordinate system is at an arbitrary point.

Prandtl's boundary layer equations governing the steady flow in the axisymmetric liquid, free and wall jets, in the absence of a pressure gradient, are

$$uu_z + vu_r = \nu(u_{rr} + \frac{u_r}{r}), \tag{1}$$

$$(ru)_z + (rv)_r = 0, \tag{2}$$

rehananaz_qau@yahoo.com

where $u(z,r)$ and $v(z,r)$ are the velocity components in the z and r directions, respectively and ν is the kinematic viscosity of the fluid.

A. Stream function

The stream function for a flow with an axis of symmetry is described by Batchelor [9]. From (2), $rudr - rvdz$ is a perfect differential, $d\psi$:

$$d\psi = ru(z,r)dr - rv(z,r)dz. \tag{3}$$

Choose any point $O(z_o, \theta_o, r_o)$ as the reference point on the line $r = r_o$ in the axial plane $\theta = \theta_o$. For the free jet $r_o = 0$ and the line is the axis of symmetry. For the wall and liquid jet $r_o = a$ where a is the radius of the cylinder and the line is a generator of the cylinder on the surface. The stream function $\psi(z,r)$ at any point $P(z, \theta_o, r)$ in the axial plane is obtained by integrating $d\psi$ along any curve in the axial plane joining O and P:

$$\psi(z,r) - \psi(z_o, r_o) = \int_O^P r\left(u(z,r)dr - v(z,r)dz\right). \tag{4}$$

Equation (3) yields

$$u = \frac{1}{r}\psi_r,\ v = -\frac{1}{r}\psi_z. \tag{5}$$

The axisymmetric free jet is symmetrical about the plane $r = r_o$ and for the liquid jet and wall jet there is no suction or blowing of fluid at the solid boundary of the cylinder $r = r_o = a$. Thus for all three jets, $v(z, r_o) = 0$ and therefore $\psi_z(z, r_o) = 0$. Thus $\psi(z, r_o) = \psi_o$ where ψ_o is a constant which we choose to be zero and (4) reduces to

$$\psi(z,r) = \int_O^P r\left(u(z,r)dr - v(z,r)dz\right), \tag{6}$$

where $r_o = 0$ for free jet and $r_o = a$ for liquid and wall jets. Integrating (6) with respect to r with z kept fixed from $r = r_o$ to $r = \infty$ gives

$$\psi(z,\infty) = \int_{r_o}^{\infty} ru(z,r)dr. \tag{7}$$

We assume that $u(z,r) \to 0$ sufficiently rapidly as $r \to \infty$ to ensure that the integral in (7) exists. Thus $\psi(z,\infty)$ is finite.

We substitute (5) into (1)-(2). Equation (2) is identically satisfied while (1) gives rise to a third-order partial differential equation for the stream function ψ:

$$\frac{1}{r}\psi_r\psi_{rz} + \frac{1}{r^2}\psi_z\psi_r - \frac{1}{r}\psi_z\psi_{rr}$$
$$-\nu(\psi_{rrr} - \frac{1}{r}\psi_{rr} + \frac{1}{r^2}\psi_r) = 0. \tag{8}$$

We derive a basis of conservation laws for the system (1)-(2) as well as for the third-order partial differential equation (8), using the multiplier approach ([5], [6], [7], [8]).

B. Conservation laws for system of equations for velocity components

Multipliers Λ_1 and Λ_2, for all functions $u(z,r)$ and $v(z,r)$, not only for the solutions of (1)-(2), satisfy

$$\Lambda_1\left[uu_z + vu_r - \nu(u_{rr} + \frac{u_r}{r})\right] + \Lambda_2\left[ru_z + rv_r + v\right] = D_zT^1 + D_rT^2, \tag{9}$$

where D_z and D_r, defined by

$$D_z = \frac{\partial}{\partial z} + u_z\frac{\partial}{\partial u} + v_z\frac{\partial}{\partial v} + u_{zz}\frac{\partial}{\partial u_z} + v_{zz}\frac{\partial}{\partial v_z} + u_{zr}\frac{\partial}{\partial u_r} + v_{zr}\frac{\partial}{\partial v_r} + \cdots, \tag{10}$$

