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The Panpositionable Hamiltonicity of k-ary n-cubes

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***Abstract*—The hypercube Q_n is one of the most well-known and popular interconnection networks and the k-ary n-cube Q_n^k is an enlarged family from Q_n that keeps many pleasing properties from hypercubes. In this article, we study the *panpositionable hamiltonicity* of Q_n^k for $k \geq 3$ and $n \geq 2$. Let x, y of $V(Q_n^k)$ be two arbitrary vertices and $\mathcal{C}$ be a hamiltonian cycle of Q_n^k. We use $d_{\mathcal{C}}(x, y)$ to denote the distance between x and y on the hamiltonian cycle $\mathcal{C}$. Define l as an integer satisfying $d(x, y) \leq l \leq \frac{1}{2}|V(Q_n^k)|$. We prove the followings:**

- **When $k = 3$ and $n \geq 2$, there exists a hamiltonian cycle C of Q_n^k such that $d_C(x, y) = l$.**
- **When $k \geq 5$ is odd and $n \geq 2$, we request that $l \notin S$ where S is a set of specific integers. Then there exists a hamiltonian cycle C of Q_n^k such that $d_C(x, y) = l$.**
- **When $k \geq 4$ is even and $n \geq 2$, we request $l - d(x, y)$ to be even. Then there exists a hamiltonian cycle C of Q_n^k such that $d_C(x, y) = l$.**

The result is optimal since the restrictions on l is due to the structure of Q_n^k by definition.

***Index Terms*—Hamiltonian, panpositionable, bipanpositionable, k-ary n-cube.**

I. Introduction

THE n-dimensional hypercube Q_n is one of the most well-known and popular interconnection networks due to its excellent properties as the following: it is vertex-symmetric and edge-symmetric; it is hamiltonian; it allows cycle/path embedding when faults occur and so on. (See [1], [2] for these results and their references). Therefore, numerous studies have been devoted to the hypercube family [3]–[6], [11], [12].

The k-ary n-cube Q_n^k is an enlarged family from Q_n that keeps many pleasing properties from hypercubes. More precisely, each vertex of Q_n^k is labeled by a n-bit finite sequence $(u_0, u_1, ..., u_{n-1})$, where $0 \leq u_i \leq k-1$ for $0 \leq i \leq n-1$, and every two vertices u and v are adjacent if and only if $|u_i - v_i| = 1$ or $k-1$ for some i and $u_j = v_j$ for any $0 \leq j \leq n-1$ with $j \neq i$. It is obviously that the hypercube Q_n is indeed a subclass of the k-ary n-cube when $k = 2$. Some properties of Q_n^k mentioned in [6] are listed here: it is known that Q_n^k is vertex-symmetric and edge-symmetric [3]; it is hamiltonian [4], [5]; it has diameter $n\lfloor \frac{k}{2} \rfloor$ [4], [5]; it has a recursive structure; and it contains many important interconnection networks such as cycles (of certain lengths) [3], meshes (of certain dimensions) [4], and even hypercubes (of certain dimensions) [5]. However, as opposed to Q_n, Q_n^k has not received enough attention. In this article, we want to prove the panpositionability of Q_n^k. Readers can refer to [7] for the concept of panpositionability. A hamiltonian graph G is *panpositionable* if for any two different vertices u and v of G and any integer l with $d_G(u, v) \leq l \leq \frac{|V(G)|}{2}$, there exists a hamiltonian cycle C of G with $d_C(u, v) = l$. Similar to the hamiltonicity for the communication between processors in an interconnection network, panpositionable hamiltonicity allows more flexible communication in a hamiltonian network. It is easy to see that the panpositionable hamiltonian property inherits the hamiltonian property and advances it further [8].

The article is organized as follows. In Section 2, we introduce the graph terminologies and notations used in this paper, the precise definition of Q_n^k, and two lemmas. In Section 3, we study the panpositionability of Q_n^k, where $k \geq 3$ is an odd integer and $n \geq 2$ is an integer. In Section 4, we study the panpositionability in the bipartite version of Q_n^k, where $k \geq 4$ is an even integer and $n \geq 2$ is an integer. Our conclusion is given in the last section.

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II. Preliminaries

For the graph definitions and notations we follow [9]. $G = (V, E)$ is a *graph* if V is a finite set and E is a subset of $\{\{u, v\} | \{u, v\}$ is an unordered pair of $V\}$. We say that V is the *vertex set* and E is the *edge set* of G. Two vertices u and v are *adjacent* if $\{u, v\} \in E$. A *path* is represented by a finite sequence of vertices $\langle v_0, v_1, v_2, ..., v_n \rangle$, where every two consecutive vertices are adjacent. If P is a path represented by $\langle v_0, v_1, v_2, ..., v_n \rangle$, then we define $\text{inv}(P) = \langle v_n, v_{n-1}, v_{n-2}, ..., v_0 \rangle$. The *length* of a path P is the number of edges in P. We write the path $\langle v_0, v_1, ..., v_n \rangle$ as $\langle v_0, v_1, ..., v_{s-1}, P_1, v_{i+1}, ..., v_{j-1}, P_2, v_{t+1}, ..., v_n \rangle$, where $P_1 = \langle v_s, v_{s+1}, ..., v_i \rangle$ and $P_2 = \langle v_j, v_{j+1}, ..., v_t \rangle$. We use $d_G(u, v)$ to denote the distance between u and v in G, i.e., the length of the shortest path between u and v in G. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of G is a cycle that visits every vertex of G exactly once. We use $d_C(u, v)$ to denote the distance between u and v in a cycle C of G, i.e., the length of the shorter path between u and v in C. A *hamiltonian graph* is a graph with a hamiltonian cycle.

A *hamiltonian path* in a graph G is a path joining two distinct vertices u and v of G that visits every vertex of G exactly once. A graph G is *hamiltonian-connected* if there is a hamiltonian path joining any two distinct vertices of G. Note that any (nontrivial) bipartite graph cannot be hamiltonian-connected, whereas a bipartite graph is *hamiltonian laceable* if there exists a hamiltonian path joining every two vertices which are in distinct partite [10].

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The concept of hamiltonian panpositionability was first proposed by S. Kao etc. [7]. A hamiltonian graph G is *panpositionable* if for any two different vertices u and v of G and any integer l with $d_G(u,v) \le l \le \frac{|V(G)|}{2}$, there exists a hamiltonian cycle C of G with $d_C(u,v) = l$. A graph $G = (V_0 \cup V_1, E)$ is *bipartite* if $V(G) = V_0 \cup V_1$ and $E(G)$ is a subset of $\{\{u,v\} | u \in V_0, v \in V_1\}$. A hamiltonian bipartite graph G is *bipanpositionable* if for any two different vertices u and v of G and any integer l with $d_G(u,v) \le l \le \frac{|V(G)|}{2}$ and $(l - d_G(u,v))$ is even, there exists a hamiltonian cycle C of G with $d_C(u,v) = l$.

