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The Game of Col on Complete K-ary Trees

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***Abstract*—Col is a classic combinatorial game played on graphs and to solve a general instance is a $\mathcal{PSPACE}$-complete problem. However, winning strategies can be found for some specific graph instances. In this paper, the solution of Col on complete k-ary trees is presented.

***Keywords*—Combinatorial game, Complete k-ary tree, Map-coloring game.

I. INTRODUCTION

COL is a map-coloring games invented by Colin Vout [1], [2]. Every instance of these games is defined as an undirected graph $G = (V, E)$ where every vertex is uncolored, black or white. The two players, First and Second, play in turn and First starts the game. In the beginning, all the vertices are uncolored. First has to paint an uncolored vertex using the color black, and Second has to paint an uncolored vertex using the color white. There exists only one restriction: two adjacent vertices cannot be painted with the same color. The first player unable to paint an uncolored vertex is the loser.

The values of some Col positions and the description of some general rule for simplifying larger positions is presented in [1], [2]. Moreover, Col is proved to be a $\mathcal{PSPACE}$-complete problem on a general graph [3].

Here, the game is analyzed on complete k-ary trees where $k \geq 2$ and $d \geq 1$ is the depth of the tree.

II. SOLVING COL ON COMPLETE k-ARY TREES

Lemma 1: Let G be a complete k-ary tree of depth 1 with $k \geq 2$.

1) If k is even, then Second has a winning strategy without to be forced to paint the root.
2) If k is odd, then First has a winning strategy without to be forced to paint the root.

Proof: Trivial. ■

Theorem 1: Let G be the graph obtained by the union of n complete k-ary trees of depth 1 where $k \geq 2$ is equal for all the trees.

1) If k is even, then Second has a winning strategy in G without to be forced to paint any root.
2) If k is odd and n is odd, then First has a winning strategy in G without to be forced to paint any root.
3) If k is odd and n is even, then Second has a winning strategy in G without to be forced to paint any root.

Proof:

1) In the beginning, First will play in one of the n trees and Second will play in the same tree because by Lemma 1 he/she has a winning strategy without to be forced to paint the root. If First decides to play in one of the remaining $n - 1$ trees, then Second will do the same because, by induction hypothesis, he/she has a winning strategy in the remaining $n-1$ trees without to be forced to paint any root. An example is shown in Fig. 1.

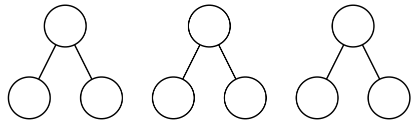

Fig. 1. An example with three complete binary trees where Second has a winning strategy.

2) By Lemma 1 First has a winning strategy without to be forced to paint the root if he/she play in any of the n trees. If Second decide to play in one of the remaining $n - 1$ trees then First will do the same because, by induction hypothesis, he/she has a winning strategy in the remaining $n - 1$ tree without to be forced to paint any root. An example is shown in Fig. 2.

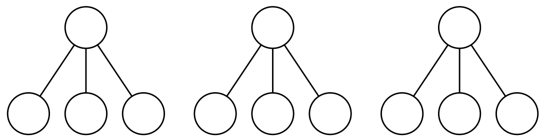

Fig. 2. An example with three complete ternary trees where First has a winning strategy.

3) Similar to the previous case.

■

Theorem 2: Let $G = (V, E)$ be the graph obtained by the union of n complete k-ary trees of depth 1 where $k \geq 2$ is equal for all the trees.

1) If k is even, then First has a winning strategy in $G' = (V \cup \{a\}, E)$ without to be forced to paint any root except a.
2) If k is odd and n is odd, then Second has a winning strategy in $G' = (V \cup \{a\}, E)$ without to be forced to paint any root except a.
3) If k is odd and n is even, then First has a winning strategy in $G' = (V \cup \{a\}, E)$ without to be forced to paint any root except a.

Proof:

1) If First plays in a then by Theorem 1 he/she has a winning strategy in G. An example is shown in Fig. 3.
2) If First plays in a, then Second has a winning strategy by Theorem 1. If First plays in one of the n trees then Second has a winning strategy in the remaining $n - 1$ trees and a by induction hypothesis. If First continues to move in the same tree where he/she played the first

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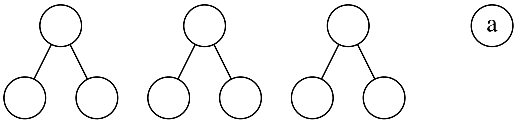

Fig. 3. An example with three complete binary trees and an extra vertex where First has a winning strategy.

time then Second has a winning strategy by Lemma 1. An example is shown in Fig. 4.

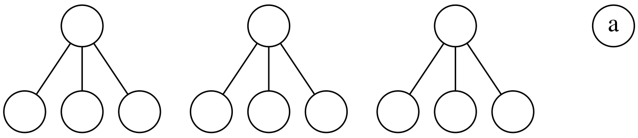

Fig. 4. An example with three complete ternary trees and an extra vertex where Second has a winning strategy.

3) Similar to the first case.

∎

Definition 1: Let G be a complete k-ary tree where $k \geq 2$ and $d \geq 1$ is the depth of the tree. We define $\sigma(G)$ as follows:

1) If d is odd, then $\sigma(G)$ is the graph obtained by G deleting all the edges which connects a vertex at depth s to a vertex at depth $s+1$ where s is odd. As a result, $\sigma(G)$ is the union of k-ary trees of depth 1. If k is odd and $d \equiv 1 \pmod 4$, then the number of trees is odd else if k is odd and $d \equiv 3 \pmod 4$, then the number of trees is even. Fig. 5 shows an example of $\sigma(G)$ where G is a complete binary tree of depth 3.

