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Complexity of Multivalued Maps

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Abstract—We consider the topological entropy of maps that in general, cannot be described by one-dimensional dynamics. In particular, we show that for a multivalued map F generated by single-valued maps, the topological entropy of any of the single-value map bounds the topological entropy of F from below.

Keywords—Multivalued maps, Topological entropy, Selectors

I. Introduction

One measure of the complexity of a system is the positivity of its topological entropy. The topological entropy for single-value maps is well-known [1], while the case for multivalued maps, in general, is not widely studied. However, it is worth noting the works of [2] and [3] on lower bounds for topological entropy using the Conley index theory, and works of [4] and [5] on branch image entropy defined from the backward iterates of a non-invertible map.

Because many systems encountered in practice cannot be described by one-dimensional dynamics (e.g. in population dynamics, where many observed units can occupy a state, and are mapped to different states [6]), it is useful to consider multivalued maps. We define in Section ll the topological entropy of such maps, and give conditions in bounding their topological entropy from below in Section lll. We gives two examples in Section lV, particularly focusing on the map studied in [6].

II. Preliminaries

We give the standard definition of the topological entropy due to Bowen [8]. Let (X, d) be a compact metric space with distance d and let $f : X \to X$ be continuous. For any integer $\ell \geq 1$, define the distance function $d_\ell : X \times X \to \mathbb{R}_{\geq 0}$ by

$$d_\ell(x, y) = \max_{0 \leq j < \ell} d(f^j(x), f^j(y)).$$

Definition 1: A finite set $E \subset X$ is called (ℓ, δ)*-separated* if $d_\ell(x, y) \geq \delta$ for all $x, y \in E$. Moreover, if E has the maximal cardinality among all the (ℓ, δ)-separated sets, then E is called a *maximal* (ℓ, δ)*-separated set*.

Definition 2: The topological entropy of f is given by

$$h_{\text{top}}(f) = \lim_{\delta \to 0} \limsup_{\ell \to \infty} \frac{\log s_f(\ell, \delta)}{\ell},$$

where $s_f(\ell, \delta)$ is the cardinality of the maximal (ℓ, δ)-separated set for f.

Definition 3: [9] Let X and Y be arbitrary sets. A *multi-valued map F from X to Y*, denoted by $F : X \rightrightarrows Y$, is such that $F(x)$ is assigned a set $Y_x \subset Y$ for all $x \in X$. Let

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1. $\text{dom}(F) = \{x \in X \mid F(x) \neq \emptyset\}$,
2. $F(X') = \bigcup\{F(x) \mid x \in X'\}$ for $X' \subset X$.

A single-valued map $f : X'(\subset X) \to Y$ is called a *selector for F over X'* if $g(x) \in F(x)$ for all $x \in X'$.

Let (X, d) be a compact metric space. Let $F : X \rightrightarrows X$ be a multivalued map, $g : X \to X$ be a continuous map, $Sel^0(F)$ be the set of all continuous selectors for F, and $\{u_m\}_{m=0}^{M}$ be a sequence of $M + 1$ points such that

$$u_{m+1} \in F(u_m) \quad \text{for } m = \{0, 1, \ldots, M - 1\}.$$

Assume that the graph of F can be viewed as a union of its continuous selectors, i.e.

$$F(x) = \bigcup\{g(x) | g \in Sel^0(F)\} \quad \forall x \in \text{dom}(F).$$

Definition 4: Let (X, d) be a compact metric space and let $F : X \rightrightarrows X$ be a multivalued map. Denote the *set of partial orbits of F of length ℓ* by

$$U_\ell = \{u = \{u_i, u_{i+1}, \ldots, u_{i+\ell}\}\}_{i=0}^{M-\ell} \subset X^{\ell+1},$$

where $u_{i+j} \in F(u_{i+j-1})$ for all $1 \leq j \leq \ell$. A set $S \subset U_\ell$ is called a (ℓ, δ)*-separated set for F* if for any $u, u' \in S$, $d(u, u') \geq \delta$.

We extend the notion of topological entropy to multivalued maps.