The k-ary n-cube, Q_n^k, is defined for all integers $k \ge 2$ and $n \ge 1$. The subclass Q_n^2 is the well-studied hypercube family. The subclass Q_1^k with $k \ge 3$ is defined as the cycle of length k. The k-ary n-cube, Q_n^k, for $k \ge 3$ and $n \ge 2$ is defined as follows. Let $u \in V(Q_n^k)$ be represented by $(u_0, u_1, ..., u_{n-1})$, where $0 \le u_i \le k-1$. u and v are adjacent if and only if $|u_i - v_i| = 1$ or $k-1$ for some i and $u_j = v_j$ for any $0 \le j \le n-1$ with $j \ne i$. It is shown that Q_n^k is bipartite if k is even [11]. Here we mention some properties of Q_n^k that will be used in this article.

It is known that Q_n^k is *vertex-symmetric* and *edge-symmetric*. Moreover, given any two distinct vertices (u_1, u_2) and (v_1, v_2) of Q_2^k, there is an automorphism of Q_2^k mapping (u_1, u_2) and (v_1, v_2) to $(m, 0)$ and $(0, n)$. Each vertex of Q_n^k is represented by a n-bit tuple, and we will call the dth-bit *the dth dimension*. We can *partition* Q_n^k *over dimension* d by fixing the dth element of any vertex tuple at some value a, for every $a \in \{0, 1, ..., k-1\}$. This results in k copies $Q_{d,n-1}^{k,0}, Q_{d,n-1}^{k,1}, ..., Q_{d,n-1}^{k,k-1}$ of Q_{n-1}^k, with corresponding vertices in $Q_{d,n-1}^{k,0}, Q_{d,n-1}^{k,1}, ..., Q_{d,n-1}^{k,k-1}$ joined in a cycle of length k (in dimension d) [6]. It is proven in [11], [12] that Q_n^k is hamiltonian connected for odd k and Q_n^k is hamiltonian laceable for even k.

Note that the length of a path between u and v in Q_n^k, where $k \ge 5$ is an odd integer, can not be arbitrary. For example, in Q_2^5, for any two vertices u and v and $d(u,v) = 1$, there exists no path P between u and v with $|P| = 2$. In fact, we have the following observation. Given two vertices $u = (u_0, u_1, ..., u_{n-1})$ and $v = (v_0, v_1, ..., v_{n-1})$ of Q_n^k. Define the number $m_i = \min\{|u_i - v_i|, k - |u_i - v_i|\}$, where $0 \le i \le n-1$. Let $s = \max\{m_i : 0 \le i \le n-1\}$. Then there exists no path between u and v with length $r = d(u,v) - s + k - s - 2l = d(u,v) + k - 2s - 2l$, where l is an integer and $1 \le l \le \frac{k}{2} - s$. Consequently, we modify the concept of panpositionability of Q_n^k by saying that Q_n^k is *nearly-panpositionable* if for any two distinct vertices x and y of Q_n^k and for any integer l' with $d(x,y) \le l' \le \frac{|V(Q_n^k)|}{2}$ and $l' \notin \{r : r = d(u,v) + k - 2s - 2l$ for $1 \le l \le \frac{k}{2} - s\}$, there exists a hamiltonian cycle C of Q_n^k with $d_C(x,y) = l'$. Therefore, in this article, we want to prove that Q_n^3 is panpositionable, Q_n^k is nearly-panpositionable if $k \ge 5$ is an odd integer, and is bipanpositionable if $k \ge 4$ is an even integer. First of all, we prove the following two lemmas.

Lemma 1. *Let k be an integer with $k \ge 3$. For any path P with length 2 in Q_2^k, there exists a hamiltonian cycle of Q_2^k that contains P.*

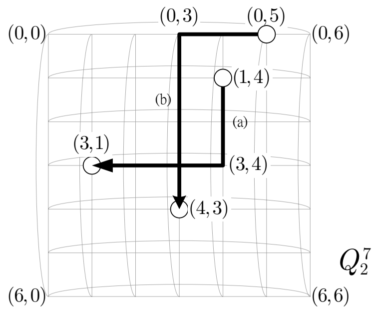

Fig. 1. (a) $f_{-3}^2(1,4)$ and (b) $h_{-2}^4(0,5)$.

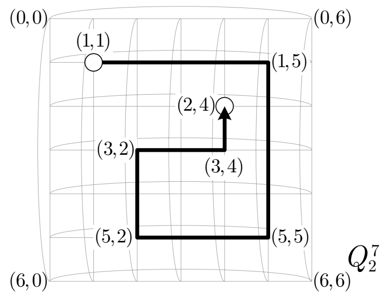

Fig. 2. $H_{\vec{b},3}^{\vec{a}}(1,1)$, where $\vec{a} = (4,-2,-1)$ and $\vec{b} = (4,-3,2)$.

Proof: Let c, r, i be nonzero integers, $\frac{c}{|c|} = s$, $\frac{r}{|r|} = t$, $\vec{a} = (a_1, a_2, ..., a_i)$ and $\vec{b} = (b_1, b_2, ..., b_i)$. If $c = 0$, then $s = 0$. Similarly, if $r = 0$, then $t = 0$. To construct the required hamiltonian cycles, we define some path patterns in the following.

$f_r^c(x,y) = \langle (x,y), (x + s \cdot 1, y), (x + s \cdot 2, y), ..., (x + c, y), (x + c, y + t \cdot 1), (x + c, y + t \cdot 2), ..., (x + c, y + r) \rangle$;

$h_r^c(x,y) = \langle f_r^0(x,y), f_0^c(x, y + r) \rangle$;

$H_{\vec{b},i}^{\vec{a}}(x,y) = \langle h_{b_1}^{a_1}(x,y), h_{b_2}^{a_2}(x + a_1, y + b_1), h_{b_3}^{a_3}(x + a_1 + a_2, y + b_1 + b_2), ..., h_{b_i}^{a_i}(x + \sum_{n=1}^{i-1} a_n, y + \sum_{n=1}^{i-1} b_n) \rangle$.