$$D_r = \frac{\partial}{\partial r} + u_r\frac{\partial}{\partial u} + v_r\frac{\partial}{\partial v} + u_{rr}\frac{\partial}{\partial u_r} + v_{rr}\frac{\partial}{\partial v_r} + u_{rz}\frac{\partial}{\partial u_z} + v_{rz}\frac{\partial}{\partial v_z} + \cdots, \tag{11}$$

are the total derivative operators. The two determining equations for the multipliers are obtained by applying standard Euler operators E_u and E_v on (9). The Euler operators E_u and E_v are defined by

$$E_u = \frac{\partial}{\partial u} - D_z\frac{\partial}{\partial u_z} - D_r\frac{\partial}{\partial u_r} + D_z^2\frac{\partial}{\partial u_{zz}} + D_zD_r\frac{\partial}{\partial u_{zr}} + D_r^2\frac{\partial}{\partial u_{rr}} - \cdots, \tag{12}$$

and

$$E_v = \frac{\partial}{\partial v} - D_z\frac{\partial}{\partial v_z} - D_r\frac{\partial}{\partial v_r} + D_z^2\frac{\partial}{\partial v_{zz}} + D_zD_r\frac{\partial}{\partial v_{zr}} + D_r^2\frac{\partial}{\partial v_{rr}} - \cdots. \tag{13}$$

Thus the determining equations for multipliers of the form $\Lambda_1 = \Lambda_1(z,r,u,v)$ and $\Lambda_2 = \Lambda_2(z,r,u,v)$ are

$$E_u[\Lambda_1(uu_z + vu_r - \nu u_{rr} - \frac{\nu u_r}{r}) + \Lambda_2(ru_z + rv_r + v)] = 0, \tag{14}$$

$$E_v[\Lambda_1(uu_z + vu_r - \nu u_{rr} - \frac{\nu u_r}{r}) + \Lambda_2(ru_z + rv_r + v)] = 0. \tag{15}$$

After expansion, (14) and (15) become

$$\Lambda_{1u}\Big(uu_z + vu_r - \nu u_{rr} - \frac{\nu u_r}{r}\Big) + \Lambda_{2u}(ru_z + rv_r + v) + \Lambda_1 u_z - D_z\left[u\Lambda_1 + r\Lambda_2\right] - D_r\left[\Lambda_1(v - \frac{\nu}{r})\right] - \nu D_r^2\left(\Lambda_1\right) = 0, \tag{16}$$

$$\Lambda_{1v}\Big(uu_z + vu_r - \nu u_{rr} - \frac{\nu u_r}{r}\Big) + \Lambda_{2v}(ru_z + rv_r + v) - D_r\left(r\Lambda_2\right) + \Lambda_1 u_r + \Lambda_2 = 0, \tag{17}$$

which must be satisfied for all functions $u(z,r)$ and $v(z,r)$. Equating the coefficients of derivatives of u and v in (16) and (17) to zero yields an overdetemined system of equations for multipliers which finally gives

$$\Lambda_1 = c_2 r,\ \Lambda_2 = c_1 + c_2 u, \tag{18}$$

where c_1 and c_2 are constants.

Equation (9) together with (18) gives

$$c_2 r\Big(uu_z + vu_r - \nu u_{rr} - \frac{\nu u_r}{r}\Big) + (c_1 + c_2 u)\,(ru_z + rv_r + v)$$
$$= D_z\left[c_1 ru + c_2 ru^2\right] + D_r\left[c_1 rv + c_2 r(uv - \nu u_r)\right], \quad (19)$$

for arbitrary $u(z,r)$ and $v(z,r)$. When $u(z,r)$ and $v(z,r)$ are the solutions of the system of differential equations (1)-(2), then

$$D_z[c_1 ru + c_2 ru^2] + D_r[c_1 rv + c_2 r(uv - \nu u_r)] = 0. \quad (20)$$

Thus

$$T^1 = ru, \quad T^2 = rv, \quad (21)$$

$$T^1 = ru^2, \quad T^2 = r(uv - \nu u_r), \quad (22)$$

form a basis of conserved vectors for the system (1)-(2) with multipliers of the form $\Lambda_1(z,r,u,v)$ and $\Lambda_2(z,r,u,v)$.

