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On fractional (k, m)-deleted graphs with constrains conditions

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Abstract—Let G be a graph of order n, and let $k \geq 2$ and $m \geq 0$ be two integers. Let $h : E(G) \rightarrow [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for each $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. A graph G is called a fractional (k, m)-deleted graph if there exists a fractional k-factor $G[F_h]$ of G with indicator function h such that $h(e) = 0$ for any $e \in E(H)$, where H is any subgraph of G with m edges. In this paper, it is proved that G is a fractional (k, m)-deleted graph if $\delta(G) \geq k + m + \frac{m}{k+1}$, $n \geq 4k^2 + 2k - 6 + \frac{(4k^2+6k-2)m-2}{k-1}$ and

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for any vertices x and y of G with $d_G(x, y) = 2$. Furthermore, it is shown that the result in this paper is best possible in some sense.

Keywords—graph, degree condition, fractional k-factor, fractional (k, m)-deleted graph.

I. Introduction

In this paper, we consider finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any $x \in V(G)$, the degree and the neighborhood of x in G are denoted by $d_G(x)$ and $N_G(x)$, respectively. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and $G - S = G[V(G) \setminus S]$. Let S and T be two disjoint vertex subsets of G, we use $e_G(S, T)$ to denote the number of edges with one end in S and the other end in T. We denote the minimum degree and the maximum degree of G by $\delta(G)$ and $\Delta(G)$, respectively. We define the distance $d_G(x, y)$ between two vertices x and y as the minimum of the lengths of the (x, y) paths of G.

Let $k \geq 1$ be an integer. Then a spanning subgraph F of G is called a k-factor if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \rightarrow [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$. Zhou [1] introduced firstly the definition of a fractional (k, m)-deleted graph, that is, a graph G is called a fractional (k, m)-deleted graph if there exists a fractional k-factor $G[F_h]$ of G with indicator function h such that $h(e) = 0$ for any $e \in E(H)$, where H is any subgraph of G with m edges. A fractional (k, m)-deleted graph is simply called a fractional k-deleted graph if $m = 1$.

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Some people studied graph factors [2–6]. Yu and Liu [7] obtained a Fan-type condition for a graph to have a fractional k-factor. Liu and Zhang [8] gave a toughness condition for a graph to have a fractional k-factor. Cai and Liu [9] showed a stability number condition for graphs to have fractional k-factors. Zhou [10] obtained some sufficient conditions for graphs to have fractional k-factors. Zhou [1,11] obtained two sufficient conditions for graphs to be fractional (k, m)-deleted graphs.

The following results on k-factors, fractional k-factors and fractional (k, m)-deleted graphs are known.

Theorem 1. ([12]). Let G be a connected graph of order n with $\delta(G) \geq k$, where k is a positive integer, kn is even and $n \geq 8k^2 + 12k + 6$. If G satisfies

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for any vertices x and y of G with $d_G(x, y) = 2$, then G has a k-factor.

Theorem 2. ([7]). Let G be a connected graph of order n with $\delta(G) \geq k$, where k is a positive integer and $n \geq 8k^2 + 12k + 6$. If G satisfies

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for any vertices x and y of G with $d_G(x, y) = 2$, then G has a fractional k-factor.

Theorem 3. ([1]). Let $k \geq 2$ and $m \geq 0$ be two integers. Let G be a connected graph of order n with $n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2} + 2(2k+1)m$, $\delta(G) \geq k + m + \frac{(m+1)^2 - 1}{4k}$. If

$$|N_G(x) \cup N_G(y)| \geq \frac{1}{2}(n + k - 2)$$

for each pair of nonadjacent vertices x, y of G, then G is a fractional (k, m)-deleted graph.

Theorem 4. ([11]). Let $k \geq 1$ and $m \geq 1$ be two integers. Let G be a graph of order n with $n \geq 4k - 5 + 2(2k+1)m$. If

$$\delta(G) \geq \frac{n}{2},$$

then G is a fractional (k, m)-deleted graph.

In this paper, we proceed to study the fractional (k, m)-deleted graphs and obtain a Fan-type condition for a graph to be a fractional (k, m)-deleted graph. Our result is the following theorem which is an extension of Theorems 1 and 2.

