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Optimal Control of Viscoelastic Melt Spinning Processes

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Abstract**—**The optimal control problem for the viscoelastic melt spinning process has not been reported yet in the literature. In this study, an optimal control problem for a mathematical model of a viscoelastic melt spinning process is considered. Maxwell-Oldroyd model is used to describe the rheology of the polymeric material, the fiber is made of. The extrusion velocity of the polymer at the spinneret as well as the velocity and the temperature of the quench air and the fiber length serve as control variables. A constrained optimization problem is derived and the first–order optimality system is set up to obtain the adjoint equations. Numerical solutions are carried out using a steepest descent algorithm. A computer program in `MATLAB` is developed for simulations.

Keywords**—**Fiber spinning, Maxwell-Oldroyd, Optimal control, First-order optimality system, Adjoint system

I. Introduction

Many kinds of synthetic textile fibers, like Nylon, Polyester, etc. are manufactured by a so–called melt spinning process. In this process, the molten polymer is extruded through a die called the spinneret to create a slender, cylindrical jet of viscous polymer, the fiber. Far away from the spinneret, the fiber is wrapped around a drum, which pulls it away at a pre–determined take–up speed. The take–up speed is much higher than the extrusion speed; in industrial processes the take–up speed is about $50\mathrm{m/s}$ and the extrusion speed is about $10\mathrm{m/s}$, see [2], [6]. The ratio between the take–up speed v_L and the extrusion speed v_0 is called draw–ratio d. Hence the filament is stretched considerably in length and therefore it decreases in diameter. The ambient atmosphere temperature is below the polymer solidification temperature such that the polymer is cooled and solidifies before the take–up, see Figure 1. In industrial processes a whole bundle of hundreds of single filaments is extruded and spun in parallel, however for the analysis we consider a single filament.

The dynamics of melt spinning processes has been studied by many research groups throughout the world during the last decades starting with early works of Kase and Matsuo [4] and Ziabicki [12]. In later works, the energy balance for the heat transfer was introduced into the model, and more and more sophisticated descriptions, including material effects, crystallization kinetics and viscoelastic behavior, were developed by several authors in order to achieve a better understanding of the fiber formation process. Up to now it is possible to use the basic models with more or less modifications in different technological aspects of the melt spinning process. Due to the complex behaviour of the polymeric material, several parameters are included in all the available models. Typically,

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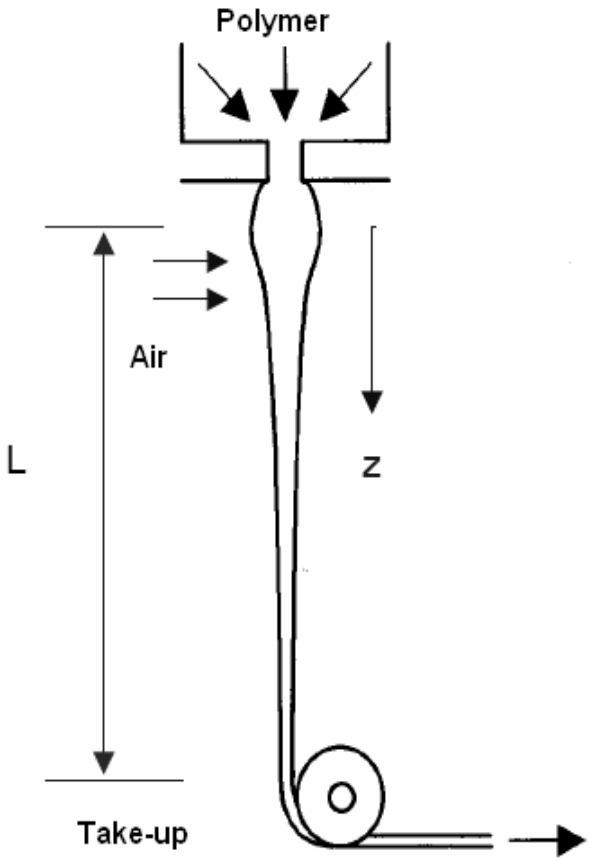

Fig. 1. Sketch of the melt spinning process.

these parameters are hard to measure. An identification based on comparing available data and simulations is one way to determine those parameters. Additionally, the outcome of the melt spinning process depends heavily on the boundary conditions, e.g. the draw ratio, the ambient temperature, the quench air velocity and temperature. The topic of optimizing the fiber production with respect to the external variables has not been widely discussed in the literature. Especially, the viscoelastic case has not yet been treated, although the viscous models were discussed [9].