Please see Fig. 1 and Fig. 2 for an illustration. Fig. 1 is examples of $f_{-3}^2(1,4)$ and $h_{-2}^4(0,5)$. Note that $f_{-3}^2(1,4) = \langle (1,4), (2,4), (3,4), (3,3), (3,2), (3,1) \rangle$ and $h_{-2}^4(0,5) = \langle f_{-2}^0(0,5), f_0^4(0,3) \rangle = \langle (0,5), (0,4), (0,3), (1,3), (2,3), (3,3), (4,3) \rangle$. Fig. 2 is an example of $H_{\vec{b},3}^{\vec{a}}(1,1)$, where $\vec{a} = (4, -2, -1)$ and $\vec{b} = (4, -3, 2)$. Note that

$H_{\vec{b},3}^{\vec{a}}(1,1) = \langle h_4^4(1,1), h_{-3}^{-2}(5,5), h_2^{-1}(3,2) \rangle$
$= \langle f_4^0(1,1), f_0^4(1,5), f_{-3}^0(5,5), f_0^{-2}(5,2), f_2^0(3,2), f_0^{-1}(3,4) \rangle$
$= \langle (1,1), (1,2), (1,3), (1,4), (1,5), (2,5), (3,5), (4,5), (5,5), (5,4), (5,3), (5,2), (4,2), (3,2), (3,3), (3,4), (2,4) \rangle$.

Let $P = \langle u, x, v \rangle$, where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in Q_2^k. We have following cases.

Case 1. k is odd.

Case 1.1. Either $u_1 = v_1$ or $u_2 = v_2$. W.L.O.G., let $u = (0,0)$, $v = (2,0)$ and $P = \langle u, (1,0), v \rangle$.
Let $a_i = (-1)^i(2-k)$, for $i \le k-1$ and $a_k = 0$; $\vec{b} = (0, -1, -1, ..., -1)$. There exists a hamiltonian cycle $C = \langle (0,0), P, (2,0), f_{k-1}^{k-3}(2,0), H_{\vec{b},k}^{\vec{a}}(0, k -$

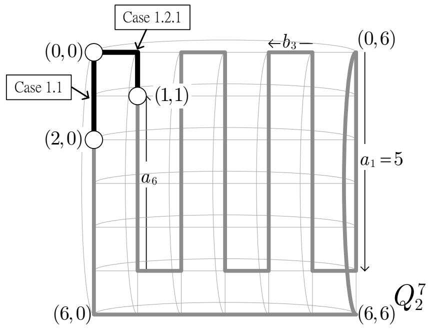

Fig. 3. Examples of Case 1.1 and Case 1.2.1 for $k = 7$.

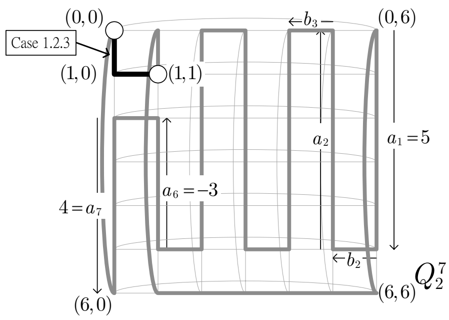

Fig. 4. An Example of Case 1.2.2 for $k = 7$.

1), (0,0)⟩. Please see Fig. 3 for an illustration. The hamiltonian cycle in Fig. 3 is $C = \langle(0,0), P, (2,0), f_6^4(2,0), H^{\vec{a}}_{\vec{b},7}(0,6), (0,0)\rangle$ and $H^{\vec{a}}_{\vec{b},7}(0,6) = \langle h_0^5(0,6), h_{-1}^{-5}(5,6), h_{-1}^{5}(0,5), h_{-1}^{-5}(5,4), h_{-1}^{5}(0,3), h_{-1}^{-5}(5,2), h_{-1}^{0}(0,1)\rangle$.

Case 1.2. $u_1 \neq v_1, u_2 \neq v_2$. W.L.O.G., let $u = (0,0)$ and $v = (1,1)$.
Case 1.2.1. $P = \langle u, (0,1), v\rangle$, where $k \geq 3$.
The hamiltonian cycle is the same as in Case 1.1. Please see Fig. 3 for an illustration.
Case 1.2.2. $P = \langle u, (1,0), v\rangle$, where $k = 3$.
The hamiltonian cycle is $C = \langle(0,0), P, (1,1), f_1^{-1}(1,1), f_{-2}^{2}(0,2), (2,0), (0,0)\rangle$.
Case 1.2.3. $P = \langle u, (1,0), v\rangle$, where $k \geq 5$.
Let $a_i = (-1)^i(2-k)$, for $i \leq k-2$, $a_{k-1} = 4-k$ and $a_k = k-3$; $\vec{b} = (0,-1,-1,...,-1)$. There exists a hamiltonian cycle $C = \langle(0,0), P, (0,1), f_{k-2}^{0}(k-1,1), H^{\vec{a}}_{\vec{b},k}(0,k-1), (0,0)\rangle$. Please see Fig. 4 for an illustration. The hamiltonian cycle in Fig. 4 is $C = \langle(0,0), P, (0,1), f_5^0(6,1), H^{\vec{a}}_{\vec{b},7}(0,6), (0,0)\rangle$ and $H^{\vec{a}}_{\vec{b},7}(0,6) = \langle h_0^5(0,6), h_{-1}^{-5}(5,6), h_{-1}^{5}(0,5), h_{-1}^{-5}(5,4), h_{-1}^{5}(0,3), h_{-1}^{-3}(5,2), h_{-1}^{4}(2,1)\rangle$.

Case 2. k is even.
Case 2.1. Either $u_1 = v_1$ or $u_2 = v_2$. W.L.O.G., let $u = (0,0)$ and $v = (2,0)$ and $P = \langle u, (1,0), v\rangle$.
Let $a_i = (-1)^i(2-k)$, for $3 \leq i \leq k$, $a_1 = k-3$, $a_2 = 1-k$ and $a_{k+1} = 0$;

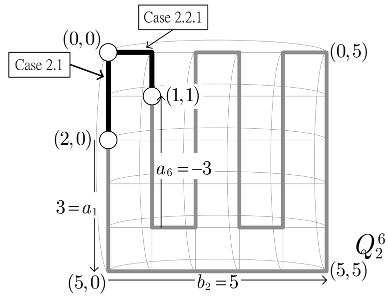

Fig. 5. Examples of Case 2.1 and Case 2.2.1 for $k = 6$.

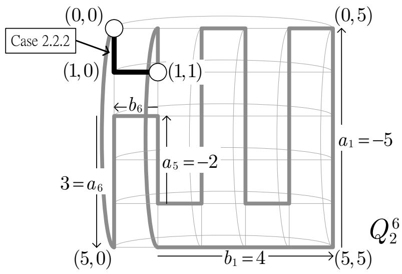

Fig. 6. An Example of Case 2.2.2 for $k = 6$.