C. Conservation laws for axisymmetric laminar jet flow in terms of stream function

A multiplier for the third-order partial differential equation (8) is a function Λ satisfying

$$\Lambda\left[\frac{1}{r}\psi_r\psi_{rz} + \frac{1}{r^2}\psi_z\psi_r - \frac{1}{r}\psi_z\psi_{rr} - \nu(\psi_{rrr} - \frac{1}{r}\psi_{rr} + \frac{1}{r^2}\psi_r)\right]$$
$$= D_z T^1 + D_r T^2, \quad (23)$$

for all functions $\psi(z,r)$. In (23), D_z and D_r are total derivative operators defined by

$$D_z = \frac{\partial}{\partial z} + \psi_z\frac{\partial}{\partial \psi} + \psi_{zz}\frac{\partial}{\partial \psi_z} + \psi_{zr}\frac{\partial}{\partial \psi_r} + \cdots, \quad (24)$$

$$D_r = \frac{\partial}{\partial r} + \psi_r\frac{\partial}{\partial \psi} + \psi_{rr}\frac{\partial}{\partial \psi_r} + \psi_{rz}\frac{\partial}{\partial \psi_z} + \cdots. \quad (25)$$

The determining equation for a multiplier of the form $\Lambda = \Lambda(z,r,\psi,\psi_z,\psi_r)$ is

$$E_\psi[\Lambda(\frac{1}{r}\psi_r\psi_{rz} + \frac{1}{r^2}\psi_z\psi_r - \frac{1}{r}\psi_z\psi_{rr}$$
$$-\nu\psi_{rrr} + \frac{\nu}{r}\psi_{rr} - \frac{\nu}{r^2}\psi_r)] = 0, \quad (26)$$

where

$$E_\psi = \frac{\partial}{\partial \psi} - D_z\frac{\partial}{\partial \psi_z} - D_r\frac{\partial}{\partial \psi_r}$$
$$+D_z^2\frac{\partial}{\partial \psi_{zz}} + D_z D_r\frac{\partial}{\partial \psi_{zr}} + D_r^2\frac{\partial}{\partial \psi_{rr}} - \cdots, \quad (27)$$

is the standard Euler operator. Expanding equation (26) and separating according to derivatives of ψ, we finally obtain

$$\Lambda = c_3 + c_4\left(\psi - \nu z\right), \quad (28)$$

where c_3 and c_4 are constants.

The substitution of the multiplier Λ from (28) in (23) gives

$$D_z\left[c_3\left(\frac{1}{r}\psi_r^2\right) + c_4\frac{1}{r}\psi_r^2\left(\psi - \nu z\right)\right]$$
$$+D_r\left[c_3\left(-\frac{1}{r}\psi_z\psi_r - \nu\psi_{rr} + \frac{\nu}{r}\psi_r\right)\right.$$
$$\left.+c_4(\frac{\nu}{2}\psi_r^2 + (\frac{\nu}{r}\psi_r - \nu\psi_{rr} - \frac{1}{r}\psi_z\psi_r)(\psi - \nu z))\right] = 0, \quad (29)$$

whenever $\psi(z,r)$ is a solution of third-order partial differential equation (8). For each arbitrary constant in (29) we obtain a conserved vector. Thus

$$T^1 = \frac{1}{r}\psi_r^2,\ T^2 = -\frac{1}{r}\psi_z\psi_r - \nu\psi_{rr} + \frac{\nu}{r}\psi_r, \quad (30)$$

$$T^1 = \frac{1}{r}\psi_r^2(\psi - \nu z),$$

$$T^2 = \frac{\nu}{2}\psi_r^2 + \left(\frac{\nu}{r}\psi_r - \nu\psi_{rr} - \frac{1}{r}\psi_z\psi_r\right)(\psi - \nu z), \quad (31)$$

are the conserved vectors for third-order partial differential equation (8) with multipliers of the form $\Lambda = \Lambda(z,r,\psi,\psi_x,\psi_y)$. We obtained two local conservation laws for the third-order partial differential equation (8). The conserved vector (31) is a local conservation law for the third-order partial differential equation (8) but it is a non-local conservation law for the system (1)-(2).

