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k-Fuzzy Ideals of Ternary Semirings

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Abstract—The notion of k-fuzzy ideals of semirings was introduced by Kim and Park in 1996. In 2003, Dutta and Kar introduced a notion of ternary semirings. This structure is a generalization of ternary rings and semirings. The main purpose of this paper is to introduce and study k-fuzzy ideals in ternary semirings analogous to k-fuzzy ideals in semirings considered by Kim and Park.

Keywords—k-ideals, k-fuzzy ideals, fuzzy k-ideals, ternary semirings

I. Introduction

The notion of ternary algebraic system was introduced by Lehmer [15] in 1932. He investigated certain ternary algebraic systems called triplexes. In 1971, Lister [16] characterized additive semigroups of rings which are closed under the triple ring product and he called this algebraic system a ternary ring. Dutta and Kar [3] introduced a notion of ternary semirings which is a generalization of ternary rings and semirings, and they studied some properties of ternary semirings ([3], [4], [5], [6], [7] and [11], etc.).

The theory of fuzzy sets was first studied by Zadeh [17] in 1965. Many papers on fuzzy sets appeared showing the importance of the concept and its applications to logic, set theory, group theory, ring theory, real analysis, topology, measure theory, etc. Fuzzy ideals of semirings were studied by some authors ([1], [2], [8], [9], [10] and [14], etc.). The notion of k-fuzzy ideals of semirings was introduced by Kim and Park [14]. Recently, Kavikumar, Khamis and Jun studied fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals in ternary semirings in [12] and [13]. The fuzzy ideal of ternary semirings is a good tool for us to study the fuzzy algebraic structure. The main purpose of this paper is to study k-fuzzy ideals in ternary semirings analogous to k-fuzzy ideals in semirings considered by Kim and Park.

II. Preliminaries

In this section, we refer to some elementary aspects of the theory of semirings and ternary semirings and fuzzy algebraic systems that are necessary for this paper.

Definition 2.1. A nonempty set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$, respectively) is called a *semiring* if $(S,+)$ is a commutative semigroup, $(S,\cdot)$ is a semigroup and multiplicative distributes over addition both from the left and from the right, i.e., $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$ for all $a,b,c\in S$.

Definition 2.2. A nonempty set S together with a binary operation and a ternary operation called addition $+$ and ternary multiplication, respectively, is said to be a *ternary semiring* if $(S,+)$ is a commutative semigroup satisfying the following conditions: for all $a,b,c,d,e\in S$,

(i) $(abc)de=a(bcd)e=ab(cde)$,
(ii) $(a+b)cd=acd+bcd$,
(iii) $a(b+c)d=abd+acd$ and
(iv) $ab(c+d)=abc+abd$.

We can see that any semiring can be reduced to a ternary semiring. However, a ternary semiring does not necessarily reduce to a semiring by this example. We consider $\mathbb{Z}_0^-$, the set of all non-positive integers under usual addition and multiplication, we see that $\mathbb{Z}_0^-$ is an additive semigroup which is closed under the triple multiplication but is not closed under the binary multiplication. Moreover, $\mathbb{Z}_0^-$ is a ternary semiring but is not a semiring under usual addition and multiplication.

Definition 2.3. Let S be a ternary semiring. If there exists an element $0\in S$ such that $0+x=x=x+0$ and $0xy=x0y=xy0=0$ for all $x,y\in S$, then 0 is called the *zero element* or simply the *zero* of the ternary semiring S. In this case we say that S is a ternary semiring with zero.

Definition 2.4. An additive subsemigroup T of S is called a *ternary subsemiring* of S if $t_1t_2t_3\in T$ for all $t_1,t_2,t_3\in T$.

Definition 2.5. An additive subsemigroup I of S is called a *left [resp. right, lateral] ideal* of S if $s_1s_2i\in I$ [resp. $is_1s_2\in I, s_1is_2\in I$] for all $s_1,s_2\in S$ and $i\in I$. If I is a left, right and lateral ideal of S, then I is called an *ideal* of S.

It is obvious that every ideal of a ternary semiring with zero contains a zero element.

Definition 2.6. Let S and R be ternary semirings. A mapping $\varphi: S\rightarrow R$ is said to be a *homomorphism* if $\varphi(x+y)=\varphi(x)+\varphi(y)$ and $\varphi(xyz)=\varphi(x)\varphi(y)\varphi(z)$ for all $x,y,z\in S$.

