

Generalized Measures of Fuzzy Entropy and their Properties

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Abstract—In the present communication, we have proposed some new generalized measure of fuzzy entropy based upon real parameters, discussed their and desirable properties, and presented these measures graphically. An important property, that is, monotonicity of the proposed measures has also been studied.

Keywords—Fuzzy numbers, Fuzzy entropy, Characteristic function, Crisp set, Monotonicity.

I. INTRODUCTION

THE tendency of the systems to become more disordered over time is described by the second law of thermodynamics, which states that the entropy of the system cannot spontaneously decrease. It was Shannon [9] who founded the subject of information theory by introducing the concept of entropy into communication theory. It was then realized that entropy is a property of any stochastic system and today it finds widespread applications in the fields of Statistics, Operations Research Techniques, Information Processing and Computing.

In practice, exact values of model parameters are rare in most engineering, data processing, and biological systems modeling and thus uncertainties arise due to incomplete information reflected in uncertain model parameters. A fruitful approach to handle parameter uncertainties is the use of fuzzy numbers and arithmetic. A feature of imperfect information known as fuzziness results from the lack of crisp distinction between the elements belonging and not belonging to a set, that is, the boundaries of the set under consideration are not sharply defined. A measure of fuzziness often used and cited in the literature of information theory, known as fuzzy entropy, was first introduced by Zadeh [12]. De Luca and Termini [1] introduced some requirements which capture our intuition for the degree of fuzziness. Kaufmann [7] proposed to measure the degree of fuzziness of any fuzzy set A by a metric distance between its membership function and the characteristic function of its nearest crisp set.

Fuzzy entropy is one of the important digital features of fuzzy sets and occupies an important place in system model and system design. Thus, keeping in view the idea of fuzzy sets, De Luca and Termini [1] introduced a measure of fuzzy

entropy corresponding to Shannon's [9] measure. This fuzzy entropy is given by

$$H(A) = - \sum_{i=1}^n \left[\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i)) \right] \quad (1)$$

After this development, a large number of measures of fuzzy entropy were discussed, characterized and generalized by various authors. Kapur [6] introduced the following measure of fuzzy entropy:

$$H_{\alpha, \beta}(A) = \frac{1}{\beta - \alpha} \log \frac{\sum_{i=1}^n \{ \mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^\alpha \}}{\sum_{i=1}^n \{ \mu_A^\beta(x_i) + (1 - \mu_A(x_i))^\beta \}} \quad (2)$$

$;\alpha \geq 1, \beta \leq 1$

Keeping in view the existing probabilistic measures, Parkash and Sharma [8] introduced two measures of fuzzy entropy, given by

$$K_a(A) = \sum_{i=1}^n \left[\begin{array}{l} \log(1 + a\mu_A(x_i)) \\ + \log(1 + a(1 - \mu_A(x_i))) \\ - \log(1 + a) \end{array} \right]; a \geq 0 \quad (3)$$

and

$$H_a(A) = - \sum_{i=1}^n \left\{ \begin{array}{l} \mu_A(x_i) \log \mu_A(x_i) \\ + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i)) \end{array} \right\} \quad (4)$$

$$- \frac{1}{a} \sum_{i=1}^n \left\{ \begin{array}{l} (1 + a\mu_A(x_i)) \log (1 + a\mu_A(x_i)) \\ + \{1 + a(1 - \mu_A(x_i))\} \\ \log \{1 + a(1 - \mu_A(x_i))\} \\ - (1 + a) \log (1 + a) \end{array} \right\}; a > 0$$

Some other measures of fuzzy entropy, their characterizations and generalizations have been provided by Zadeh [11,12], Ebanks [2], Kapur [6], Emptoz [3], Guo and Xin [4], Singpurwalla and Booker [10], Hu, Yu [5] etc. In section II, we have proposed some new generalized measure of fuzzy entropy based upon real parameters, discussed their properties and presented these measures graphically. A

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desirable property, that is, monotonicity of the proposed measures has been studied in section III.

II. GENERALIZED MEASURES OF FUZZY ENTROPY AND THEIR VALIDITY

A. Firstly, we propose a new parametric measure of fuzzy entropy as given by the following mathematical expression:

$$H_\alpha(A) = - \sum_{i=1}^n \left[\begin{array}{l} \mu_A^\alpha(x_i) \log \mu_A(x_i) \\ + (1 - \mu_A(x_i))^\alpha \\ \log(1 - \mu_A(x_i)) \end{array} \right] \quad (5)$$

$, \alpha \neq 1, \alpha > 1$

To prove that the measure introduced in equation (5) is a correct measure of fuzzy entropy, we study its essential properties as follows:

1. $H_\alpha(A)$ is a concave function of $\mu_A(x_i)$.