The multiplier approach gives only those multipliers which yield local conservation laws. We cannot obtain the multipliers for non-local conservation laws using the multiplier approach. It is of interest to observe that we can derive a non-local conservation law for the third-order partial differential equation (8) by multiplying with ψ. If we multiply equation (8) by ψ, we obtain

$$\psi\left[\frac{1}{r}\psi_r\psi_{rz} + \frac{1}{r^2}\psi_z\psi_r - \frac{1}{r}\psi_z\psi_{rr} - \nu\left(\psi_{rrr} - \frac{1}{r}\psi_{rr} + \frac{1}{r^2}\psi_r\right)\right]$$

$$= D_z\left[\frac{1}{r}\psi\psi_r^2 - \nu\int^z\frac{1}{r}\psi_r^2 dz\right]$$

$$+D_r\left[\frac{\nu}{2}\psi_r^2 + \left(\frac{\nu}{r}\psi_r - \nu\psi_{rr} - \frac{1}{r}\psi_z\psi_r\right)\psi\right], \quad (32)$$

for arbitrary functions $\psi(z,r)$. When $\psi(z,r)$ is solution of partial differential equation (8), we have

$$D_z\left[\frac{1}{r}\psi\psi_r^2 - \nu\int^z\frac{1}{r}\psi_r^2 dz\right]$$

$$+D_r\left[\frac{\nu}{2}\psi_r^2 + \left(\frac{\nu}{r}\psi_r - \nu\psi_{rr} - \frac{1}{r}\psi_z\psi_r\right)\psi\right] = 0, \quad (33)$$

which yields

$$T^1 = \frac{1}{r}\psi\psi_r^2 - \nu\int^z\frac{1}{r}\psi_r^2 dz,$$

$$T^2 = \frac{\nu}{2}\psi_r^2 + \left(\frac{\nu}{r}\psi_r - \nu\psi_{rr} - \frac{1}{r}\psi_z\psi_r\right)\psi. \quad (34)$$

Thus we have obtained two local conserved vectors (30), (31) and one non-local conserved vector (34) for the third-order partial differential equation (8).

III. CONSERVED QUANTITIES FOR AXISYMMETRIC LIQUID, FREE AND WALL JETS

In this section we derive the conserved quantities for the axisymmetric liquid jet, free jet and wall jet by a new method. The conserved vectors (21) and (22) for the system (1)-(2) give the conserved quantities for the axisymmetric liquid and free jets. The conserved vectors for the partial differential equation (8) for the stream function equation are (30), (31) and (34). The conserved vector (30) is used to give an alternative derivation of the conserved quantity for the axisymmetric free jet and the conserved vector (34) gives a new derivation of the conserved quantity for the wall jet. The conserved vector (31) may give the conserved quantity for some other flow.

The conserved vectors (T^1, T^2) depend on $u(z,r),\ v(z,yr)$ or $\psi(z,r)$ and can therefore be expressed in terms of z and r as follows:

$$D_zT^1 + D_rT^2 = \frac{\partial T^1(z,r)}{\partial z} + \frac{\partial T^2(z,r)}{\partial r}. \tag{35}$$

But for a conserved vector, $D_zT^1 + D_rT^2 = 0$ and (35) becomes

$$\frac{\partial T^1(z,r)}{\partial z} + \frac{\partial T^2(z,r)}{\partial r} = 0. \tag{36}$$

Expression (36) is the basis of the derivation of conserved quantities for the axisymmetric liquid jet, free jet and wall jet.