Let $\varphi: S\rightarrow R$ be an onto homomorphism of ternary semirings. Note that if I is an ideal of S, then $\varphi(I)$ is an ideal of R. If S and R be ternary semirings with zero 0, then $\varphi(0)=0$.

Definition 2.7. Let S be a non-empty set. A mapping $f: S\rightarrow[0,1]$ is called a *fuzzy subset* of S.

Definition 2.8. Let A be a subset of a non-empty set S. The *characteristic function* χ_A of A is a fuzzy subset of S defined

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as follows:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Definition 2.9. Let f be a fuzzy subset of a non-empty subset S. For $t \in [0,1]$, the set $f_t = \{x \in S \mid f(x) \geq t\}$ is called a *level subset* of S with respect to f.

III. Main Result

Definition 3.1. An ideal I of a ternary semiring S is said to be a *k-ideal* if for $x, y \in S, x+y, y \in I \Rightarrow x \in I$.

Example 3.1. Consider the ternary semiring $\mathbb{Z}_0^-$ under usual addition and ternary multiplication, let $I = \{0, -3\} \cup \{-5, -6, -7, \ldots\}$. It is easy to prove that I is an ideal of $\mathbb{Z}_0^-$ but not a k-ideal of $\mathbb{Z}_0^-$ because $-3, (-2)+(-3) \in I$ but $-2 \notin I$.

Example 3.2. Consider the ternary semiring $\mathbb{Z}_0^-$ under usual addition and ternary multiplication, let $I = \{-3k \mid k \in \mathbb{N} \cup \{0\}\}$. It is easy to show that I is a k-ideal of $\mathbb{Z}_0^-$.

Definition 3.2. For each ideal I of a ternary semiring S, the *k-closure* $\overline{I}$ of I is defined by

$$\overline{I} = \{x \in S \mid a + x = b \text{ for some } a, b \in I\}.$$

The next theorem holds.

Theorem 3.1. Let I be an ideal of a ternary semiring S with zero. Then I is a k-ideal of S if and only if $I = \overline{I}$.

Definition 3.3. A fuzzy subset f of a ternary semiring S is called a *fuzzy ideal* of S if for all $x, y, z \in S$,

(i) $f(x+y) \geq \min\{f(x), f(y)\}$ and
(ii) $f(xyz) \geq \max\{f(x), f(y), f(z)\}$.

By the definitions of ideals and fuzzy ideals of ternary semirings, the following lemma holds.

Lemma 3.2. Let I be a non-empty subset of a ternary semiring S. Then I is an ideal of S if and only if the characteristic function χ_I is a fuzzy ideal of S.

Lemma 3.3. Let f be a fuzzy ideal of a ternary semiring S with zero 0. Then $f(x) \leq f(0)$ for all $x \in S$.

Proof. For any $x \in S, f(0) = f(00x) \geq \max\{f(0), f(x)\} \geq f(x)$. □

Definition 3.4. A fuzzy ideal f of a ternary semiring S with zero 0 is said to be a *k-fuzzy ideal* of S if

$$f(x+y) = f(0) \text{ and } f(y) = f(0) \Rightarrow f(x) = f(0)$$

for all $x, y \in S$.

Example 3.3. Consider the ternary semiring $\mathbb{Z}_0^-$ under usual addition and ternary multiplication. Define a fuzzy subset f on $\mathbb{Z}_0^-$ by

$$f(x) = \begin{cases} 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

It is easy to prove that f is a fuzzy ideal of $\mathbb{Z}_0^-$. However, f is not a k-fuzzy ideal of $\mathbb{Z}_0^-$ because $f((-1)+(-2)) = f(-3) = 0.5 = f(0)$ and $f(-2) = 0.5 = f(0)$ but $f(-1) = 0 \neq 0.5 = f(0)$.

Example 3.4. Let f be a fuzzy subset of a ternary semiring $\mathbb{Z}_0^-$ under usual addition and ternary multiplication defined by

$$f(x) = \begin{cases} 0.3 & \text{if } x \text{ is odd} \\ 0.5 & \text{if } x \text{ is even.} \end{cases}$$

It is easy to show that f is a fuzzy ideal of $\mathbb{Z}_0^-$. Let $x, y \in \mathbb{Z}_0^-$ such that $f(x+y) = f(0)$ and $f(x) = f(0)$. So $f(x+y) = 0.5$ and $f(y) = 0.5$. Thus $x+y$ and y are even. Hence x is even, this implies $f(x) = 0.5 = f(0)$. Therefore f is a k-fuzzy ideal of $\mathbb{Z}_0^-$.

