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A New Iterative Method for Solving Nonlinear Equations

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***Abstract*—** In this study, a new root-finding method for solving nonlinear equations is proposed. This method requires two starting values that do not necessarily bracketing a root. However, when the starting values are selected to be close to a root, the proposed method converges to the root quicker than the secant method. Another advantage over all iterative methods is that; the proposed method usually converges to two distinct roots when the given function has more than one root, that is, the odd iterations of this new technique converge to a root and the even iterations converge to another root. Some numerical examples, including a sine-polynomial equation, are solved by using the proposed method and compared with results obtained by the secant method; perfect agreements are found.

***Keywords*—** Iterative method, root-finding method, sine-polynomial equations, nonlinear equations.

I. Formulation of the Problem

ONE classical problem in numerical analysis is the solution of nonlinear equations $f(x)=0$. To approximate a solution to one of these equations we can use iterative methods. An iterative method starts from two initial guesses x_0 and x_1, which are then improved by means of a sequence $\{x_{k+1}\}_{k=1}^{\infty}=\Phi(x_{k-1},x_k)$, $k\ge 1$, is known as a two-point iterative method. Conditions are imposed on x_0, x_1 and, eventually, on f or Φ or both, in order to ensure the convergence of the sequence to $\{x_{k+1}\}$ for $k\ge 1$ to a solution α of the nonlinear equation $f(x)=0$, then proceed to find the order of convergence of the sequence.

In this paper, we shall study a new iterative approach that requires two starting values, but the order of convergence and the convergence criterion of the proposed method will be put off to a later extended work.

Assume that $f(x)$, $f'(x)$ and $f''(x)$ are continuous near a root α, then the graphs of the functions $f(x)$ and $-f(x)$ intersect the x-axis at the point $(\alpha,0)$. Furthermore, assume that initial approximations x_0 and x_1 are near the root α and $x_0\neq x_1$, then the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ lie on the curve of $y=f(x)$ near the point $(\alpha,0)$, see Figure 1. The information regarding the nature of $f(x)$ and $-f(x)$ can be used to develop an algorithm that will produce a sequence $\{x_{k+1}\}$ that converges to the root α. We shall first introduce the algorithm of this sequence graphically and then a more rigorous treatment based on the Taylor series will be given. Now, define x_2 to be the x-coordinate of the point where the straight line L_1 intersects the graph of $y=f(x)$, where L_1 is the line that emanates from the point $(x_0,0)$ and passes through the point $(x_1,-f(x_1))$. And x_3 is defined as the x-coordinate of the point where the line L_2 intersects the graph of $y=f(x)$, where L_2 is the line that emanates from $(x_1,0)$ and passes through the point $(x_2,-f(x_2))$. Now, referring to figure 1 the slopes m_1 and m_2 of the lines L_1 and L_2 can be written, respectively, as

$$
\begin{aligned}
m_1 &= \tan\theta = \frac{-f(x_1)}{(x_0-x_1)} = \frac{f(x_2)}{(x_0-x_2)},\\
m_2 &= \tan\beta = \frac{-f(x_2)}{(x_2-x_1)} = \frac{f(x_3)}{(x_3-x_1)}.
\end{aligned} \tag{1}
$$

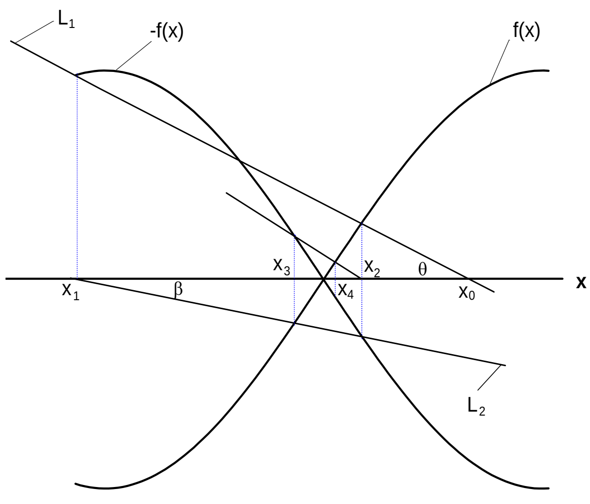

Figure 1: Geometric construcrtion of the proposed method.