Definition 5: Let $S_F(\ell, \delta)$ be the maximal (ℓ, δ)-separated set for F with cardinality $s_F(\ell, \delta)$. We define the topological entropy of F by

$$h_{\text{top}}(F) = \lim_{\delta \to 0} \limsup_{\ell \to \infty} \frac{\log s_F(\ell, \delta)}{\ell}.$$

Definition 6: [10] Let X be any set and let $F : X \rightrightarrows X$ be a multivalued map. Suppose $\mathcal{P} = \{P_0, P_1, \ldots, P_{L-I}\}$ is a partition of X into L disjoint regions and suppose that the intersection of any element of $\mathcal{P}$ with the image under F of another is either itself or is empty. The structure of $\mathcal{P}$ can be described a transition matrix $T = (T_{ij})$ defined by

$$T_{ij} = \begin{cases} 1 & \text{if } P_j \cap F(P_i) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

T is called the transition matrix for F.

From the definiton above, if $x \in P_i$, then $F(x) \in P_j$.

Definition 7: [11] Given a transition matrix $T = (T_{ij})$ with $T_{ij} \in \{0, 1\}$, let

$$\Sigma_T^+ = \{w = (w_0 w_1 \ldots) \mid T_{w_m, w_{m+1}} = 1 \ (\forall m \geq 0)\}.$$

The shift map $\sigma_T : \Sigma_T^+ \to \Sigma_T^+$ is such that $(\sigma_T(w))_m = w_{m+1}$ $(\forall m \geq 0)$. The pair (σ_T, Σ_T^+) is called the *one-sided subshift of finite type (SFT) for the matrix T*.

Theorem 1: [11] Let T be a transition matrix and let $\sigma_T : \Sigma_T^+ \to \Sigma_T^+$ be the associated SFT. Then

$$h_{\text{top}}(\sigma_T) = \ln(\lambda_{\max})$$

where $\lambda_{\max}$ is the maximum eigenvalue of T.

III. Results

Theorem 2: Let (X, d) be a compact metric space and let $F : X \rightrightarrows X$ be a multivalued map. For any continuous selector $g : X \to X$ of F, the following inequality holds

$$h_{\text{top}}(F) \geq \sup\{h_{\text{top}}(g) | g \in Sel^0(F)\}.$$

Proof. Take an arbitrary $g \in Sel^0(\mathcal{F})$. Let S_g be the maximal (k, δ)-separated set for g. We show that the maximal (ℓ, δ)-separated set for any $g \in Sel^0(\mathcal{F})$ is a (ℓ, δ)-separated set for F. For all $x \in S_g$, denote by

$$S_g^\ell = \{x_g^\ell = (x, g(x), \ldots, g^\ell(x))\} \subset U_\ell$$

the subset of the partial orbits of g of length ℓ. Clearly, $\#S_g = \#S_g^\ell$. Since

$$d_\ell(u, v) = \max_{0 \leq j < \ell}(u_j, v_j)$$

for $u, v \in X^{\ell+1}$ and $u_j, v_j \in X$, then for any two distinct $x, y \in S_g$,

$$d(x, y) \geq \delta \Rightarrow d_\ell(x_g^\ell, y_g^\ell) \geq \delta,$$

where $x_g^\ell, y_g^\ell \in S_g^\ell$. Thus, S_g^ℓ is a (ℓ, δ)-separated set for F. Note however that S_g may not be maximal for F so

$$s_F(\ell, \delta) \geq s_g(\ell, \delta).$$

Passing to the limits, we establish the claim.

Corollary 1: Let (X, d) and F be as in Theorem 2. If $g_1, g_2 : X \to X$ are continuous selectors of F, then $g = g_1 \circ g_2$ is also a continuous selector of F, and that

$$h_{\text{top}}(F) \geq h_{\text{top}}(g).$$

IV. Examples

Example A. Consider the case on the interval $X = [0, 1]$, where the multivalued map $F : X \rightrightarrows X$ is generated by two self-maps

$$g_1 = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

and

$$g_2 = \begin{cases} 2ax & 0 \leq x \leq \frac{1}{2} \\ 2a - 2ax & \frac{1}{2} \leq x \leq 1 \end{cases}, \quad 1/2 \leq a < 1.$$

It is well-known that the topological entropy of g_1 and g_1 are $\log 2$ and $\log 2a$ respectively. Hence,

$$h_{\text{top}}(F) \geq \sup\{\log 2, \log 2a\}.$$

Example B. We consider the method in analysing longitudinal data, studied in [6]. Longitudinal data is simply a repeated measurement of the same variables (observed units) in time. Let $\mathbb{N}_0$ be the set of non-negative integers, and let the integer $n \geq 1$. Let $I_n = \{0, 1, \ldots, n-1\}$, $\mathcal{C}_n = \mathcal{P}(I_n)$ be the power set of I_n, and x_i^* be such that