From the condition of Definition 3.4 and Lemma 3.2, the following theorem holds.

Theorem 3.4. Let S be a ternary semiring with zero 0 and I is a non-empty subset of S. Then I is a k-ideal of S if and only if the characteristic function χ_I is a k-fuzzy ideal of S.

Proof. Assume I is a k-ideal of S. By Lemma 3.2, χ_I is a fuzzy ideal of S. Next, let $x, y \in S$ and assume $\chi_I(x+y) = \chi_I(0)$ and $\chi_I(y) = \chi_I(0)$. Since I is an ideal of S, $0 \in I$. Thus $\chi_I(0) = 1$, this implies $\chi_I(x+y) = 1$ and $\chi_I(y) = 1$. Then $x+y, y \in I$. Since I is a k-ideal of S, $x \in I$. Hence $\chi_I(x) = 1 = \chi_I(0)$. Therefore χ_I is a k-fuzzy ideal of S. Conversely, assume characteristic function χ_I is a k-fuzzy ideal of S. By Lemma 3.2, I is an ideal of S. So $0 \in I$, this implies $\chi_I(0) = 1$. Let $x, y \in S$ such that $x+y, y \in I$. So $\chi_I(x+y) = \chi_I(0)$ and $\chi_I(y) = \chi_I(0)$. Then $\chi_I(x) = \chi_I(0) = 1$. So $x \in I$. Hence I is a k-ideal of S. □

Theorem 3.5. Let f be a fuzzy subset of a ternary semiring S. Then f is a fuzzy ideal of S if and only if for any $t \in [0,1]$ such that $f_t \neq \emptyset, f_t$ is an ideal of S.

Proof. Let f be a fuzzy ideal of S. Let $t \in [0,1]$ such that $f_t \neq \emptyset$. Let $x, y \in f_t$. Then $f(x), f(y) \geq t$. Then $f(x+y) \geq \min\{f(x), f(y)\} \geq t$. Next, let $x, y \in S$ and $a \in f_t$. We have $f(xya) \geq \max\{f(x), f(y), f(a)\} \geq f(a) \geq t$. Thus $xya \in f_t$. Similarly, $xay, axy \in f_t$. Therefore f_t is an ideal of S. Conversely, let $x, y, z \in S$ and $t = \min\{f(x), f(y)\}$. Then $f(x), f(y) \geq t$. Thus $x, y \in f_t$. By assumption, $x+y \in f_t$. So $f(x+y) \geq t = \min\{f(x), f(y)\}$. Next, let $s = \max\{f(x), f(y), f(z)\}$. Then $f(x) = s$ or $f(y) = s$ or $f(z) = s$. Thus $x \in f_s$ or $y \in f_s$ or $z \in f_s$. By assumption, $xyz \in f_s$. So $f(xyz) \geq s = \max\{f(x), f(y), f(z)\}$. Therefore f is a fuzzy ideal of S. □

However, it is not true in general that f is a fuzzy ideal of a ternary semiring S with zero 0, then for any $t \in [0,1]$ such that $f_t \neq \emptyset, f_t$ is a k-ideal of S. We can see this example.

Example 3.5. Consider the ternary semiring $\mathbb{Z}_0^-$ under usual addition and ternary multiplication. Define a fuzzy subset f on $\mathbb{Z}_0^-$ by

$$f(x) = \begin{cases} 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

Then f is a fuzzy ideal of $\mathbb{Z}_0^-$ but $f_{0.5} = \mathbb{Z}_0^- \setminus \{-1\}$ is not a k-ideal of $\mathbb{Z}_0^-$ because $(-1)+(-2) = -3 \in f_{0.5}$ and $-2 \in f_{0.5}$ but $-1 \notin f_{0.5}$.

Theorem 3.6. Let f be a fuzzy subset of a ternary semiring S with zero 0. If for any $t \in [0,1]$ such that $f_t \neq \emptyset, f_t$ is a k-ideal of S, then f is a k-fuzzy ideal of S.