The main goal of this study is to control the temperature profile of the fiber, such that the final temperature is below the fiber solidification point. We consider Maxwell-Oldroyd model to describe the viscoelastic behaviour of the

polymeric material. The optimal control problem is considered as a constrained minimization problem, see [3], and derived formally the corresponding first–order optimality system via the Lagrange functional. For the numerical computation of the optimal control variables, a steepest descent algorithm is presented using the adjoint variables.

II. The Optimal Control Problem

A. Simple Melt Spinning Model

By considering basic conservations laws (mass, momentum and energy) one can obtain the simple model for the fiber spinning process, see [6], [7], [8], [9].

$$\rho A v = W_0 \,, \tag{1}$$
$$\rho A v \frac{dv}{dz} = \frac{dA\tau}{dz} - \sqrt{A\pi} C_d \rho_{air} v^2 + \rho A g \,, \tag{2}$$
$$\rho C_p v \frac{dT}{dz} = -\frac{2\alpha\sqrt{\pi}}{\sqrt{A}}(T - T_\infty) \,, \tag{3}$$

where A represent spinline cross-sectional area, ρ is the density of the fluid, g is the gravity, W_0 mass throughput, v is the spinline velocity, T represent the spinline temperature, T_0 is the initial temperature, T_∞ is the air temperature, C_p is the fluid heat capacity, z is the distance coordinate from the spinneret. The axial stress τ is given by

$$\tau + \lambda\left(v\frac{d\tau}{dz} - 2\tau\frac{dv}{dz}\right) = 3\eta\frac{dv}{dz} \,, \tag{4}$$

where η and λ denote the viscosity and the fluid characteristic relaxation time respectively. The viscosity and the characteristic relaxation time are given below

$$\eta = \eta_0 \exp\left[\frac{E_a}{R_G}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right] \,, \tag{5}$$
$$\lambda = \frac{\eta_0}{G} \exp\left[\frac{E_a}{R_G}\left(\frac{1}{T} - \frac{1}{T_0}\right)\right] \,. \tag{6}$$

Here $\eta_0 > 0$ is the zero shear viscosity at the initial temperature T_0, E_a denotes the activation energy, R_G represents the gas constant and G denotes the fluid shear modulus.

According to [6], we assume the following relations for the air drag coefficient

$$C_d = 0.37\,\mathrm{Re}_{\mathrm{air}}^{-0.61}$$

and the heat transfer coefficient

$$\alpha = \frac{0.21}{R_0}\kappa \mathrm{Re}_{\mathrm{air}}^{\frac{1}{3}}\left[1 + \frac{64 v_c^2}{v^2}\right]^{\frac{1}{6}}$$

depending on the Reynolds–number of the quench air flow

$$\mathrm{Re}_{\mathrm{air}} = \frac{2 v \rho_{\mathrm{air}}}{\eta_{\mathrm{air}}}\sqrt{\frac{A}{\pi}} \,.$$

Here R_0 is the radius of the spinneret, $\rho_{\mathrm{air}}, \eta_{\mathrm{air}}$ and κ represent the density, viscosity and heat conductivity of the air and v_c is the velocity of the quench air.

The equations (1)—(3) are subject to the boundary conditions

$$v = v_0 \quad \text{and} \quad T = T_0 \quad \text{at} \quad z = 0 \tag{7}$$
$$v = v_L \quad \text{at} \quad z = L \tag{8}$$

where L denotes the length of the spinline.