Proof: We have

$$\frac{dH_\alpha(A)}{d\mu_A(x_i)} = \left[\begin{array}{l} \mu_A^{\alpha-1}(x_i) + \alpha \mu_A^{\alpha-1}(x_i) \log \mu_A(x_i) \\ - (1 - \mu_A(x_i))^{\alpha-1} \\ - \alpha (1 - \mu_A(x_i))^{\alpha-1} \log(1 - \mu_A(x_i)) \end{array} \right]$$

Also

$$\frac{d^2H_\alpha(A)}{d\mu_A^2(x_i)} = - \left[\begin{array}{l} (\alpha-1) \left(\mu_A^{\alpha-2}(x_i) + (1 - \mu_A(x_i))^{\alpha-2} \right) \\ + \alpha \mu_A^{\alpha-2}(x_i) \left(1 + \log \mu_A^{(\alpha-1)}(x_i) \right) \\ + \alpha (1 - \mu_A(x_i))^{\alpha-2} \left(1 + \log(1 - \mu_A(x_i))^{\alpha-1} \right) \end{array} \right]$$

< 0 for $\alpha > 1$

2. $H_\alpha(A)$ does not change when $\mu_A(x_i)$ is replaced by $1 - \mu_A(x_i)$

3. $H_\alpha(A)$ is an increasing function of $\mu_A(x_i)$ for $0 \leq \mu_A(x_i) \leq \frac{1}{2}$

4. $H_\alpha(A)$ is a decreasing function of $\mu_A(x_i)$ for $\frac{1}{2} \leq \mu_A(x_i) \leq 1$

5. $H_\alpha(A) = 0$ when $\mu_A(x_i) = 0$ or 1

Since $H_\alpha(A)$ satisfies all the essential properties of being a measure of fuzzy entropy, it is a correct measure of fuzzy entropy. Next, with the help of the data, we have presented the measure (5) graphically. For this purpose, we have computed different values of $H_\alpha(A)$ corresponding to different fuzzy values $\mu_A(x_i)$ as shown in the following Table-I:

TABLE I

$H_\alpha(A)$ VERSUS $\mu_A(x_i)$

$\mu_A(x_i)$	$H_\alpha(A)$
0.0	0.00000
0.1	0.10836
0.2	0.20718
0.3	0.28312
0.4	0.33050
0.5	0.34657
0.6	0.33050
0.7	0.28312
0.8	0.20718
0.9	0.10836
1.0	0.00000

Next, we have presented the values of $H_\alpha(A)$ graphically and obtained the following Fig.-1 which shows that the measure introduced in equation (5) is a concave function.

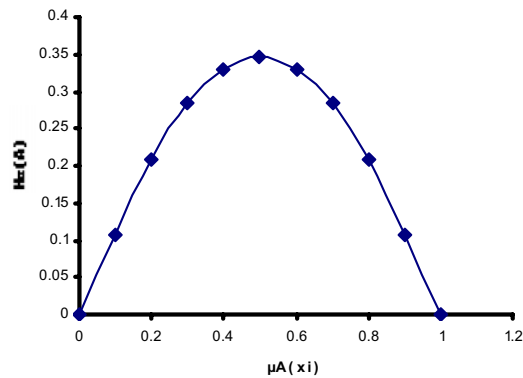


Fig.1 Graph $H_\alpha(A)$ Versus $\mu_A(x_i)$

B. Next, we propose a new parametric measure of fuzzy entropy of order , given by the following expression:

$$\begin{aligned}
 H^\alpha(A) = & -\sum_{i=1}^n \left[\mu_A(x_i) \log \mu_A(x_i) \right. \\
 & \left. + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i)) \right] \\
 & + \frac{2^{\alpha-1}}{1-\alpha} \sum_{i=1}^n \left[\frac{\mu_A^\alpha(x_i)}{+(1 - \mu_A(x_i))^\alpha - 1} \right] \quad (6) \\
 & ; \alpha \neq 1, \alpha > 1
 \end{aligned}$$

We shall prove that (6) as a measure of fuzzy entropy. We have

$$\frac{d^2 H^\alpha(A)}{d\mu_A^2(x_i)} = - \left[\frac{1}{\mu_A(x_i)(1 - \mu_A(x_i))} \right. \\
 \left. + 2^{\alpha-1} \alpha \left[\frac{\mu_A^{\alpha-2}(x_i)}{+(1 - \mu_A(x_i))^{\alpha-1}} \right] \right]$$

< 0

Thus $H^\alpha(A)$ is concave.