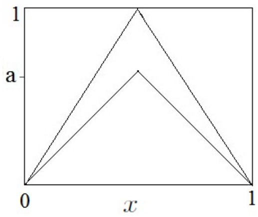

Fig. 1. The graph of the multivalued map F as a union of g_1 and g_2.

$$x_i^* = \begin{cases} 1 & \text{if } x_i = 0 \\ 0 & \text{if } x_i = 1. \end{cases}$$

Let

$$\Gamma_2^n = \left\{(x_j)_{j=0}^{n-1} : x_j \in \{0, 1\}\right\}$$

and let

$$\Gamma_n^n = \left\{(i_j)_{j=0}^{n-1} : i_j \in I_n,\ i_j'\text{s distinct}\right\},$$

where the subscripts 2 and n are the cardinalities of the sets $\{0, 1\}$ and I_n, respectively. Let

$$\begin{aligned} \mathcal{S}_n &= \{p = (x, y) : x \in \Gamma_2^n,\ y \in \Gamma_n^n\} \\ &= \Gamma_2^n \times \Gamma_n^n. \end{aligned}$$

Given n, we have $|\Gamma_2^n| = 2^n$, $|\Gamma_n^n| = n!$, and so $\mathcal{S}_n$ is a finite space composed of $L = 2^n \times n!$ states.

Definition 8: Let $\Delta \in \mathcal{C}_n$ and let $j \in \Delta$. The *change map* $\phi_\Delta : \mathcal{S}_n \to \mathcal{S}_n$ is defined by

$$\phi_\Delta(x_0 x_1 \cdots x_j \cdots x_{n-1},\ y) = (x_0 x_1 \cdots x_j^* \cdots x_{n-1},\ y).$$

The *jump map* $\phi_J : \mathcal{S}_n \to \mathcal{S}_n$ is defined by

$$\phi_J(x_0 x_1 \cdots x_j \cdots x_{n-1},\ i_0 i_1 \ldots i_j \ldots i_{n-1}) = (x_0 x_1 \ldots x_{j-1} x_{j+1} \ldots x_{n-1} x_j,\ i_0 i_1 \ldots i_{j-1} i_{j+1} \ldots i_{n-1} i_j).$$

Let $j, j' \in \Delta$. Then x_j and $x_{j'}$ both change values under ϕ_Δ. If $j < j'$, then ϕ_J is first applied to j'. That is,

$$\begin{aligned} \phi_J(x_0 x_1 \cdots x_j \cdots x_{j'} \cdots x_{n-1},\ i_0 i_1 \cdots i_j \cdots i_{j'} \cdots i_{n-1}) = \\ (x_0 x_1 \cdots x_{j-1} x_{j+1} \cdots x_{j'-1} x_{j'+1} \cdots x_{n-1} x_{j'} x_j, \\ i_0 i_1 \ldots i_{j-1} i_{j+1} \ldots i_{n-1} i_{j'} i_j). \end{aligned}$$

Consider an observed unit (variable) k in the longitudinal data, and questionnaire $\mathcal{Q}$ of $n \geq 1$ questions. Denote by

$$\begin{aligned} \mathcal{Q}_t^k &= \left\{Q_{i_{0t}^k}, Q_{i_{1t}^k}, \ldots, Q_{i_{(n-1)t}^k}\right\} \qquad (*) \\ &:= \{Q_{i_0}, Q_{i_1}, \ldots, Q_{i_{n-1}}\}_t^k, \quad i_j, i_{jt} \in I_n \end{aligned}$$

any reordering of questions of unit k at time t,

$\mathcal{A}_t^k = \left\{x_{0t}^k, x_{1t}^k, \cdots, x_{(n-1)t}^k\right\} := \{x_0, x_1, \cdots, x_{n-1}\}_t^k$
the set of coded answers to $\mathcal{Q}_t^k$,

$x_t^k = x_{0t}^k x_{1t}^k \cdots x_{(n-1)t}^k := \left(x_0 x_1 \cdots x_{(n-1)}\right)_t^k = \left(x_{jt}^k\right)_{j=0}^{n-1}$
the concatenation of elements of $\mathcal{A}_t^k$, and

$y_t^k = i_{0t}^k i_{1t}^k \cdots i_{(n-1)t}^k := \left(i_0 i_1 \cdots i_{(n-1)}\right)_t^k = \left(i_{jt}^k\right)_{j=0}^{n-1}$
the concatenation of indices in $\mathcal{Q}_t^k$.