To complete the graphical description of the proposed method, we need to apply the backward and forward finite divided difference formulas at x_1 which can be written, respectively, as [1]

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$$f'(x_0) \cong \frac{f(x_0) - f(x_2)}{(x_0 - x_2)}, \quad f'(x_1) \cong \frac{f(x_3) - f(x_1)}{(x_3 - x_1)}. \tag{2}$$

Now substituting (1) into (2) and rearranging the resultant equations we can, respectively, write

$$x_2 = x_0 - \frac{f(x_0)(x_1 - x_0)}{f(x_1) + f'(x_0)(x_1 - x_0)}, \quad x_3 = x_1 - \frac{f(x_1)(x_2 - x_1)}{f(x_2) + f'(x_1)(x_2 - x_1)}. \tag{3}$$

Obviously, figure 1 shows that the new estimates x_2 and x_3 are closer to the root α than the initial guesses x_0 and x_1. The process discussed above can be repeated to obtain a sequence $\{x_{k+1}\}$ that converges to the root α. The following theorem summarizes the above numerical scheme when x_{k-1} is used in place of x_0 in the first equation of (3) or when x_{k-1} is used in place of x_1 in the second equation of (3).

Theorem (The Proposed Method)

Assume that $f \in C^2[a,b]$ and there exists a number $\alpha \in [a,b]$ where $f(\alpha) = 0$. Then there exists a real number $\varepsilon > 0$ such that the sequence $\{x_{k+1}\}_{k=1}^{\infty}$ defined by the two-point iteration formula for

$$x_{k+1} = x_{k-1} - \frac{f(x_{k-1})(x_k - x_{k-1})}{f(x_k) + f'(x_{k-1})(x_k - x_{k-1})}, \tag{4}$$

will converge to α for the initial guesses $(x_0, x_1) \in [\alpha - \varepsilon, \alpha + \varepsilon]$. If there exist numbers (α_1 and $\alpha_2) \in [a,b]$ where both $f(\alpha_1) = 0$ and $f(\alpha_2) = 0$, then (4) may converge to α_1 when k is odd and it may converge to α_2 when k is even, or visa versa.

Proof: The geometric descriptions of finding the new estimates x_2 and x_3 are shown in Figure 1, but this graphical approach does not help us in understanding why the continuity of $f''(x)$ is essential and why we need x_0 and x_1 to be close to the root α ? Our proof starts with the Taylor polynomial of degree one and its remainder term:

$$f(x) = f(x_{k-1}) + f'(x_{k-1})(x - x_{k-1}) + \frac{f''(\xi)(x - x_{k-1})^2}{2!}, \tag{5}$$

where $x \le \xi \le x_{k-1}$. Using the fact that, the right triangle with vertices x_{k-1}, x_{k+1}, $f(x_{k+1})$ is similar to the right triangle with vertices x_{k-1}, x_k, $-f(x_k)$, that is:

$$f(x_{k+1})(x_{k-1} - x_k) + (x_{k-1} - x_{k+1}) f(x_k) = 0. \tag{6}$$

Now, setting $x = x_{k+1}$ in (5) and substituting the resultant equation into (6), the latest equation can be rearranged as:

$$(x_{k+1} - x_{k-1})[-f'(x_{k-1})(x_k - x_{k-1}) - f(x_k)] = (x_k - x_{k-1}) f(x_{k-1}) + \frac{f''(\xi)(x_{k+1} - x_{k-1})^2}{2}(x_k - x_{k-1}),$$

which can be reduced to:

$$x_{k+1} = x_{k-1} - \frac{f(x_{k-1})(x_k - x_{k-1})}{f(x_k) + f'(x_{k-1})(x_k - x_{k-1})} - R, \tag{7}$$

where

$$R = \frac{f''(\xi)(x_{k+1} - x_{k-1})^2 (x_k - x_{k-1})}{2[f'(x_{k-1})(x_k - x_{k-1}) + f(x_k)]}.$$

Thus, if x_{k+1} is enough close to x_{k-1}, the last term on the right side of (7) will be small compared to the other terms of that equation. Hence, the remainder term (R) can be neglected and we can see that the general iterative equation (4) is established. Furthermore, equation (7) shows that the continuity of $f'(x)$ and $f''(x)$ are essential. For most root-finding applications this is all that needs to be understood. However, in an extended copy of this paper our aim will be to prove the other part of the above theorem and to study the conditions needed for convergence and to study the speed of convergence of the iterative equation proposed in (4).