$\vec{b} = (0, k-1, -1, -1, ..., -1)$. There exists a hamiltonian cycle $C = \langle(0,0), P, (2,0), H^{\vec{a}}_{\vec{b},k+1}(2,0), (0,0)\rangle$. Please see Fig. 5 for an illustration. The hamiltonian cycle in Fig. 5 is $C = \langle(0,0), P, (2,0), H^{\vec{a}}_{\vec{b},7}(2,0), (0,0)\rangle$ and $H^{\vec{a}}_{\vec{b},7}(2,0) = \langle h_0^3(2,0), h_5^{-5}(5,0), h_{-1}^{4}(0,5), h_{-1}^{-4}(4,4), h_{-1}^{4}(0,3), h_{-1}^{-4}(4,2), h_{-1}^{0}(0,1)\rangle$.

Case 2.2. $u_1 \neq v_1, u_2 \neq v_2$. W.L.O.G., let $u = (0,0)$ and $v = (1,1)$.
Case 2.2.1. $P = \langle u, (0,1), v\rangle$
The hamiltonian cycle is the same as in Case 2.1. Please see Fig. 5 for an illustration.
Case 2.2.2. $P = \langle u, (1,0), v\rangle$
Let $a_i = (-1)^i(k-2)$, for $2 \leq i \leq k-2$, $a_1 = 1-k$, $a_{k-1} = 4-k$ and $a_k = k-3$; $\vec{b} = (k-2,-1,-1,...,-1)$. There exists a hamiltonian cycle $C = \langle(0,0), P, (1,1), (0,1), H^{\vec{a}}_{\vec{b},k}(k-1,1), (0,0)\rangle$. Please see Fig. 6 for an illustration. The hamiltonian cycle in Fig. 6 is $C = \langle(0,0), P, (1,1), (0,1), H^{\vec{a}}_{\vec{b},6}(5,1), (0,0)\rangle$ and $H^{\vec{a}}_{\vec{b},6}(5,1) = \langle h_4^{-5}(5,1), h_{-1}^{4}(0,5), h_{-1}^{-4}(4,4), h_{-1}^{4}(0,3), h_{-1}^{-2}(4,2), h_{-1}^{3}(2,1)\rangle$.

The lemma is proved. ■

To facilitate our derivation in the following, we define some path patterns. We shall use $x_0^i, x_1^i, x_2^i, ..., x_{k^{n-1}-1}^i$ to denote the k^{n-1} vertices of $Q^{k,i}_{d,n-1}$ for some d. For simplicity, denote $Q^{k,i}_{d,n-1}$ as $Q^{k,i}_{n-1}$. Let the path $p(x_a^i, x_b^i) = \langle x_a^i, x_{a_1}^i, x_{a_2}^i, ..., x_b^i\rangle$ and $a_i = (a + i \textbf{ mod } k^{n-1})$. For example, if $k^{n-1} = 64$, then $p(x_{60}^i, x_2^i) = \langle x_{60}^i, x_{61}^i, x_{62}^i, x_{63}^i, x_0^i, x_1^i, x_2^i\rangle$. It is known that there exists a hamiltonian cycle in Q^k_{n-1} [4]. Thus x_a^i and x_{a+1}^i are adjacent and so are x_a^i and x_a^{i+1}.

Lemma 2. *Let k be an integer with $k \geq 3$. For any path P*

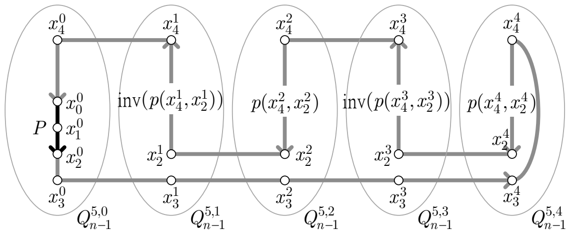

Fig. 7. An Example of Case 1 with $k = 5$.

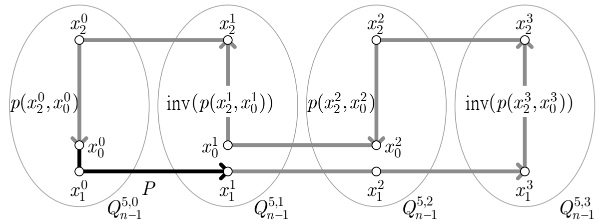

Fig. 8. An Example of Case 2 with $k = 4$.

with length 2 in Q_n^k, there exists a hamiltonian cycle of Q_n^k that contains P.

Proof: The lemma will be proved by mathematical induction. By Lemma 1, the statement holds for Q_2^k. Using the induction hypothesis, we assume that the statement holds for Q_{n-1}^k,where $n \geq 3$. Now we want to prove that the lemma is true for Q_n^k. There are three cases.

Case 1. P is in $Q_{n-1}^{k,i}$. W.L.O.G., let $i = 0$.
By the induction hypothesis, there exists a hamiltonian cycle C^0 of $Q_{n-1}^{k,0}$ that contains P. Let $P = \langle x_0^0, x_1^0, x_2^0 \rangle$ and $C^0 = \langle x_0^0, P, x_2^0, x_3^0, ..., x_{k^{n-1}-1}^0, x_0^0 \rangle$. Since Q_{n-1}^k is hamiltonian [4], let the hamiltonian cycles in Q_{n-1}^k be $C^i = \langle x_0^i, x_1^i, x_2^i, x_3^i, ..., x_{k^{n-1}-1}^i, x_0^i \rangle$.

1) k is odd. Then the hamiltonian cycle is
$C = \langle x_0^0, P, x_2^0, x_3^0, x_3^1, x_3^2, ..., x_3^{k-1}, p(x_4^{k-1}, x_2^{k-1}),$ $\mathrm{inv}(p(x_4^{k-2}, x_2^{k-2})), p(x_4^{k-3}, x_2^{k-3}), \mathrm{inv}(p(x_4^{k-4}, x_2^{k-4})),$ $..., p(x_4^2, x_2^2), \mathrm{inv}(p(x_4^1, x_2^1)), p(x_4^0, x_0^0), x_0^0 \rangle$.
2) k is even. Then the hamiltonian cycle is
$C = \langle x_0^0, P, x_2^0, x_3^0, x_3^1, x_3^2, ..., x_3^{k-1}, \mathrm{inv}(p(x_4^{k-1}, x_2^{k-1})),$ $p(x_4^{k-2}, x_2^{k-2}), \mathrm{inv}(p(x_4^{k-3}, x_2^{k-3})), p(x_4^{k-4}, x_2^{k-4}), ...,$ $p(x_4^2, x_2^2), \mathrm{inv}(p(x_4^1, x_2^1)), p(x_4^0, x_0^0), x_0^0 \rangle$.

Please see Fig. 7 for an illustration, where the hamiltonian cycle in Fig. 7 is $C = \langle x_0^0, P, x_2^0, x_3^0, x_3^1, x_3^2, x_3^3, x_3^4, p(x_4^4, x_2^4),$ $\mathrm{inv}(p(x_4^3, x_2^3)), p(x_4^2, x_2^2), \mathrm{inv}(p(x_4^1, x_2^1)), p(x_4^0, x_0^0), x_0^0 \rangle$.