A. Conserved quantity for an axisymmetric liquid jet

The surface of the cylinder is $r = a$. There is no slip or suction and blowing on the surface $r = a$. The boundary conditions for the axisymmetric liquid jet therefore are

$$r = a: \quad u(z,a) = 0,\ v(z,a) = 0, \tag{37}$$

$$r = \phi(z): \quad u_r(z,\phi(z)) = 0. \tag{38}$$

The r-component of the fluid velocity on the free surface $r = \phi(z)$ is

$$v(z,\phi(z)) = \frac{D}{Dt}\left[\phi(z)\right] = u(z,\phi(z))\frac{d\phi(z)}{dz}, \tag{39}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u(z,r)\frac{\partial}{\partial z} + v(z,r)\frac{\partial}{\partial r} \tag{40}$$

is the material time derivative. The conserved vector (21) gives the conserved quantity for the axisymmetric liquid jet. Integrating (36) with respect to r from $r = a$ to $r = \phi(z)$, keeping z fixed during the integration, gives for the conserved vector (21)

$$\int_a^{\phi(z)} \left[\frac{\partial}{\partial z}\left(ru(z,r)\right) + \frac{\partial}{\partial r}(rv(z,r))\right] dr = 0. \tag{41}$$

Applying the result for differentiating under an integration sign [10] to the first term in (41), we obtain

$$\frac{d}{dz}\int_a^{\phi(z)} ru(z,r)dr$$
$$-\phi(z)u(z,\phi(z))\frac{d\phi(z)}{dz} + \left[rv(z,r)\right]_0^{\phi(z)} = 0. \tag{42}$$

Using (39) and the boundary condition (37) for $v(z,0)$, we have

$$\int_a^{\phi(z)} ru(z,r)dr = \text{constant, independent of } z. \tag{43}$$

Therefore

$$F = \int_a^{\phi(z)} ru(z,r)dr \tag{44}$$

is the conserved quantity for the axisymmetric liquid jet which gives the total volume flux as constant along the jet.

B. Conserved quantity for axisymmetric free jet

The boundary conditions for the axisymmetric free jet are

$$r = 0: v(z,0) = 0,\ u_r(z,0) = 0, \tag{45}$$

$$r \to \infty: u(z,\infty) = 0,\ u_r(z,\infty) = 0. \tag{46}$$

The conserved vector (22) gives the conserved quantity for the axisymmetric free jet. Integrating (36) with respect to r from $r = 0$ to $r = \infty$ keeping z fixed, we have for the conserved vector (22),

$$\frac{d}{dz}\int_0^{\infty} ru^2(z,r)dr$$
$$+\left[r(u(z,r)v(z,r) - \nu u_r(z,r))\right]_0^{\infty} = 0. \tag{47}$$

The boundary conditions (45) and (46) and the fact that $v(z,\infty)$ is finite yield

$$\int_0^{\infty} ru^2(z,r)dr = \text{constant, independent of } z. \tag{48}$$

Thus the conserved quantity is

$$F = \int_0^{\infty} ru^2(z,r)dr. \tag{49}$$

Goldstein [2] used $2\pi\rho F$ as the conserved quantity for the axisymmetric free jet where ρ is the constant density of the fluid.

The conserved quantity (49) can also be constructed for the stream function formulation using the conserved vector (30). In terms of the stream function the boundary conditions (45) and (46) take the following form:

$$r = 0: \frac{1}{r}\psi_z(z,r) = 0,\ \frac{\partial}{\partial r}\left(\frac{1}{r}\psi_r(z,r)\right) = 0, \tag{50}$$

$$r = \infty: \frac{1}{r}\psi_r(z,r) = 0,\ r\frac{\partial}{\partial r}\left(\frac{1}{r}\psi_r(z,r)\right) = 0. \tag{51}$$

The conserved quantity for the axisymmetric free jet is obtained by integrating (36) with respect to r from $r = 0$ to $r = \infty$ keeping z fixed. Thus

$$\frac{d}{dz}\int_0^{\infty} \frac{1}{r}\psi_r^2(z,r)dr$$
$$+\left[-\frac{1}{r}\psi_z(z,r)\psi_r(z,r) - \nu\psi_{rr}(z,r) + \frac{\nu}{r}\psi_r(z,r)\right]_0^{\infty} = 0. \tag{52}$$

We assume that $\psi_z(z,\infty)$ is finite. The boundary conditions (50) and (51) finally yield

$$\int_0^{\infty} \frac{1}{r}\psi_r^2(z,r)dr = \text{constant, independent of } z, \tag{53}$$

which is equivalent to (48) and hence we obtain the conserved quantity (49). Thus the conserved quantity is the same if we use the stream function formulation or the system of equations for the velocity components.