Theorem 5. Let $k \geq 2$ and $m \geq 0$ be two integers, and let G be a graph of order n with $n \geq 4k^2+2k-6+\frac{(4k^2+6k-2)m-2}{k-1}$. If $\delta(G) \geq k+m+\frac{m}{k+1}$ and

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for any vertices x and y of G with $d_G(x,y)=2$, then G is a fractional (k,m)-deleted graph.

Our result is stronger than Theorem 4 if $k \geq 2$ and the order n is sufficiently large. Set $m=0$ in Theorem 5. Then we get the following corollary.

Corollary 1. Let $k \geq 2$ be an integer, and let G be a graph of order n with $n \geq 4k^2+2k-6$. If $\delta(G) \geq k$ and

$$\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$$

for any vertices x and y of G with $d_G(x,y)=2$, then G has a fractional k-factor.

Obviously, the result of Corollary 1 is stronger than Theorem 2 if $k \geq 2$.

II. The Proof of Theorem 5

The proof of Theorem 5 relies heavily on the following lemma.

Lemma 2.1. ([1]). Let $k \geq 1$ and $m \geq 0$ be two integers, and let G be a graph and H a subgraph of G with m edges. Then G is a fractional (k,m)-deleted graph if and only if for any subset S of $V(G)$

$$k|S| + \sum_{x \in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \geq 0,$$

where $T=\{x : x \in V(G)\backslash S, d_{G-S}(x) - d_H(x) + e_H(x,S) \leq k-1\}$.

Proof of Theorem 5. Suppose that G satisfies the assumption of Theorem 5, but is not a fractional (k,m)-deleted graph. Then from Lemma 2.1, there exists some subset S of $V(G)$ such that

$$k|S| + \sum_{x \in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \leq -1, \quad (1)$$

where $T=\{x : x \in V(G)\backslash S, d_{G-S}(x) - d_H(x) + e_H(x,S) \leq k-1\}$ and H is some subgraph of G with m edges. It is easy to see that $d_{G-S}(x) - d_H(x) + e_H(x,S) \geq 0$ for any $x \in V(G)$. Since $|E(H)| = m$, we have $\sum_{x \in T} d_H(x) - e_H(S,T) \leq 2m$.

Now, we prove the following claims.

Claim 1. $1 \leq |S| < \frac{n}{2}$.

Proof. If $S=\emptyset$, then from (1), $\delta(G) \geq k+m+\frac{m}{k+1} \geq k+m$ and $d_H(x) \leq m$ for any $x \in V(G)$, we get $-1 \geq \sum_{x \in T}(d_G(x) - d_H(x) - k) \geq \sum_{x \in T}(\delta(G) - m - k) \geq 0$, a contradiction. Hence, $|S| \geq 1$.

On the other hand, according to (1), $|S|+|T| \leq n$ and $d_{G-S}(x) - d_H(x) + e_H(x,S) \geq 0$ for any $x \in V(G)$, we obtain

$$\begin{aligned}
-1 &\geq k|S| + \sum_{x \in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\
&\geq k|S| - k|T| \geq k|S| - k(n-|S|) = 2k|S| - kn,
\end{aligned}$$

which implies $|S| < \frac{n}{2}$. This completes the proof of Claim 1.

Claim 2. $|T| > |S|$.

Proof. In terms of (1) and $d_{G-S}(x) - d_H(x) + e_H(x,S) \geq 0$ for any $x \in V(G)$, we have that $-1 \geq k|S| + \sum_{x \in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \geq k|S| - k|T|$, which implying $|T| > |S|$. The proof of Claim 2 is complete.

Claim 3. $|T| \geq k+1$.

Proof. Suppose that $|T| \leq k$. Then from (1), $\delta(G) \geq k+m+\frac{m}{k+1} \geq k+m$ and $d_H(x) \leq m$ for any $x \in V(G)$, we obtain

$$\begin{aligned}
-1 &\geq k|S| + \sum_{x \in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\
&\geq |T||S| + \sum_{x \in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\
&= \sum_{x \in T}(|S| + d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\
&\geq \sum_{x \in T}(\delta(G) - m - k) \geq 0,
\end{aligned}$$

which is a contradiction. This completes the proof of Claim 3.