Case 2. P passes through two $Q_{n-1}^{k,i}$. W.L.O.G., those two are $Q_{n-1}^{k,0}$ and $Q_{n-1}^{k,1}$.
Let $P = \langle x_0^0, x_1^0, x_1^1 \rangle$. In [11], [12], it has been shown that there exists a hamiltonian path $\langle x_1^i, p(x_1^i, x_0^i), x_0^i \rangle$ in $Q_{n-1}^{k,i}$.

1) k is odd. Then the hamiltonian cycle is
$C = \langle x_0^0, P, x_1^1, x_1^2, x_1^3, x_1^4, ..., x_1^{k-1}, p(x_2^{k-1}, x_0^{k-1}),$ $\mathrm{inv}(p(x_2^{k-2}, x_0^{k-2})), p(x_2^{k-3}, x_0^{k-3}), \mathrm{inv}(p(x_2^{k-4}, x_0^{k-4})),$ $..., p(x_2^2, x_0^2), \mathrm{inv}(p(x_2^1, x_0^1)), p(x_2^0, x_0^0), x_0^0 \rangle$.
2) k is even. Then the hamiltonian cycle is
$C = \langle x_0^0, P, x_1^1, x_1^2, x_1^3, x_1^4, ..., x_1^{k-1}, \mathrm{inv}(p(x_2^{k-1}, x_0^{k-1})),$ $p(x_2^{k-2}, x_0^{k-2}), \mathrm{inv}(p(x_2^{k-3}, x_0^{k-3})), p(x_2^{k-4}, x_0^{k-4}), ...,$ $p(x_2^2, x_0^2), \mathrm{inv}(p(x_2^1, x_0^1)), p(x_2^0, x_0^0), x_0^0 \rangle$.

Please see Fig. 8 for an illustration, where the hamiltonian cycle in Fig. 8 is $C = \langle x_0^0, P, x_1^1, x_1^2, x_1^3, \mathrm{inv}(p(x_2^3, x_0^3)),$ $p(x_2^2, x_0^2), \mathrm{inv}(p(x_2^1, x_0^1)), p(x_2^0, x_0^0), x_0^0 \rangle$.

Case 3. P passes through three $Q_{n-1}^{k,i}$.
It is known that we can partition Q_n^k over dimension d by fixing the dth element of any vertex tuple at some value a, for every $a \in \{0, 1, ..., k-1\}$. In this case, $P = \langle u, x, v \rangle$ passes through three $Q_{n-1}^{k,i}$, i.e., u, x and v have the same value in at least one element of vertex tuple. Hence this case is equivalent to Case 1.

By the mathematical induction, the lemma is proved. ■

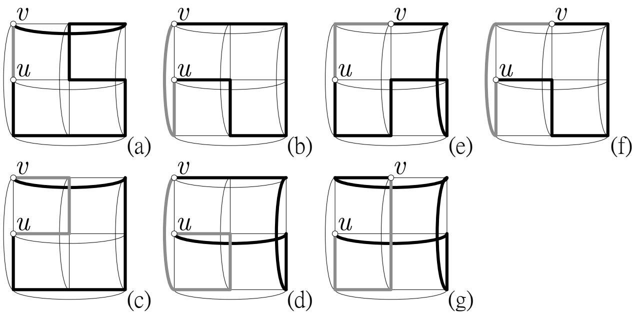

Fig. 9. Illustrations of Lemma 3.

III. The panpositionability of Q_n^k, where $k \geq 3$ is an odd integer and $n \geq 2$ is an integer.

Lemma 3. *Q_2^3 is a panpositionable hamiltonian graph.*

Proof: There are two cases: Case 1. $u = (0, 0)$ and $v = (1, 0)$; Case 2. $u = (1, 0)$ and $v = (0, 1)$. By brute force, we construct the required hamiltonian cycles. Please see Fig. 9. ■

Theorem 1. *Q_n^3 is a panpositionable hamiltonian graph.*

Proof: The theorem is proved by mathematical induction using Lemma 3 as base case. The detailed derivation is skipped. ■

Lemma 4. *Let k be an odd integer with $k \geq 5$. Then Q_2^k is nearly-panpositionable.*

Proof: The proof is by brute force and hence is skipped. ■

Theorem 2. *Let k be an odd integer with $k \geq 5$. Q_n^k is nearly-panpositionable hamiltonian.*

Proof: We will prove the theorem using the mathematical induction. By Lemma 4, Q_2^k is nearly-panpositionable hamiltonian. With the induction hypothesis, we assume that Q_{n-1}^k is nearly-panpositionable hamiltonian for some $n \geq 3$. We need to show that Q_n^k is nearly-panpositionable hamiltonian. Let $u, v \in Q_n^k$ and l be an integer with

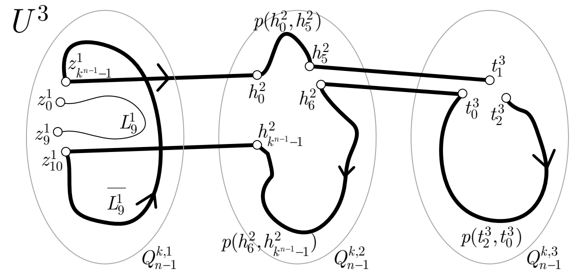

Fig. 10. U^3 for $r=3$ and $l'=9$.

$d \le l \le \frac{|Q_n^k|}{2}$, where $d = d_{Q_n^k}(u,v)$.

Case 1. $u, v \in Q_{n-1}^{k,i}$. W.L.O.G., let $i=0$.

Obviously, $d_{Q_{n-1}^k}(u,v) = d$.

Case 1.1. $d \le l \le \frac{k^{n-1}-1}{2}$.
By the induction hypothesis, there exist a hamiltonian cycle $C_l^i = \langle x_0^i, x_1^i, ..., x_l^i, ..., x_{k^{n-1}-1}^i, x_0^i\rangle$ in $Q_{n-1}^{k,i}$ for $u = x_0^0$ and $v = x_l^0$. Then we have the hamiltonian cycle $C = \langle x_0^0, p(x_0^0, x_l^0), x_l^1, x_l^2, ..., x_l^{k-1}, p(x_{l+1}^{k-1}, x_{l-1}^{k-1}), \mathrm{inv}(p(x_{l+1}^{k-2}, x_{l-1}^{k-2})), p(x_{l+1}^{k-3}, x_{l-1}^{k-3}), \mathrm{inv}(p(x_{l+1}^{k-4}, x_{l-1}^{k-4})), ..., p(x_{l+1}^2, x_{l-1}^2), \mathrm{inv}(p(x_{l+1}^1, x_{l-1}^1)), p(x_{l+1}^0, x_{k^{n-1}-1}^0), x_0^0\rangle$.