The dimensional equations of (1)-(3) end (4) are now rendered into dimensionless form by introducing following quantities

$$v^* = \frac{v}{v_0}, \;\; z^* = \frac{z}{L}, \;\; T^* = \frac{T}{T_0}, \;\; A^* = \frac{A}{A_0}, \;\; \text{and} \;\; q^* = \frac{q}{q_0},$$

where $q = \frac{\rho A \tau}{W_0}$ and $q_0 = \frac{\rho A_0 v_0 \eta_0}{L W_0}$. Dropping the star, rearranging the terms, the system reads as

$$\frac{dv}{dz} = \frac{1}{3\eta + \vartheta v q}\left(qv + \vartheta v^2 \frac{dq}{dz}\right) \,, \tag{9}$$
$$\frac{dq}{dz} = \mathrm{Re}\left(\frac{dv}{dz} - \frac{\mathrm{Fr}^{-1}}{v} + C v^{3/2}\right) \,, \tag{10}$$
$$\frac{dT}{dz} = -\gamma\frac{T - T_\infty}{\sqrt{v}} \,, \tag{11}$$
$$v(0) = 1, \;\; v(1) = d(= \frac{v_L}{v_0} > 1), \;\; T(0) = T_0, \tag{12}$$

where $\mathrm{Re} = \frac{\rho L v_0}{\eta_0}$ is the Reynolds number, $\mathrm{Fr}^{-1} = \frac{gL}{v_0^2}$ is the inverse of the Froude number, $C = \frac{C_d \rho_{\mathrm{air}} L}{\rho R_0}$ is the scaled drag coefficient and $\gamma = \frac{2\alpha L}{\rho C_p v_0 R_0}$ denotes the scaled heat transfer coefficient.

The viscosity η and ϑ are given by

$$\eta = \exp\left[\frac{E_a}{R_G T_0}\left(\frac{1}{T} - 1\right)\right] \,, \tag{13}$$
$$\vartheta = \frac{\eta_0 v_0}{GL}\exp\left[\frac{E_a}{R_G T_0}\left(\frac{1}{T} - 1\right)\right] \,. \tag{14}$$

B. Cost Functional

We want to control the temperature profile of the fiber, such that the final temperature is below the solidification point $T_s^* = T_s/T_0$. The air temperature T_∞, the air velocity v_c and the extrusion velocity v_0 can be influenced and hence serve as control variables. In addition to the air temperature, air velocity and the extrusion velocity the fiber length is also considered as a control variable. Therefore, the following two cost functionals are considered

$$J_1(y,u) = -\omega_1 u_3 + \omega_2(y_3(1) - T_s^*) + \frac{\omega_3}{2}\int_0^1 (u_2(z) - T_{\mathrm{R}})^2 dz + \frac{\omega_4}{2}\int_0^1 u_1(z)^2 dz \tag{15a}$$

$$J_2(y,u) = -\omega_1 u_3 + \omega_2(y_3(1) - T_s^*) + \frac{\omega_3}{2}\int_0^1 (u_2(z) - T_{\mathrm{R}})^2 dz + \frac{\omega_4}{2}\int_0^1 u_1(z)^2 dz + \frac{\omega_5}{2}\int_0^1 u_4(z)^2 dz \tag{15b}$$

where $y = (v, q, T) \in Y$ denotes the vector of state variables and control variables $u \in U$ are given by

$$u = \begin{cases} (v_c, T_\infty, v_0) & \text{for } J_1 \\ (v_c, T_\infty, v_0, L) & \text{for } J_2. \end{cases}$$

Here T_{R} denotes the reference temperature.

The weighting coefficients $\omega_i > 0$, $i = 1..5$ allow to adjust the cost functionals to different scenarios.

Summarizing, the following constrained optimization problems are considered

$$\text{minimize } \ J_1(y, u) \ \text{ w.r.t } u, \text{subject to } 9 - 11. \tag{16}$$

$$\text{minimize } \ J_2(y, u) \ \text{ w.r.t } u, \text{subject to } 9 - 11. \tag{17}$$

III. The First–Order Optimality System

In this section we introduce the Lagrangian associated to the constrained minimization problems 16 and 17 and derive the system of first–order optimality conditions. Eventhough, two constrained minimization problems are need to be handled, it is enough to consider one problem to derive the necessary theory. So we consider the problem (16) and in generally J denotes the cost functional.