II. Numerical Examples and Discussion

In this study, two numerical examples will be solved by using the proposed method, (4), and they will be compared with the secant method. The secant method is selected here, because both the proposed and secant method require two initial guesses to start the iterative process. In these examples, when equation (4) is used then the absolute relative percent error is determined according to [2] as

$$|\varepsilon_c| = \left| \frac{x_{k+1} - x_{k-1}}{x_{k+1}} \right| 100\%. \tag{8}$$

The nominator of (8) represents the difference between the current approximation and the approximation before the previous one, because the two-point iterative formula (4) sometimes converges to two different roots**.** This is due to the fact that, the first term presented on the right hand side of equation (4) does not refer to the previous estimate but it refers to the estimate before it. However, in all iterative root-finding methods this term is referred directly to the previous estimate. This main advantage of the proposed method will be seen clearly in studying the following numerical examples.

A. Example 1

In the first example, our aim is to find the roots of

TABLE I
COMPARISON BETWEEN THE PROPOSED AND THE SECANT METHOD FOR THE FIRST EXAMPLE

k	Secant method with $x_0 = 2$ and $x_1 = -1$	$\lvert\varepsilon_c\rvert = \left\lvert\frac{x_{k+1} - x_k}{x_{k+1}}\right\rvert 100\%$	Proposed method with $x_0 = 2$ and $x_1 = -1$	$\lvert\varepsilon_c\rvert$, equation (8)
1	-4.99043285366	79.961658050068	0.765004156887	161.43648790
2	-.854271287101	484.17424640371	-0.600632138116	66.491257550
⋮	⋮	⋮	⋮	⋮
11	-.458962267537	0.0002E-10	.9100075724887089	0.08604E-10
12	-.458962267537	0.0	-.458962267536948	0.00022E-10
13			.9100075724887092	0.00033E-10
14			-.458962267536948	0.0

$f(x) = e^x - 3x^2$. This nonlinear equation has three real simple roots; one is near -0.5, the other is near 1.0, while the last one is near 4.0. This problem is solved by both the proposed and secant methods for two different initial guesses, Tables I and II. The numerical results given in these tables are considered to be correct to at least 12 decimals.

TABLE II
COMPARISON BETWEEN THE PROPOSED AND THE SECANT METHODS FOR THE FIRST EXAMPLE

k	Secant method with $x_0 = 4$ and $x_1 = 1$	$\lvert\varepsilon_c\rvert = \left\lvert\frac{x_{k+1} - x_k}{x_{k+1}}\right\rvert 100\%$	Proposed method with $x_0 = 4$ and $x_1 = 1$	$\lvert\varepsilon_c\rvert$, equation (8)
1	1.12284457921	10.9404793401	3.785020918321	5.67973298743
2	0.91895886495	22.18659855605	0.9029237302942	10.7513255493
⋮	⋮	⋮	⋮	⋮
9	0.910007572489	0.0	3.733079028633	0.630927E-9
10			0.910007572489	0.0
11			3.7330790286328	0.0268E-12

B. *Example 2*

Definition A sine polynomial equation $f = f(x, \sin x)$ is an analytic function which can be considered as a polynomial with real coefficients depending on the variables x and the function $\sin x$,[3].

Let us remark that a sine-polynomial is defined on the real axis and the set of its roots is discrete. Moreover, there are only a finite number of roots in a bounded interval. Computer algebra systems like MATLAB or MAPLE provide the numerical equation solvers *fzero* and *fsolve*, respectively. However, in general these solvers do not return all solutions of a sine-polynomial equation. For example, the solver *fzero* finds a zero of the function which is near x_0 and the solver fsolve find the required root at a specified interval. For example, the sine-polynomial equation:

$$\begin{aligned} f(x) = -2x^2 \sin^5 x - (x^5 - 2 + 4x^2) \sin^3 x + \\ (2x^5 - x - 2x^4) \sin^2 x + \\ (3x + 2x^4 - 4 - x^2 - 4x^5) \sin x + \\ \left(6 + 8x^5 - 4x^4 - 7x^3 - x + 5x^2\right) = 0, \end{aligned} \tag{9}$$

has three real simple roots; If the solution is required at a specified interval then the *fsolve* in MAPLE 7.0 can find them as:

```
>fsolve(f(x), x, x=-4..-1); -1.11285139990809
> fsolve(f(x), x, x=-1..1.5); 1.0191074744305
> fsolve(f(x), x, x=1.5..4); 1.63954852796097
```

This problem has been solved in [3], where an algorithm that determines whether a sine-polynomial has a finite number of real roots or not is proposed. Now let us solve this problem by the proposed method and compare the solutions with that of the secant method. The numerical results obtained by these two methods for two different initial guesses are shown in Tables III and IV.