Definition 9: For each observed unit k, define $\Delta_t^k \in \mathcal{C}_n$ by $\Delta_t^k = \{j : i_j$ is a question index of unit k at time t that changes answer value at $t+1$, ordered in ascending order, as in (*)$\}$.

Definition 10: Fix observed unit k and let p_0^k be the initial state of k. Define the map $\varphi : (\mathbb{N}_0, \mathcal{C}_n, \mathcal{S}_n) \to \mathcal{S}_n$ such that

$$\begin{aligned} \varphi(t, \Delta_t^k, p_t^k) &= \varphi_{[\Delta_t^k]}(p_t^k) \\ &= (\phi_J \circ \phi_{\Delta_t^k})(p_t^k) \\ &= p_{t+1}^k. \end{aligned}$$

The set $\Delta_t^k \in \mathcal{C}_n$ is given by the longitudinal data for all t. The action of $\phi_{\Delta_t^k}$ is data dependent while ϕ_J is a strictly deterministic reordering of the position of the question indices and answer values in Δ_t^k. At any time t, we can always trace the answer to its corresponding question such that the coded answer value x_j corresponds to question i_j. For each k, the nonautonomous map $\varphi_{[\Delta_t^k]}$ displaces the most frequently changing answers to the right, while slowly changing answers displace to the left.

Given k, if $p_t^k \in p$ and $p_{t+1}^k \in p'$, then we say that there is a transition from p to p' under $\Delta_t \in \mathcal{C}_n$. A *self-transition* is under the empty set $\Delta = \emptyset$ (i.e. there is no change in answer). A transition from p to p', and from p' to p, under the same set Δ is *reversible*. A way to visualize the state transitions of unit k in $\mathcal{S}_n$ is by a *directed graph* (digraph) $\mathcal{G}$ whose vertices are points in $\mathcal{S}_n$, with an edge from p to p' if there is a transition from p to p'. A *path of length* m in $\mathcal{G}$ is a sequence of vertices $v_0, v_1, \ldots, v_m$ such that there is a directed edge from v_j to v_{j+1}.

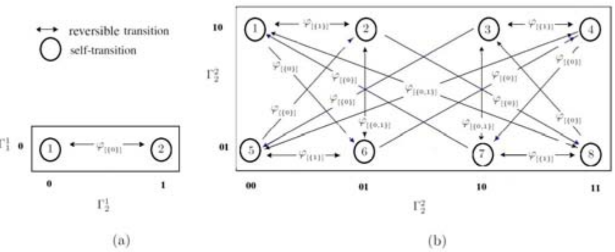

Fig. 2. All possible transitions between states in (a) $\mathcal{S}_1$ and (b) $\mathcal{S}_2$.

Fig. 2 illustrates all possible transitions between states in $\mathcal{S}_1$ and $\mathcal{S}_2$. Transitions alternating between states 1 and 2 in Fig. 2(a) denote alternating answer between 0 and 1, to question 0. On the other hand, transitions alternating between states 1 and 2 in Fig. 2(b) denote constant answer=0 to question 1, and alternating answer between 0 and 1, to question 0. Because question 1 has constant answer, it is positioned on the left of question order '10'.

Consider our state space $\mathcal{S}_n$. In [6], the map $\varphi_{[\Delta_t^k]}$ is constructed for each observed unit k. By defining a multivalued map $F : \mathcal{S}_n \rightrightarrows \mathcal{S}_n$, we can describe all possible paths of observed units in $\mathcal{S}_n$. Denote by $\varphi_{[\Delta]} : \mathcal{S}_n \to \mathcal{S}_n$ the case where Δ_t in Definition 9 is constant. Let

$$G = \{\varphi_{[\Delta]} : \Delta \in \mathcal{C}_n\}.$$

It is clear that $\varphi_{[\Delta]}$ is a selector for F over $\mathcal{S}_n$, and is continuous if $\mathcal{S}_n$ is endowed with the discrete metric. Define

$$F(p) = \{p' : g(p) = p',\ g \in G\}.$$

Observe that $|F(p)| = 2^n$, i.e. F is a 1 to 2^n map. Define

$$G_t = \{g_{\ell_{t-1}} \circ \cdots \circ g_{\ell_0} : \mathcal{S}_n \to \mathcal{S}_n |\ g_{\ell_i} \in G\}.$$