Case 1.2. $\frac{k^{n-1}-1}{2}+1 \le l \le \frac{|Q_n^k|}{2}$.
By the induction hypothesis, for any two vertices $x, y \in V(Q_{n-1}^k)$ and $1 \le l' \le k^{n-1}-1$ there exists a hamiltonian cycle C of $Q_{n-1}^{k,i}$ with $d_C(x,y) = l'$. We set $x = z_0^i$ and $y = z_{l'}^i$, then the hamiltonian cycle will be $\langle z_0^i, p(z_0^i, z_{k^{n-1}-1}^i), z_0^i\rangle$. Split the hamiltonian cycle into two pathes $L_{l'}^i$ and $\overline{L_{l'}^i}$ by letting $L_{l'}^i = p(z_0^i, z_{l'}^i)$ and $\overline{L_{l'}^i} = p(z_{l'+1}^i, z_{k^{n-1}-1}^i)$.

In [12], it is shown that for all $x, y \in Q_{n-1}^{k,i}$, there exists a hamiltonian path H^i of $Q_{n-1}^{k,i}$ between x and y. Define $H^i = p(h_0^i, h_{k^{n-1}-1}^i)$ with $x = h_0^i$ and $y = h_{k^{n-1}-1}^i$. By Lemma 2, for any path with length 2 denoted by $\langle t_0^i, t_1^i, t_2^i\rangle$, there exists a hamiltonian cycle $T^i = \langle t_0^i, p(t_0^i, t_{k^{n-1}-1}^i), t_0^i\rangle$ of $Q_{n-1}^{k,i}$. Let $t_0^i = h_{j+1}^i$, $t_1^i = h_j^i$, $z_{l'+1}^i = h_{k^{n-1}-1}^i$, $h_0^i = z_{k^{n-1}-1}^i$ and . Then there is a unique path $U^i = \langle t_2^i, p(t_2^i, t_0^i), h_{j+1}^{i-1}, p(h_{j+1}^{i-1}, h_{k^{n-1}-1}^{i-1}), z_{l'+1}^{i-2}, \overline{L_{l'}^{i-2}}, z_{k^{n-1}-1}^{i-2}, h_0^{i-1}, p(h_0^{i-1}, h_j^{i-1}), t_1^i\rangle$. For example, let $r=3$ and $l'=9$, then $U^3 = \langle t_2^3, p(t_2^3, t_0^3), h_6^2, p(h_6^2, h_{k^{n-1}-1}^2), z_{10}, \overline{L_9^1}, z_{k^{n-1}-1}^1, h_0^2, p(h_0^2, h_5^2), t_1^3\rangle$. Please see Fig. 10 for an illustration.

Let m and r be integers and $0 \le r \le \frac{k-1}{2}$ such that $\frac{k^{n-1}-1}{2} - m + r \cdot k^{n-1} + l' + 1 = l$. W.L.O.G., let $u = x_0^0$ and $v = x^0_{\frac{k^{n-1}-1}{2}-m}$. For simplicity, denote $x^i_{\frac{k^{n-1}-1}{2}-m}$ as v^i, $x^i_{\frac{k^{n-1}-1}{2}-m-1}$ as v_0^i and $x^i_{\frac{k^{n-1}-1}{2}-m+1}$ as v_1^i. If r is even, let $t_1^i = v_0^i$, $t_2^i = v^i$, $z_0^{1+r} = x_0^{1+r}$ and $z_{l'}^{1+r} = x_1^{1+r}$. There is

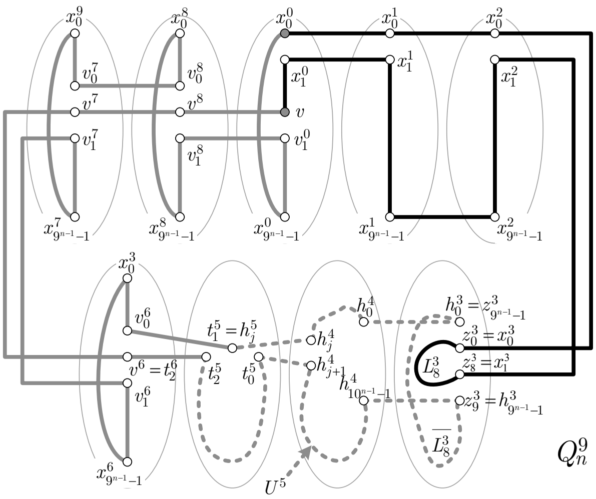

Fig. 11. An Example of Case 1.2 with $k=9$ and $l = \frac{5 \cdot 9^{n-1}-17}{2}$.

a hamiltonian cycle

$$C = \langle x_0^0, x_0^1, ..., x_0^r, L_{l'}^{r+1}, p(x_1^r, x_{k^{n-1}-1}^r), \mathrm{inv}(p(x_1^{r-1}, x_{k^{n-1}-1}^{r-1})), p(x_1^{r-2}, x_{k^{n-1}-1}^{r-2}), \mathrm{inv}(p(x_1^{r-3}, x_{k^{n-1}-1}^{r-3})), ..., p(x_1^2, x_{k^{n-1}-1}^2), \mathrm{inv}(p(x_1^1, x_{k^{n-1}-1}^1)), p(x_1^0, v_0^0), v, v^{k-1}, v^{k-2}, ..., v^{r+4}, U^{r+3}, v_0^{r+4}, \mathrm{inv}(p(v_1^{r+4}, v_0^{r+4})), p(v_1^{r+5}, v_0^{r+5}), \mathrm{inv}(p(v_1^{r+6}, v_0^{r+6})), p(v_1^{r+7}, v_0^{r+7}), ..., p(v_1^{k-2}, v_0^{k-2}), \mathrm{inv}(p(v_1^{k-1}, v_0^{k-1})), v_1^0, p(x^0_{\frac{k^{n-1}-1}{2}-m+2}, x^0_{k^{n-1}-1}), x_0^0\rangle.$$

Please see Fig. 11 for an illustration, where $m=0$, $r=2$, $l'=8$ and the hamiltonian cycle is

$$C = \langle x_0^0, x_0^1, x_0^2, L_8^3, p(x_1^2, x_{9^{n-1}-1}^2), \mathrm{inv}(p(x_1^1, x_{9^{n-1}-1}^1)), p(x_1^0, v_0^0), v, v^8, v^7, v^6, U^5, v_0^6, \mathrm{inv}(p(v_1^6, v_0^6)), p(v_1^7, v_0^7), \mathrm{inv}(p(v_1^8, v_0^8)), v_1^0, p(x^0_{\frac{9^{n-1}-1}{2}+2}, x^0_{9^{n-1}-1}), x_0^0\rangle.$$