Proof. By Theorem 3.5, f is a fuzzy ideal of S. Next, let $x, y \in S$ such that $f(x+y) = f(0)$ and $f(y) = f(0)$. Then $x+y, y \in f_{f(0)}$. By assumption, $x \in f_{f(0)}$. Hence $f(x) \geq f(0)$. Since f is a fuzzy ideal of S, by Lemma 3.3, $f(x) = f(0)$. Therefore f is a k-fuzzy ideal of S. □

However, the converse of Theorem 3.6 does not hold. We can see this example.

Example 3.6. Consider the ternary semiring $\mathbb{Z}_0^-$ under usual addition and ternary multiplication. Let f be a fuzzy subset of $\mathbb{Z}_0^-$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is even,} \\ 0 & \text{if } x = -1, \\ 0.5 & \text{otherwise.} \end{cases}$$

Then f is a fuzzy ideal of $\mathbb{Z}_0^-$. Let $x, y \in \mathbb{Z}_0^-$ such that $f(x+y) = f(0)$ and $f(x) = f(0)$. So $f(x+y) = 1$ and $f(y) = 1$. Thus $x+y$ and y are even. Hence x is even, this implies $f(x) = 1 = f(0)$. Therefore f is a k-fuzzy ideal of $\mathbb{Z}_0^-$. However, $f_{0.5} = \mathbb{Z}_0^- \setminus \{-1\}$ is not a k-ideal of $\mathbb{Z}_0^-$ because $(-1)+(-2) = -3 \in f_{0.5}$ and $-2 \in f_{0.5}$ but $-1 \notin f_{0.5}$.

Definition 3.5. Let S be a ternary semiring with zero 0 and f a fuzzy ideal of S. the *k-fuzzy closure* $\overline{f}$ of f is defined by

$$\overline{f}(x) = \begin{cases} f(x) & \text{if } x \notin \overline{f_{f(0)}}, \\ f(0) & \text{if } x \in \overline{f_{f(0)}}. \end{cases}$$

The next theorem holds.

Theorem 3.7. Let S be a ternary semiring with zero 0 and f a fuzzy ideal of S. Then f is a k-fuzzy ideal of S if and only if $f = \overline{f}$.

Proof. Assume f is a k-fuzzy ideal of S and let $x \in \overline{f_{f(0)}}$. Then $\overline{f}(x) = f(0)$. Since $x \in \overline{f_{f(0)}}$, there exist $a, b \in f_{f(0)}$ such that $a + x = b$. Thus $f(a) = f(0)$ and $f(x+a) = f(b) = f(0)$. Then $f(x) = f(0)$. Then $f = \overline{f}$. Conversely, assume $f = \overline{f}$. So $f_{f(0)} = \overline{f_{f(0)}}$, by Theorem 3.1, $f_{f(0)}$ is a k-ideal of S. Let $x, y \in S$ such that $f(x+y) = f(0) = f(y)$. So $x+y, y \in f_{f(0)}$. Then $x \in f_{f(0)}$. So $f(x) = f(0)$. Hence f is a k-fuzzy ideal of S.

Definition 3.6. Let $\varphi : S \to R$ be a homomorphism of ternary semirings. Let f be a fuzzy subset of R. We define a fuzzy subset $\varphi^{-1}(f)$ of S by

$$\varphi^{-1}(f)(x) = f(\varphi(x)) \text{ for all } x \in S.$$

We call $\varphi^{-1}(f)$ the *preimage* of f under φ.

Theorem 3.8. Let $\varphi : S \to R$ be an onto homomorphism of ternary semirings. If f be a fuzzy ideal of R, then $\varphi^{-1}(f)$ is a fuzzy ideal of S.