Let $Y = C^1([0, 1]; \mathbb{R}^3)$ be the state space consisting of triples of differentiable functions $y = (v, q, T)$ denoting velocity, stress and temperature of the fiber. Further, let $U = C^1([0, 1]; \mathbb{R}^2) \times \mathbb{R}$ be the control space consisting of a pair $(u_1, u_2) = (v_c, T_\infty)$ of differentiable functions, i.e. air velocity and temperature, and a scalar $u_3 = v_0$ interpreted as the inflow velocity.

The operator $e = (e_v, e_q, e_T) : Y \times U \to Y^*$is defined via the weak formulation of the state system (9)-(11):

$$\langle e(y, u), \xi \rangle_{Y,Y^*} = 0 \qquad \forall \xi \in Y^*$$

where $\langle \cdot, \cdot \rangle_{Y,Y^*}$ denotes the duality pairing between Y and its dual space Y^*. Now, the minimization problem (16) reads as

$$\text{minimize } J(y, u) \text{ w.r.t } u \in U, \text{ subject to } e(y, u) = 0. \tag{18}$$

Introducing the Lagrangian $\mathcal{L} : Y \times U \times Y^* \to \mathbb{R}$ defined as

$$\mathcal{L}(y, u, \xi) = J(y, u) + \langle e(y, u), \xi \rangle_{Y,Y^*} \ ,$$

the first–order optimality system reads as

$$\nabla_{y,u,\xi} \mathcal{L}(y, u, \xi) = 0 \ .$$

Considering the variation of $\mathcal{L}$ with respect to the adjoint variable ξ, one can get the state system

$$e(y, u) = 0$$

or in the classical form

$$\frac{dy}{dz} = f(y, u), \ \text{ with } v(0) = 1, \ v(1) = d, \ T(0) = 1 \tag{19}$$

Rearranging the equations (9)–(11), $f(y, u)$ can be obtained.

Taking variations of $\mathcal{L}$ with respect to the state variable y, the adjoint system can be obtained

$$J_y(y, u) + e_y^*(y, u)\xi = 0$$

or in classical form

$$\begin{aligned} -\frac{d\xi}{dz} &= F(y, u, \xi) \ , \\ \text{with } \xi_q(0) = 0, \ \xi_q(1) &= 0, \xi_T(1) = -\omega_2 \ , \end{aligned} \tag{20}$$

where

$$F(y, u, \xi) = \left(\frac{\partial f}{\partial y}\right)^\top \xi \ .$$

Finally, considering variations of $\mathcal{L}$ with respect to the control variable u in a direction of δu the optimality condition can be obtained

$$\langle J_u(y, u), \delta u \rangle + \langle e_u(y, u)\delta u, \xi \rangle = 0 \ . \tag{21}$$

IV. Algorithm

To solve the nonlinear first–order optimality system (19), (20) and (21), an iterative steepest–descent method is proposed.

1) *Set $k = 0$ and choose initial control $u^{(0)} \in U$.*
2) *Given the control $u^{(k)}$, solve the state system* (19) *with a shooting method to obtain $y^{(k)}$.*
3) *Solve the adjoint system* (20) *with a shooting method to obtain $\xi^{(k)}$.*
4) *Compute the gradient $g^{(k)}$ of the cost functional.*
5) *Update the control $u^{(k+1)} = u^{(k)} - \beta g^{(k)}$ for a step size $\beta > 0$.*
6) *Compute the cost functional $J^{(k)} = J(y^{(k)}, u^{(k)})$.*
7) *If $\left|g^{(k)}\right| \geq$ Tol, goto 2.*

Here, Tol is some prescribed tolerance for the termination of the optimization procedure. In each iteration step, we need to solve two boundary value problems, i.e. the state system (19) and the adjoint system (20). Both systems are solved using a shooting method based on a Newton–iteration.