If r is odd, let $t_1^i = v_1^i$, $t_2^i = v^i$, $x_0^{1+r} = z_0^{1+r}$ and $x_{k^{n-1}-1}^{1+r} = z_{l'}^{1+r}$. There is a hamiltonian cycle

$$C = \langle x_0^0, x_0^1, ..., x_0^r, L_{l'}^{r+1}, \mathrm{inv}(p(x_1^r, x_{k^{n-1}-1}^r)), p(x_1^{r-1}, x_{k^{n-1}-1}^{r-1}), \mathrm{inv}(p(x_1^{r-2}, x_{k^{n-1}-1}^{r-2})), p(x_1^{r-3}, x_{k^{n-1}-1}^{r-3}), ..., \mathrm{inv}(p(x_1^3, x_{k^{n-1}-1}^3)), p(x_1^2, x_{k^{n-1}-1}^2), \mathrm{inv}(p(x_1^1, x_{k^{n-1}-1}^1)), p(x_1^0, v_0^0), v, v^{k^{n-1}-1}, v^{k^{n-1}-2}, ..., v^{r+4}, U^{r+3}, v_0^{r+4}, p(v_1^{r+4}, v_0^{r+4}), \mathrm{inv}(p(v_1^{r+5}, v_0^{r+5})), p(v_1^{r+6}, v_0^{r+6}), \mathrm{inv}(p(v_1^{r+7}, v_0^{r+7})), ..., p(v_1^{k^{n-1}-2}, v_0^{k^{n-1}-2}), \mathrm{inv}(p(v_1^{k^{n-1}-1}, v_0^{k^{n-1}-1})), v_1^0, p(x^0_{\frac{k^{n-1}-1}{2}-m+2}, x^0_{k^{n-1}-1}), x_0^0\rangle.$$

Case 2. $u \in Q_{n-1}^{k,i}$, $v \in Q_{n-1}^{k,j}$ and $i \ne j$. W.L.O.G., let $i=0$. For any vertex x_a^j in $Q_{n-1}^{k,j}$, there exists a corresponding vertex x_a^0. Set $u = x_0^0$ and $v = x_{d'}^j$, where d' is the length

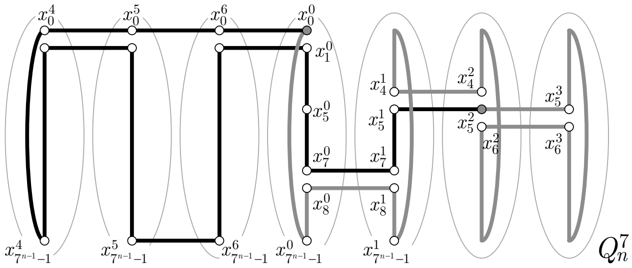

Fig. 12. An Example of Case 2.1 with $k = 7$ and $l = 3 \cdot 7^{n-1} + 11$.

of the shortest path between x_0^0 and $v = x_{d'}^0$ in $Q_{n-1}^{k,0}$. Note that for all $i \leq k-1$, there is a hamiltonian cycle $C^i = \langle x_0^i, x_1^i, ..., x_{d'}^i, ..., x_{k^{n-1}}^i \rangle$ in $Q_{n-1}^{k,i}$.

Case 2.1. $l - d$ is even.
Let $0 \leq r \leq \frac{k-1}{2}$ be an integer, $d + 2(t - d') + r \cdot k^{n-1} = l$, $d' \leq t \leq k^{n-1} - 1$ and $e = k - 1 - r$.
Let r be an odd integer. We have the hamiltonian cycle $C = \langle x_0^0, x_0^{k-1}, x_0^{k-2}, ..., x_0^{k-r}, \text{inv}(p(x_1^{k-r}, x_{k^{n-1}-1}^{k-r})), p(x_1^{k-r+1}, x_{k^{n-1}-1}^{k-r+1}), \text{inv}(p(x_1^{k-r+2}, x_{k^{n-1}-1}^{k-r+2})), p(x_1^{k-r+3}, x_{k^{n-1}-1}^{k-r+3}), ..., \text{inv}(p(x_1^{k-3}, x_{k^{n-1}-1}^{k-3})), p(x_1^{k-2}, x_{k^{n-1}-1}^{k-2}), \text{inv}(p(x_1^{k-1}, x_{k^{n-1}-1}^{-1})), x_1^0, p(x_2^0, x_{d'-1}^0), p(x_{d'}^0, x_t^0), \text{inv}(p(x_{d'}^1, x_t^1)), x_{d'}^2, ..., x_{d'}^j, ..., x_{d'}^e, \text{inv}(p(x_{d'+1}^e, x_{d'-1}^e)), p(x_{d'+1}^{e-1}, x_{d'-1}^{e-1}), \text{inv}(p(x_{d'+1}^{e-2}, x_{d'-1}^{e-2})), p(x_{d'+1}^{e-3}, x_{d'-1}^{e-3}), ..., \text{inv}(p(x_{d'+1}^3, x_{d'-1}^3)), p(x_{d'+1}^2, x_{d'-1}^2), \text{inv}(p(x_{t+1}^1, x_{d'-1}^1)), p(x_{t+1}^0, x_0^0), x_0^0 \rangle$.
Please see Fig. 12 for an illustration, where $r = 3$, $d' = 5$, $t = 7$ and the hamiltonian cycle is $C = \langle x_0^0, x_0^6, x_0^5, x_0^4, \text{inv}(p(x_1^4, x_{7^{n-1}-1}^4)), p(x_1^5, x_{7^{n-1}-1}^5), \text{inv}(p(x_1^6, x_{7^{n-1}-1}^6)), x_1^0, p(x_2^0, x_7^0), \text{inv}(p(x_5^1, x_7^1)), x_5^2, x_5^3, \text{inv}(p(x_6^3, x_4^3)), p(x_6^2, x_4^2), \text{inv}(p(x_8^1, x_4^1)), p(x_8^0, x_0^0), x_0^0 \rangle$.
Let r be an even integer. We have the hamiltonian cycle $C = \langle x_0^0, x_0^{k-1}, x_0^{k-2}, ..., x_0^{k-r}, p(x_1^{k-r}, x_{k^{n-1}-1}^{k-r}), \text{inv}(p(x_1^{k-r+1}, x_{k^{n-1}-1}^{k-r+1})), p(x_1^{k-r+2}, x_{k^{n-1}-1}^{k-r+2}), \text{inv}(p(x_1^{k-r+3}, x_{k^{n-1}-1}^{k-r+3})), ..., p(x_1^{k-2}, x_{k^{n-1}-1}^{k-2}), \text{inv}(p(x_1^{k-1}, x_{k^{n-1}-1}^{-1})), x_1^0, p(x_2^0, x_{d'-1}^0), p(x_{d'}^0, x_t^0), \text{inv}(p(x_{d'}^1, x_t^1)), x_{d'}^2, ..., x_{d'}^j, ..., x_{d'}^e, p(x_{d'+1}^e, x_{d'-1}^e), \text{inv}(p(x_{d'+1}^{e-1}, x_{d'-1}^{e-1})), p(x_{d'+1}^{e-2}, x_{d'-1}^{e-2}), \text{inv}(p(x_{d'+1}^{e-3}, x_{d'-1}^{e-3})), ..., p(x_{d'+1}^2, x_{d'-1}^2), \text{inv}(p(x_{t+1}^1, x_{d'-1}^1)), p(x_{t+1}^0, x_0^0), x_0^0 \rangle$.