Hence, the expression $H^\alpha(A)$ satisfies the following properties:

- i. $H^\alpha(A)$ is a concave function of $\mu_A(x_i)$.
- ii. $H^\alpha(A)$ doesn't change when $\mu_A(x_i)$ is replaced by $1 - \mu_A(x_i)$.
- iii. $H^\alpha(A)$ is an increasing function of $\mu_A(x_i)$ for $0 \leq \mu_A(x_i) \leq \frac{1}{2}$.
- iv. $H^\alpha(A)$ is a decreasing function of $\mu_A(x_i)$ for $0 \leq \mu_A(x_i) \leq \frac{1}{2}$.

v. $H^\alpha(A) = 0$ when $\mu_A(x_i) = 0$ or 1 .

Hence, ${}_\alpha H(A)$ is a valid measure of fuzzy entropy.

C. Next, we propose another new parametric measure of fuzzy entropy of order α , given by:

$$\begin{aligned}
 {}_\alpha H(A) = & -\sum_{i=1}^n \left[\mu_A(x_i) \log \mu_A(x_i) \right. \\
 & \left. + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i)) \right] \\
 & + \frac{1}{1-\alpha} \sum_{i=1}^n \log \left[\frac{\mu_A^\alpha(x_i)}{+(1 - \mu_A(x_i))^\alpha} \right], \quad (7) \\
 & \alpha \neq 1, \alpha > 0
 \end{aligned}$$

Proceeding as above, we have verified that ${}_\alpha H(A)$ satisfies all the essential properties for being a measure of

fuzzy entropy. Hence, ${}_\alpha H(A)$ is a valid measure of fuzzy entropy.

Note: As $\alpha \rightarrow 1$, the measure (7) becomes

$${}_\alpha H(A) = 2H(A) \text{ where}$$

$$H(A) = -\sum_{i=1}^n \left[\mu_A(x_i) \log \mu_A(x_i) \right. \\
 \left. + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i)) \right]$$

The above is De Luca and Termini's [1] measure of fuzzy entropy.

III. MONOTONICITY OF THE PROPOSED MEASURES OF FUZZY ENTROPY

In this section, we study the monotonic character of the different measures of fuzzy entropy introduced in the above section.

A. Monotonicity of $H_\alpha(A)$

Differentiating equation (5) w.r.t α , we get

$$\begin{aligned}
 \frac{dH_\alpha(A)}{d\alpha} = & -\sum_{i=1}^n \left[\mu_A^\alpha(x_i) \{ \log \mu_A(x_i) \}^2 \right. \\
 & \left. + (1 - \mu_A(x_i))^\alpha \{ \log(1 - \mu_A(x_i)) \}^2 \right] \\
 & < 0
 \end{aligned}$$

which shows that the measure (5) is monotonically decreasing function of α .

Next, with the help of the data, we have presented the measure (5) graphically. For this purpose, we have computed different values of $H_\alpha(A)$ corresponding to different values of the parameter α and obtained the following Fig.2.

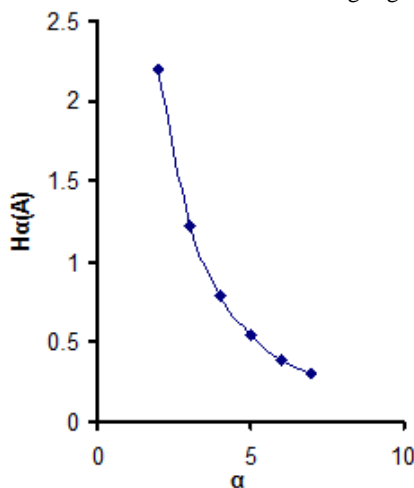


Fig.2 Graph $H_\alpha(A)$ Versus α

B. Monotonicity of $H^\alpha(A)$

Differentiating equation (6) w.r.t α , we get

$$\frac{dH^\alpha(A)}{d\alpha} = -\frac{2^{\alpha-1}}{1-\alpha} H_\alpha(A) + {}^1H_\alpha(A) \left\{ \frac{\log 2^{1-\alpha} + 1}{1-\alpha} \right\} 2^{\alpha-1} \quad (8)$$

where

$$H_\alpha(A) = -\sum_{i=1}^n \left[\begin{array}{l} \mu_A^\alpha(x_i) \log \mu_A(x_i) \\ + (1-\mu_A(x_i))^\alpha \\ \log(1-\mu_A(x_i)) \end{array} \right]$$

is a fuzzy entropy introduced in the above section and

$${}^1H_\alpha(A) = \frac{1}{1-\alpha} \sum_{i=1}^n \left[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha - 1 \right]$$

The above is a well-known measure of fuzzy entropy introduced by Kapur [6].

Now, since $H_\alpha(A) \geq 0$, the first term on the R.H.S. of (8) is non-negative for $\alpha > 1$.