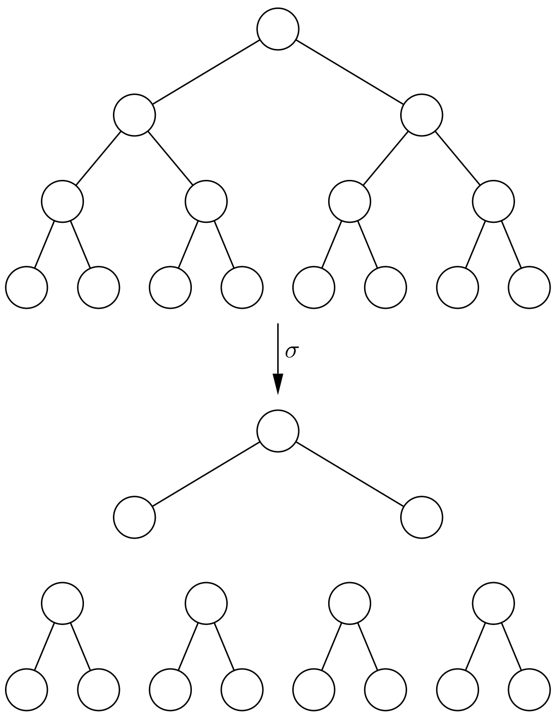

Fig. 5. An example of $\sigma(G)$ where G is a complete binary tree of depth 3.

2) If d is even, then $\sigma(G)$ is the graph obtained by G deleting all the edges which connects a vertex at depth s to a vertex at depth $s+1$ where s is even or 0. As a result, $\sigma(G)$ is the union of k-ary trees of depth 1 and a single vertex. If k is odd and $d \equiv 0 \pmod 4$, then the number of trees is even else if k is odd and $d \equiv 2 \pmod 4$, then the number of trees is odd. Fig. 6 shows an example of $\sigma(G)$ where G is a complete ternary tree of depth 2.

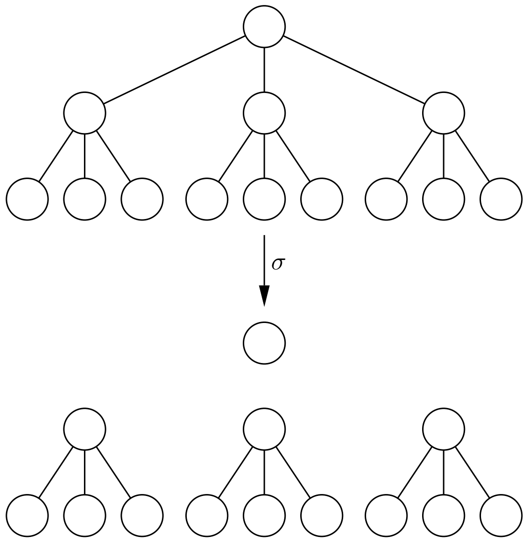

Fig. 6. An example of $\sigma(G)$ where G is a complete ternary tree of depth 2.

Theorem 3: Let G be a complete k-ary tree where $k \geq 2$ and $d \geq 1$ is the depth of the tree. First (Second) has a winning strategy in G if and only if he/she has a winning strategy in $\sigma(G)$.

Proof: If First (Second) has a winning strategy on $\sigma(G)$, then he/she can use exactly the same strategy in G because by Theorem 1 and Theorem 2 he/she will paint only the leaves of the trees (and the single vertex when d is even) in $\sigma(G)$ and these vertices are not connected each other in G.

Conversely, if First (Second) has a winning strategy in G and Second (First) has a winning strategy in $\sigma(G)$, then it is a contradiction because even Second (First), by the first part of this theorem, should have a winning strategy in G. ∎

Corollary 1: Let G be a k-ary tree where $k \geq 2$ and $d \geq 1$ is the depth of the tree.

1) If k is even and d is odd, then Second has a winning strategy.
2) If k is even and d is even, then First has a winning strategy.
3) if k is odd and $d \equiv 1 \pmod 4$, then First has a winning strategy.
4) if k is odd and $d \equiv 3 \pmod 4$, then Second has a winning strategy.
5) if k is odd and $d \equiv 0 \pmod 4$, then First has a winning strategy.
6) if k is odd and $d \equiv 2 \pmod 4$, then Second has a winning strategy.

Proof:

1) It follows by Theorem 1 (case 1) and Theorem 3.

2) It follows by Theorem 2 (case 1) and Theorem 3.
3) It follows by Theorem 1 (case 2) and Theorem 3.
4) It follows by Theorem 1 (case 3) and Theorem 3.
5) It follows by Theorem 2 (case 3) and Theorem 3.
6) It follows by Theorem 2 (case 2) and Theorem 3.

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REFERENCES

[1] E. R. Berlekamp, J. H. Conway, R. K. Guy, *Winning Ways For Your Mathematical Plays*, 2nd ed. Natick, Massachusetts: A K Peters, 2001, vol. 1, ch. 2, pp. 37-39.
[2] J. H. Conway, *On Numbers and Games*, 2nd ed. Natick, Massachusetts: A K Peters, 2001, ch. 8, pp. 91-95.
[3] T. J. Schaefer, "On the complexity of some two-person perfect information games," *J. Comput. Systems Sci.*, vol. 16, pp. 185-225, 1978.