We observe that the conserved vector (30) which establishes the conserved quantity for the stream function formulation is equivalent to the conserved vector (22) for the system of equations.

C. Conserved quantity for axisymmetric wall jet

For an axisymmetric wall jet the boundary conditions are

$$r = a : u(z, a) = 0,\ v(z, a) = 0, \tag{54}$$

$$r = \infty : u = O(\frac{1}{r^2}). \tag{55}$$

In terms of the stream function, we obtain

$$r = a : \psi_z(z, a) = 0,\ \psi_r(z, a) = 0, \tag{56}$$

$$r = \infty : \psi_r = O(\frac{1}{r}) \tag{57}$$

and the stream function is zero at $r = a$, that is $\psi(z, a) = 0$. The local conserved vector (31) obtained by the multiplier approach cannot give a conserved quantity for the axisymmetric wall jet because it is not compatible with the boundary conditions. The non-local conserved vector (34) gives the conserved quantity for the axisymmetric wall jet. Integrating (36) with respect to r from $r = a$ to $r = \infty$ keeping z as fixed and by considering the conserved vector (34), we have

$$\int_a^\infty \frac{\partial}{\partial z}\left[\frac{1}{r}\psi(z,r)\psi_r^2(z,r) - \nu\int_0^z \frac{1}{r}\psi_r^2(z^*,r)dz^*\right]dr$$
$$+\left[\frac{\nu}{2}\psi_r^2(z,r)+\left(\frac{\nu}{r}\psi_r(z,r)-\nu\psi_{rr}(z,r)-\frac{1}{r}\psi_z(z,r)\psi_r(z,r)\right)\right.$$
$$\left.\times\psi(z,r)\right]_a^\infty = 0. \tag{58}$$

However $\psi(z, a) = 0$, $\psi(z, \infty)$ and $\psi_z(z, \infty)$ are assumed to be finite. The second term in (58) vanishes due the to the boundary conditions (56) and (57) and we obtain

$$\int_a^\infty \left[\frac{1}{r}\psi(z,r)\psi_r^2(z,r) - \nu\int_0^z \frac{1}{r}\psi_r^2(z^*,r)dz^*\right]dr$$
$$= \text{constant, independent of } z. \tag{59}$$

We obtain the conserved quantity for the axisymmetric wall jet

$$F = \int_a^\infty [\frac{1}{r}\psi(z,r)\psi_r^2(z,r) - \nu\int_0^z \frac{1}{r}\psi_r^2(z^*,r)dz^*]dr. \tag{60}$$

Duck and Bodoyni [3] established the conserved quantity (60) by integrating the momentum equation and using the continuity equation and boundary conditions.

IV. Conclusion

The conserved quantities for the axisymmetric liquid, free and wall jets can be constructed with the help of conservation laws. This is more systematic than integrating the momentum equation. The liquid jet, the free jet and the wall jet satisfy the same partial differential equations but the boundary conditions for each jet are different. The derivation of the conserved quantity depends on the boundary conditions as well as on the differential equations. The boundary conditions therefore determine which conserved vector is associated with which jet. The multiplier approach gave two local conservation laws for the system of equations for the velocity components. One of the conserved vectors gave the conserved quantity for the axisymmetric liquid jet and the second conserved vector gave the conserved quantity for the axisymmetric free jet.

For the third-order partial differential equation for the stream function two local conserved vectors were obtained, one of which was the non-local conserved vector for the system of equations for the velocity components. One of the local conserved vectors for the third-order partial differential equation for the stream function was used to give an alternative derivation of the conserved quantity for the axisymmetric free jet but the other local conserved vector cannot be used to derive the conserved quantity for the axisymmetric wall jet. The conserved quantity for the axisymmetric wall jet was established with the help of a non-local conserved vector for the third-order partial differential equation for the stream function. That non-local conservation law for the third-order partial differential equation for the stream function was not obtained by the multiplier approach. The reason is that the multiplier approach only gives multipliers for local conservation laws.

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