To find the sign of second term, we proceed as follows:

We know ${}^1H_\alpha(A) \geq 0$

Also, we have $\log x - x + 1 \leq 0$

Thus $\frac{\{\log 2^{1-\alpha} + 1\}}{1-\alpha} 2^{\alpha-1} \geq 1$

This proves that the second term on the R.H.S. of equation (8) is non-negative. Hence, we have

$$\frac{dH^\alpha(A)}{d\alpha} \geq 0$$

Which proves that the measure (6) is monotonically increasing function of its parameter α . Next, with the help of the data, we have presented the measure (6) graphically and obtained the Fig.3.

C. Monotonicity of ${}_\alpha H(A)$

Differentiating equation (7) w.r.t α , we get

$$\{1-\alpha\}^2 \frac{d_\alpha H(A)}{d\alpha} = \sum_{i=1}^n \left[\frac{A_i}{B_i} \right] \quad (9)$$

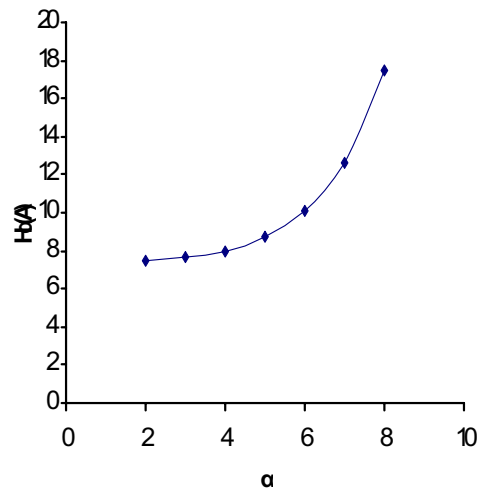


Fig.3 Graph $H_\alpha(A)$ Versus α

In the equation (9)

$$A_i = \mu_A^\alpha(x_i) \log \mu_A^{1-\alpha}(x_i) + (1-\mu_A(x_i))^\alpha \log(1-\mu_A(x_i))^{1-\alpha} + \{ \mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha \} \log [\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha] \quad (10)$$

and

$$B_i = \mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha \quad (11)$$

Let

$$f(x) = x^\alpha \log x^{1-\alpha} + (1-x)^\alpha \log(1-x)^{1-\alpha} + \{ x^\alpha + (1-x)^{1-\alpha} \} \log \{ x^\alpha + (1-x)^{1-\alpha} \}$$

Now, we have the following results:

- (i) $f(x) = f(1-x)$
- (ii) $f'(x) = -f'(1-x)$
- (iii) $f(0) = 0$ and
- (iv) $f(1) = 0$

Next, the function $f(x)$ can be written as

$$f(x) = \{ \lambda x_1 + (1-\lambda)x_2 \} \log \{ \lambda x_1 + (1-\lambda)x_2 \} - \lambda x_1 \log x_1 - (1-\lambda)x_2 \log x_2 \quad (12)$$

where

$$\lambda = x, \quad 1-\lambda = 1-x, \quad x_1 = x^{\alpha-1}, \quad x_2 = (1-x)^{\alpha-1}$$

Since $x \log x$ is a convex function of x , equation (12) is the sum of three convex functions and hence $f(x)$ is convex.

Thus, we must have $f(x) \leq 0, \quad 0 \leq x \leq 1$

Hence $f(\mu_A(x_i)) \leq 0 \quad \forall i$

Thus, the numerator of every term on the R.H.S. of equation

(9) is ≤ 0 and the denominator of every term is positive. Hence, from equation (9), we have

$$\frac{d}{d\alpha} {}_a H(A) \leq 0$$

Which proves that the measure introduced in (7) is monotonically decreasing function of its parameter α . Next, with the help of the data, we have presented the measure (7) graphically and obtained the Fig.4.

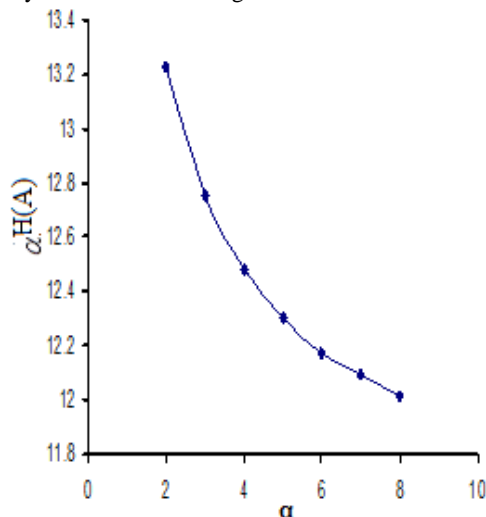


Fig.4 Graph ${}_a H(A)$ Versus α

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