Claim 4. $|S| < \frac{n}{2} - (k+m-1)$.

Proof. Suppose that $|S| \geq \frac{n}{2} - (k+m-1)$. Then using (1), $|S|+|T| \leq n$ and $\sum_{x \in T} d_H(x) - e_H(S,T) \leq 2m$, we have

$$\begin{aligned}
-1 &\geq k|S| + \sum_{x \in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\
&= k|S| + \sum_{x \in T} d_{G-S}(x) - (\sum_{x \in T} d_H(x) - e_H(S,T)) \\
&\quad -k|T| \\
&\geq k|S| + \sum_{x \in T} d_{G-S}(x) - 2m - k|T| \\
&\geq k|S| + \sum_{x \in T} d_{G-S}(x) - 2m - k(n-|S|) \\
&= 2k|S| + \sum_{x \in T} d_{G-S}(x) - 2m - kn \\
&\geq 2k(\frac{n}{2} - (k+m-1)) + \sum_{x \in T} d_{G-S}(x) - 2m - kn \\
&= -2k(k+m-1) + \sum_{x \in T} d_{G-S}(x) - 2m,
\end{aligned}$$

that is,

$$\sum_{x \in T} d_{G-S}(x) \leq 2k(k+m-1) + 2m - 1.$$

Consequently, it follows from Claim 2 and $n \geq 4k^2+2k-6+\frac{(4k^2+6k-2)m-2}{k-1}$ that

$$\begin{aligned}
\frac{\sum_{x \in T} d_{G-S}(x)}{|T|} &\leq \frac{2k(k+m-1)+2m-1}{|S|+1} \\
&\leq \frac{2k(k+m-1)+2m-1}{\frac{n}{2} - (k+m-1) + 1} \\
&\leq 1 - \frac{1}{k}.
\end{aligned}$$

Combining the inequalities above with Claim 3, we obtain

$$\sum_{x\in T} d_{G-S}(x) \le (1-\frac{1}{k})|T| = |T| - \frac{1}{k}|T| < |T|-1. \quad (2)$$

Set $T_0 = \{x : x \in T, d_{G-S}(x) = 0\}$. Note that $|T_0| \ge 2$ holds by (2). For each $x \in T_0$, $d_G(x) \le |S| < \frac{n}{2}$ by Claim 1. Since T_0 is an independent set of G and G satisfies the assumption of Theorem 5, the neighborhoods of the vertices in T_0 are disjoint. Therefore, we obtain

$$\begin{aligned} |S| &\ge |\bigcup_{x\in T_0} N_G(x)| \ge \delta(G)|T_0| \\ &\ge (k+m+\frac{m}{k+1})|T_0| \ge (k+m)|T_0|. \quad (3) \end{aligned}$$

Using (2) and the definition of T_0, we have

$$(1-\frac{1}{k})|T| \ge \sum_{x\in T} d_{G-S}(x) \ge |T| - |T_0|,$$

which implies

$$|T_0| \ge \frac{1}{k}|T|. \quad (4)$$

According to (3) and (4), we get

$$|S| \ge (k+m)|T_0| \ge (1+\frac{m}{k})|T| \ge |T|.$$

That contradicts Claim 2. This completes the proof of Claim 4.

Claim 5. $e_G(S,T) \le (k+m)|S|$.

Proof. Since $d_{G-S}(x) - d_H(x) + e_H(x,S) \le k-1$ for each $x \in T$ and $d_H(x) \le m$, we have $d_{G-S}(x) \le k+m-1$ for each $x \in T$. Combining this with Claim 4, we obtain

$$d_G(x) \le d_{G-S}(x) + |S| < \frac{n}{2} \quad (5)$$

for each $x \in T$. From (5) and the assumption of Theorem 5, $G[N_G(s)\cap T]$ is a complete induced subgraph of G for each $s \in S$. Note that $S \ne \emptyset$ by Claim 1. Thus, by $d_{G-S}(x) \le k+m-1$ for each $x \in T$, we have

$$e_G(s,T) \le \Delta(G[T]) + 1 \le k+m.$$

Hence, we obtain

$$e_G(S,T) \le (k+m)|S|.$$

The proof of Claim 5 is complete.