TABLE III
COMPARISON BETWEEN THE PROPOSED AND THE SECANT METHOD FOR THE SECOND EXAMPLE

k	Secant method with $x_0 = 2.5$ and $x_1 = -2$	$\lvert\varepsilon_c\rvert = \left\lvert\frac{x_{k+1} - x_k}{x_{k+1}}\right\rvert 100\%$	Proposed method with $x_0 = 2.5$ and $x_1 = -2$	$\lvert\varepsilon_c\rvert$, equation (8)
1	.4641080950723	530.93409083683	2.1900968237884	14.1502044
2	.4392960717554	5.6481322989649	-1.5957687947762	25.33143941
⋮	⋮	⋮	⋮	⋮
17	1.003294260910	27.750187319251	1.6395485279609	0.5489E-12
18	1.018440421454	1.4871916142650	-1.1128513999081	0.0
19	1.019115216293	0.0662137929575	1.6395485279609	0.0
⋮	⋮	⋮	⋮	⋮
22	1.019107474430	0.119E-10		
23	1.019107474430	0.0		

TABLE IV
COMPARISON BETWEEN THE PROPOSED AND THE SECANT METHOD FOR THE SECOND EXAMPLE

k	Secant method with $x_0 = -0.5$ and $x_1 = -1$	$\lvert\varepsilon_c\rvert = \left\lvert\frac{x_{k+1} - x_k}{x_{k+1}}\right\rvert 100\%$	Proposed method with $x_0 = -0.5$ and $x_1 = -1$	$\lvert\varepsilon_c\rvert$, equation (8)
1	-2.46254948889	59.391679050109	-.1231564900349	305.98753655479
2	-1.01049912303	143.69635091830	-1.139338133525	12.229743692853
⋮	⋮	⋮	⋮	⋮
10	-1.11285139991	0.88871E-10	-1.112851399798	0.331859462E-4
11	-1.11285139991	0.0	1.019118235571	0.3808372354441
12			-1.112851399908	0.98952E-8
13			1.019107474512	0.001055929784
14			-1.112851399908	0.0
15			1.019107474431	0.8004161E-8
16			-1.112851399908	0.0
17			1.019107474431	0.0

The results given in Tables I to IV show that: the proposed method converges to two different roots when the given function has more than one root, while this is not the usual case of other iterative methods. Thus, when the proposed method converges to two different roots it needs almost half the number of iterations that are needed for the secant method. For example, as seen in Table V, for the same starting values x_0 and x_1, the secant method needs 18 iterations to converge to the negative root of the equation given in example 1, while the proposed method needs 22 iterations to get the other two roots correct to at least 500 decimal places. On the other hand, to get the results presented in Table III correct to 500 decimal places, the secant method requires 30 iterations, while the proposed method needs only 28 iterations to catch two roots of the sine-polynomial function given in (9), see Table V. Another important advantage of the proposed method is that the denominator of the ratio term presented in equation (4) becomes zero if and only if $f(x_k) + f'(x_{k-1})(x_k - x_{k-1}) = 0$, which is not an easy mission, specially when the slope of the secant line connecting the starting points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is zero or approaches zero, or when the slope of the secant line is oscillated around a local maximum or minimum point. Finally, we further note that the initial guesses x_0 and x_1 are not required to bracket one or two roots of a given nonlinear equation, see Table IV. However, we note that when the initial guesses bracket roots, the proposed method work well.

In this paper, a new iterative method for solving nonlinear equations is proposed. The main advantage of the proposed method is its convergence to two different roots when the problem has more than one root. The order and speed of convergence of this method will be discussed in another extended work.

TABLE V
COMPARISON BETWEEN THE NUMBERS OF ITERATIONS NEEDE TO GET THE RESULTS PRESENTED IN TABLES I TO IV CORRECT TO AT LEAST 500 DECIMAL PLACES

	One root by secant method.	Two roots by the proposed method.
Table I	18	22
Table II	15	20
Table III	30	28
Table IV	18	26

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