For $g_{\ell_t} = \varphi_{[\Delta_t^k]}$ for all $t \geq 1$, we define the orbit of $p \in \mathcal{S}_n$ under F by

$$\begin{aligned} \mathcal{O}_F(p_0^k) &= \{p_t^k\}_{t \geq 0} \\ &= \{G_t(p_0^k)\}_{t \geq 1}, \end{aligned}$$

where under F, the state $p_{t+1}^k \in F(p_t^k)$. The multivalued map F can be interpreted as a digraph whose $N = n!2^n$ vertices are the points in $\mathcal{S}_n$, and edge $p \to p'$ if $p' \in F(p)$. Equivalently, F can be defined as a square matrix of size N. Orbits of the nonautonomous map $\varphi_{[\Delta_t^k]}$ respect paths in the digraph F.

Consider $\mathcal{S}_n$ as a union of L disjoint states labeled $i = 1, 2, \ldots N = 2^n n!$, i.e.,

$$\mathcal{S}_n = \bigcup_{i=1}^{N} s_i.$$

A way of labeling s_i is via the map

$$\psi : \quad \mathcal{S}_n \to \{1, 2, \ldots, N\}, \quad s_i \mapsto \psi(s_i) = i.$$

Definition 11: Let $n \geq 1$, $V = \{1, 2, \ldots, N\}$, and $i, j, \in V$. For $p, p' \in \mathcal{S}_n$, let $\psi(p) = i$, and $\psi(p') = j$. The transition matrix for F is denoted by $T_n^{(F)} = (T_{ij}^{(F)})$, where

$$T_{ij}^{(F)} = \begin{cases} 1 & \text{if } p' \in F(p) \\ 0 & \text{otherwise.} \end{cases}$$

Remarks.

(i) From Fig. 2, the transition matrices for $n = 1$ and $n = 2$ are

$$T_1^{(F)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad T_2^{(F)} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 1 & 0 & 1 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 1 & 0 & 1 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{1} & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 0 & 1 & 0 & 1 & \mathbf{1} & \mathbf{1} & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \\ 1 & 0 & 1 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} \end{bmatrix}$$

In [6], the analysis of longitudinal data from household units is studied in $\mathcal{S}_3$. We do not give the matrix here, however, we illustrate in the Appendix A all possible transitions in $\mathcal{S}_3$.

(ii) Because $F(p)$ is a 1 to 2^n map, then by properties of non-negative matrices, $T_n^{(F)}$ has $\lambda_{max} = 2^n$. By Theorem 1, $h_{\text{top}}(\sigma_{T_n^{(F)}}) = \ln(2^n)$.

(iii) For constant Δ, we have

$$h_{\text{top}}(\varphi_{[\Delta]}) = 0.$$

We illustrate in Appedix B the case where $\Delta = I_3$, i.e. where all three answers are constantly changing. The transition from any point p to $p' = \varphi_{[\Delta]}(p)$ is reversible. For example, $\varphi_{[I_3]}$ takes state 1 to state 48, and state 48 back to state 1. Note that the strict inequality in Theorem 2 is satisfied.

V. Conclusions

We have presented a definition of the topological entropy of multivalued maps. For the case where the multivalued map F is generated by single-valued maps, then we are able to give a lower bound for the complexity of F. If we can find a selector g for a multivalued map F, and find that the topological entropy of g is positive, then we can say that F is at least as complicated as g. In the case that we can define a partition for the state space of F (as in Example B), then we can encode F as a transition matrix T, and analysis of F is through the subshift associated to T. Then any selector, or composition of selectors of F, has complexity bounded by the complexity of F. We note that this paper does not suggest a general techniques for computing selectors, but will be persued in a following paper.

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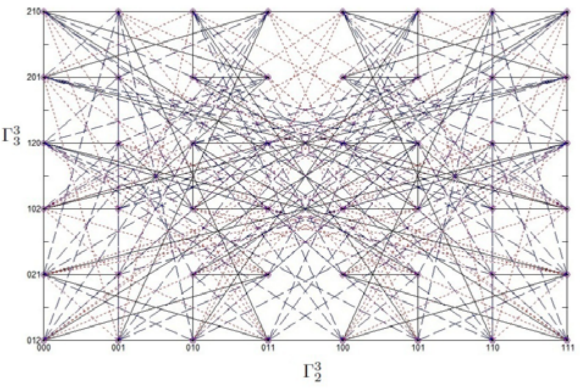
Appendix A
Transitions in $\mathcal{S}_3$



Appendix B
Reversible transitions under constant $\Delta = I_3$.