According to (1), $\sum_{x\in T} d_H(x) - e_H(S,T) \le 2m$, $\delta(G) \ge k+m+\frac{m}{k+1}$, Claim 2, Claim 3 and Claim 5, we have

$$\begin{aligned} -1 &\ge k|S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\ &\ge k|S| + \sum_{x\in T}(d_{G-S}(x) - k) - 2m \\ &= k|S| + \sum_{x\in T}(d_G(x) - k) - e_G(S,T) - 2m \\ &\ge k|S| + \sum_{x\in T}(\delta(G) - k) - (k+m)|S| - 2m \\ &\ge k|S| + (k+m+\frac{m}{k+1} - k)|T| \\ &\quad -(k+m)|S| - 2m \\ &= m(|T|-|S|) + \frac{m}{k+1}|T| - 2m \\ &\ge 0, \end{aligned}$$

which is a contradiction. This completes the proof of Theorem 5.

III. Remark

In Theorem 5, the bound in the assumption

$$\max\{d_G(x), d_G(y)\} \ge \frac{n}{2}$$

is best possible in the sense that we cannot replace $\frac{n}{2}$ by $\frac{n}{2}-1$. We can show this by constructing a graph $G = ktK_1 \vee (kt+1)K_1$, where $k \ge 2$ and $m \ge 0$ are two integers and t is enough large positive integer. Then it follows that $|V(G)| = n = 2kt+1$ and

$$\frac{n}{2} > \max\{d_G(x), d_G(y)\} = kt > \frac{n}{2} - 1$$

for any two vertices vertices x, y of $(kt+1)K_1 \subset G$ with $d_G(x,y) = 2$. Let $S = V(ktK_1) \subseteq V(G)$, $T = V((kt+1)K_1) \subseteq V(G)$ and H is any subgraph of G with m edges. Then $|S| = kt$, $|T| = kt+1$, $d_{G-S}(T) = 0$ and $\sum_{x\in T} d_H(x) - e_H(S,T) = 0$. Thus, we get

$$\begin{aligned} & k|S| + \sum_{x\in T}(d_{G-S}(x) - d_H(x) + e_H(x,S) - k) \\ = \ & k^2t - k(kt+1) = -k < 0. \end{aligned}$$

In terms of Lemma 2.1, G is not a fractional (k,m)-deleted graph.

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References

[1] S. Zhou, *A neighborhood condition for graphs to be fractional (k,m)-deleted graphs*, Glasgow Mathematical Journal 52(1)(2010), 33–40.
[2] S. Zhou, *Independence number, connectivity and (a,b,k)-critical graphs*, Discrete Mathematics 309(12)(2009), 4144–4148.
[3] S. Zhou and Q. Shen, *On fractional (f,n)-critical graphs*, Information Processing Letters 109(14)(2009), 811–815.
[4] H. Matsuda, *Fan-type results for the existence of $[a,b]$-factors*, Discrete Mathematics 306(2006), 688–693.
[5] J. R. Correa and M. Matamala, *Some remarks about factors of graphs*, Journal of Graph Theory 57(2008), 265–274.
[6] H. Liu and G. Liu, *Binding number and minimum degree for the existence of (g,f,n)-critical graphs*, Journal of Applied Mathematics and Computing 29(1-2)(2009), 207–216.
[7] J. Yu and G. Liu, *Fractional k-factors of graphs*, Chinese Journal of Engineering Mathematics 22(2)(2005), 377–380.
[8] G. Liu and L. Zhang, *Toughness and the existence of fractional k-factors of graphs*, Discrete Mathematics 308(2008), 1741–1748.
[9] J. Cai and G. Liu, *Stability number and fractional f-factors in graphs*, Ars Combinatoria 80(2006), 141–146.
[10] S. Zhou, *Some results on fractional k-factors*, Indian Journal of Pure and Applied Mathematics 40(2)(2009), 113–121.
[11] S. Zhou, *A minimum degree condition of fractional (k,m)-deleted graphs*, Comptes rendus Mathematique 347 (21-22)(2009), 1223–1226.
[12] T. Niessen, *A Fan-type result for regular factors*, Ars combinatoria, 46(1997), 277–285.