A. Step Size Control with Polynomial Models

Crucial for the convergence of the algorithm is the choice of the step size β (in step 5 of the algorithm) in the direction of the gradient. Clearly, the best choice would be the result of a line search

$$\beta^* = \text{argmin}_{\beta > 0} J(u_k - \beta g_k).$$

However this is numerically quite expensive although it is a one dimensional minimization problem. Instead of the exact line search method, an approximation based on a quadratic polynomial method is used [5] in order to find β^* which minimize $J(u_k - \beta g_k)$. The quadratic polynomial for

$$\Phi(\beta) = J(u_k - \beta g_k),$$

is constructed using following data points,

$$\begin{aligned} &\Phi(0) = J(u_k), \quad \Phi(1) = J(u_k - g_k), \\ &\Phi'(0) = \nabla J(u_k)^{\mathrm{T}} g_k < 0. \end{aligned}$$

Then the quadratic polynomial of β reads as follows,

$$\Lambda(\beta) = \Phi(0) + \Phi'(0)\beta + (\Phi(1) - \Phi(0) - \Phi'(0))\beta^2.$$

The global minimum of Φ is ,

$$\beta^* = \frac{-\Phi'(0)}{2(\Phi(1) - \Phi(0) - \Phi'(0))} \in (0, 1).$$

B. Numerics

Both state and adjoint system of ODE were solved using the `MATLAB` routine `ode23tb`. This routine uses an implicit method with backward differentiation to solve stiff differential equations. It is an implementation of `TR-BDF2` [11], an implicit two stage Runge-Kutta formula where the first stage is a trapezoidal rule step and the second stage is a backward differentiation formula of order two.

V. Results

First, we consider the results for the cost functional J_2. Figure 2 shows spinline velocity, temperature, air velocity and air temperature profiles before optimization and after optimization for the cost functional J_2. Corresponding cost functional is shown in Figure 3 for two different tolerance values (10^{-2} and 10^{-3}).

It can be seen that in an optimal state final temperature is below $50°$C and the extrusion velocity increases from $16.67\,\mathrm{m/s}$ to $17.00\,\mathrm{m/s}$. The fiber length increases from 1 m to 1.45 m. The optimal air temperature profile is more or less close to $22.5°$C. The optimal air velocity profile is very high near the spinneret and just after the spinneret it almost constant and very close to zero. These details are summarized in Table I

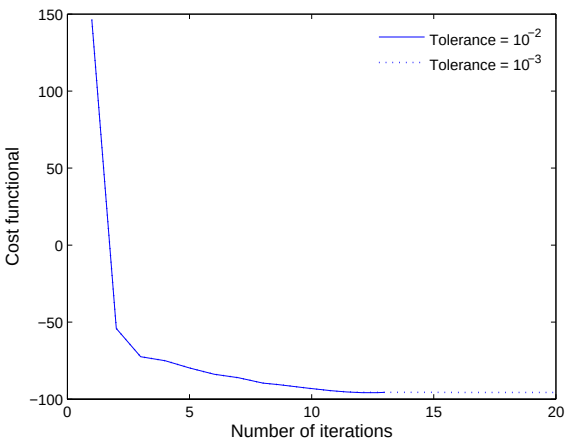

Fig. 3. Cost functional J_2.

The constrained optimization problem (16) does not yield the reasonable solution. This is because of the Maxwell-Oldroyd model has an upper bound for the final velocity and this upper bound value is given below [10]

TABLE I
SUMMARY OF OPTIMAL VALUES FOR COST FUNCTIONAL J_2.

Value / Parameter	Initial	Optimal
Final Temperature °C	71.77	45.50
Fiber length L m	1.0	1.45
Extrusion velocity m/s	16.67	17.00

$$v(L) \leq v(0) + \int_0^L \frac{1}{\lambda(T)} dz\,. \tag{23}$$

Since the fluid characteristic relaxation time depends on temperature profile, the upper bound value also depends on the temperature profile. The temperature profile varies with respect to parameters, e.g. the air velocity and temperature. Hence the upper bound of the final velocity varies with respect to the parameters. Therefore, it can be easily understood that in optimal control problem (16), after few iterations the upper bound gets lower value than the prescribed take-up velocity. Further, we have noticed that when the final temperature is close to $55°$C, the upper bound is also close to 50 m/s (which we considered as the final take-up velocity). This situation visualize in Figure 4. Hence the constrained minimization problem (16) does not generate suitable solutions. Unlike the viscous models [9], in the viscoelastic models we need to handle optimal control problem carefully due to the viscoelastic effect [10]. Introducing the fiber length L as a new control variable (cost functional J_2) we overcome these difficulties. Now each iteration the upper bound value increases since it depends on fiber length L.