Case 2.2. $l - d$ is odd.
By the induction hypothesis, there exists a hamiltonian cycle $D^i = \langle y_0^i, y_1^i, ..., y_{k^{n-1}}^i \rangle$ in $Q_{n-1}^{k,i}$ such that $x_0^i = y_0^i$, $x_{d'}^i = y_{l'}^i$ and the length l' of the path joining y_0^i to $y_{l'}^i$ in $Q_{n-1}^{k,i}$ is the smallest integer when $l' - d'$ is odd. Let $0 \leq r \leq \frac{k-1}{2}$ be an integer, $d + l' - d' + 2(t - l') + r \cdot k^{n-1} = l$, $d' \leq t \leq k^{n-1} - 1$ and $e = k - 1 - r$.
Let r be an odd integer. We have the hamiltonian cycle $C = \langle y_0^0, y_0^{k-1}, y_0^{k-2}, ..., y_0^{k-r}, \text{inv}(p(y_1^{k-r}, y_{k^{n-1}-1}^{k-r})), p(y_1^{k-r+1}, y_{k^{n-1}-1}^{k-r+1}), \text{inv}(p(y_1^{k-r+2}, y_{k^{n-1}-1}^{k-r+2})), p(y_1^{k-r+3}, y_{k^{n-1}-1}^{k-r+3}), ..., \text{inv}(p(y_1^{k-3}, y_{k^{n-1}-1}^{k-3})), p(y_1^{k-2}, y_{k^{n-1}-1}^{k-2}), \text{inv}(p(y_1^{k-1}, y_{k^{n-1}-1}^{k-1})), y_1^0, p(y_2^0, y_{l'-1}^0), p(y_{l'}^0, y_t^0), \text{inv}(p(y_{l'}^1, y_t^1)), y_{l'}^2, ..., y_{l'}^j, ..., y_{l'}^e, \text{inv}(p(y_{l'+1}^e, y_{l'-1}^e)), p(y_{l'+1}^{e-1}, y_{l'-1}^{e-1}), \text{inv}(p(y_{l'+1}^{e-2}, y_{l'-1}^{e-2})), p(y_{l'+1}^{e-3}, y_{l'-1}^{e-3}), ..., \text{inv}(p(y_{l'+1}^3, y_{l'-1}^3)), p(y_{l'+1}^2, y_{l'-1}^2), \text{inv}(p(y_{t+1}^1, y_{l'-1}^1)), p(y_{t+1}^0, y_0^0), y_0^0 \rangle$.
Let r be an even integer. We have the hamiltonian cycle $C = \langle y_0^0, y_0^{k-1}, y_0^{k-2}, ..., y_0^{k-r}, p(y_1^{k-r}, y_{k^{n-1}-1}^{k-r}), \text{inv}(p(y_1^{k-r+1}, y_{k^{n-1}-1}^{k-r+1})), p(y_1^{k-r+2}, y_{k^{n-1}-1}^{k-r+2}), \text{inv}(p(y_1^{k-r+3}, y_{k^{n-1}-1}^{k-r+3})), ..., p(y_1^{k-2}, y_{k^{n-1}-1}^{k-2}), \text{inv}(p(y_1^{k-1}, y_{k^{n-1}-1}^{-1})), y_1^0, p(y_2^0, y_{l'-1}^0), p(y_{l'}^0, y_t^0), \text{inv}(p(y_{l'}^1, y_t^1)), y_{l'}^2, ..., y_{l'}^j, ..., y_{l'}^e, p(y_{l'+1}^e, y_{l'-1}^e), \text{inv}(p(y_{l'+1}^{e-1}, y_{l'-1}^{e-1})), p(y_{l'+1}^{e-2}, y_{l'-1}^{e-2}), \text{inv}(p(y_{l'+1}^{e-3}, y_{l'-1}^{e-3})), ..., p(y_{l'+1}^2, y_{l'-1}^2), \text{inv}(p(y_{t+1}^1, y_{l'-1}^1)), p(y_{t+1}^0, y_0^0), y_0^0 \rangle$.
By the mathematical induction, the theorem is proved. ■

IV. Q_n^k IS BIPANPOSITIONABLE, WHERE $k \geq 4$ IS AN EVEN INTEGER AND $n \geq 2$ IS AN INTEGER.

Lemma 5. *Let k be an even integer with $k \geq 4$. Then Q_2^k is bipanpositionable.*

Proof: The proof is by brute force and hence is skipped. ■

Theorem 3. *Let k be an even integer with $k \geq 4$. Q_n^k is bipanpositionable hamiltonian.*

Proof: We will prove the theorem using the mathematical induction. By Lemma 5, Q_2^k is bipanpositionable hamiltonian. With the induction hypothesis, we assume that Q_{n-1}^k is bipanpositionable hamiltonian for some $n \geq 4$. We need to show that Q_n^k is bipanpositionable hamiltonian. Note that Q_n^2 is bipanpositionable [7]. Therefore, $Q_n^4 = Q_{2n}^2$ is bipanpositionable. It suffices to prove the cases for $k \geq 6$.

The proof is similar to Theorem 2, so we'll skip it. Readers can follow the similar techniques in Theorem 2 to construct the required hamiltonian cycles. ■

V. CONCLUSIONS

In this paper, we prove that the k-ary n-cube Q_n^3 is panpositionable hamiltonian and Q_n^k is nearly-papositionable for any odd integer $k \geq 5$. Moreover, we prove that Q_n^k is bipanpositionable hamiltonian for any even integer $k \geq 4$. It is known that the hypercube, Q_n^2, is bipanpositionable [7]. Thus Q_n^k is bipanpositionable for all even integers $k \geq 2$.

The panpositionability of any k-ary n-cube has been completely studied and the result is optimal in the sense that given any two vertices u and v, there exists no more hamiltonian cycle on which $d_C(u, v)$ equals any of the numbers we miss in the nearly-panpositionable Q_n^k when k is odd.

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