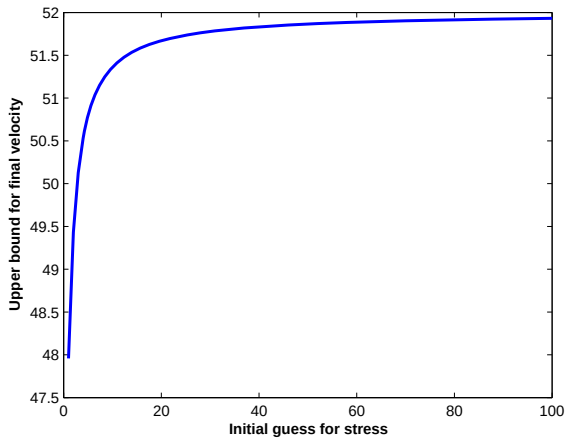

Fig. 4. The upper bound of the final velocity versus initial guess for the stress when the final temperature close to $55°$C.

VI. Conclusion

We studied an optimal control problem for a melt spinning process. The aim was to control the temperature profile such that final temperature is below fiber solidification point. In addition to that we tried to maximize the outflow and minimize the air velocity and air temperature. In this study, the

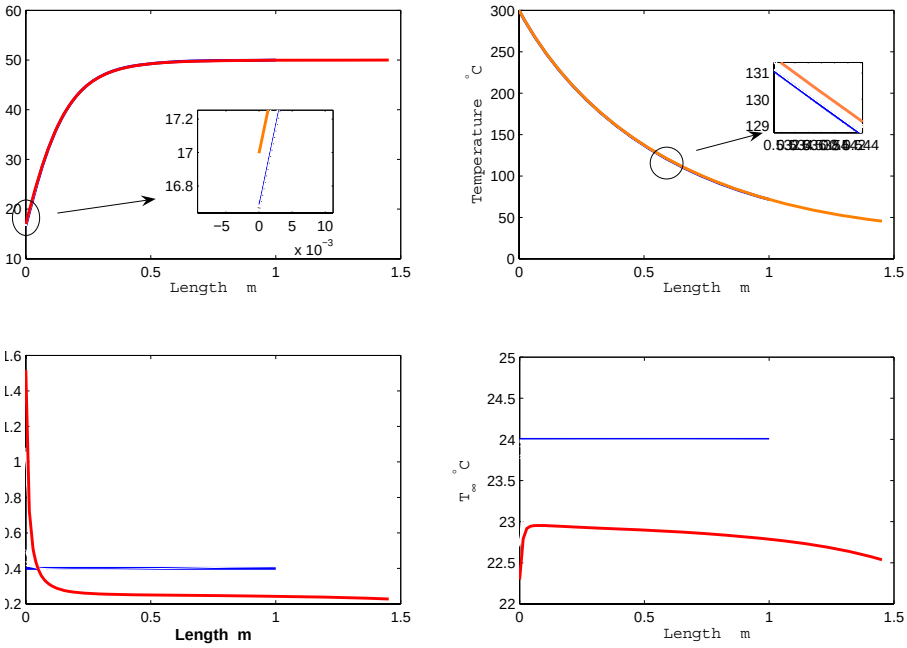

Fig. 2. For cost functional J_2: Spinline velocity (up-left) and temperature (up-right) profiles, initial (blue) optimal (red). Air velocity (down-left) and temperature (down-right) profiles, initial (blue) optimal (red).

Maxwell-Oldroyd model was considered. Based on the control variables, two cost functionals were considered. Defining the cost functional, we converted the optimal control problem into the constrained optimization problem and derived the first order optimality system. For the numerical solution, we proposed the steepest descent algorithm based on adjoint variable method. For the step size control, polynomial type model was considered.

It can be seen that the optimal control problem (16) does not produce the solution. From that we have experienced that Maxwell-Oldroyd model has an upper bound for the final take-up velocity.

In an optimal profile, final temperature was below $50°$C in the cost functional J_2 where the air temperature is also reduced to more or less equal to $22.5°$C. The optimal air velocity profile is high at the spinneret and just after the spinneret it is almost constant and very close to zero. It can be noticed that the extrusion velocity increased. Clearly, this is successful concerning the cost.

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