Proof. Let f be a fuzzy ideal of R. Then for any $x, y, z \in S$,

$$\begin{aligned} \varphi^{-1}(f)(x+y) &= f(\varphi(x+y)) \\ &= f(\varphi(x)+\varphi(y)) \\ &\geq \min\{f(\varphi(x)), f(\varphi(y))\} \\ &= \min\{\varphi^{-1}(f)(x), \varphi^{-1}(f)(y)\} \end{aligned}$$

and

$$\begin{aligned} \varphi^{-1}(f)(xyz) &= f(\varphi(xyz)) \\ &= f(\varphi(x)\varphi(y)\varphi(z)) \\ &\geq \max\{f(\varphi(x)), f(\varphi(y)), f(\varphi(z))\} \\ &= \max\{\varphi^{-1}(f)(x), \varphi^{-1}(f)(y), \varphi^{-1}(f)(z)\}. \end{aligned}$$

This shows that $\varphi^{-1}(f)$ is a fuzzy ideal of S. □

Theorem 3.9. Let S and R be ternary semirings with zero 0 and $\varphi : S \to R$ an onto homomorphism. Let f be a fuzzy ideal of R. Then f is a k-fuzzy ideal of R if and only if $\varphi^{-1}(f)$ is a k-fuzzy ideal of S.

Proof. Suppose that f is a k-fuzzy ideal of R. Let $x, y \in S$. Assume $\varphi^{-1}(f)(x+y) = \varphi^{-1}(f)(0)$ and $\varphi^{-1}(f)(y) = \varphi^{-1}(f)(0)$. Then $f(\varphi(x+y)) = f(\varphi(0)) = f(0)$ and $f(\varphi(y)) = f(\varphi(0)) = f(0)$. Since f is a k-fuzzy ideal of R, $f(\varphi(x)) = f(0) = f(\varphi(0))$. Thus $\varphi^{-1}(f)(x) = \varphi^{-1}(f)(0)$. Hence $\varphi^{-1}(f)$ is a k-fuzzy ideal of S. Conversely, assume $\varphi^{-1}(f)$ is a k-fuzzy ideal of S. Let $x, y \in R$ such that $f(x+y) = f(0)$ and $f(y) = f(0)$. Since φ is onto, there exist $a, b \in S$ such that $f(a) = x$ and $f(b) = y$. So $f(\varphi(a)+\varphi(b)) = f(\varphi(0))$ and $f(\varphi(b)) = f(\varphi(0))$. Hence $\varphi^{-1}(f)(a+b) = \varphi^{-1}(f)(0)$ and $\varphi^{-1}(f)(b) = \varphi^{-1}(f)(0)$. Since $\varphi^{-1}(f)$ is a k-fuzzy ideal of S, $\varphi^{-1}(f)(a) = \varphi^{-1}(f)(0)$, this implies $f(x) = f(\varphi(a)) = f(\varphi(0)) = f(0)$. Hence f is a k-fuzzy ideal of R. □

Definition 3.7. Let $\varphi : S \to R$ be a homomorphism of ternary semirings. Let f be a fuzzy subset of S. We define a fuzzy subset $\varphi(f)$ of R by

$$\varphi(f)(y) = \begin{cases} \sup\limits_{x \in \varphi^{-1}(y)} f(x) & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We call $\varphi(f)$ the *image* of f under φ.

The following lemma is case $L = [0,1]$ of Proposition 8 in [10].

Lemma 3.10. ([10]) Let φ be a mapping from a set X to a set Y and f a fuzzy subset of X. Then for every $t \in (0,1]$,

$$(\varphi(f))_t = \bigcap_{0<s<t} \varphi(f_{t-s}).$$

Lemma 3.11. The intersection of arbitrary set of ideals of a ternary semiring S is either empty or an ideal of S.

Theorem 3.12. Let $\varphi : S \to R$ be an onto homomorphism of ternary semirings. If f is a fuzzy ideal of S, then $\varphi(f)$ is a fuzzy ideal of R.

Proof. By Theorem 3.5, it is sufficient to show that each nonempty level subset of $\varphi(f)$ is an ideal of R. Let $t \in [0,1]$ such that $(\varphi(f))_t \neq \emptyset$. If $t = 0$, then $(\varphi(f))_t = R$. Assume that $t \neq 0$. By Lemma 3.10,

$$(\varphi(f))_t = \bigcap_{0<s<t} \varphi(f_{t-s}).$$

Then $\varphi(f_{t-s}) \neq \emptyset$ for all $0 < s < t$, and so $f_{t-s} \neq \emptyset$ for all $0 < s < t$. By Theorem 3.5, f_{t-s} is an ideal of S for all

$0 < s < t$. Since φ is an onto homomorphism, $\varphi(f_{t-s})$ is an ideal of R for all $0 < s < t$. By Lemma 3.11, $(\varphi(f))_t = \bigcap_{0<s<t} \varphi(f_{t-s})$ is an ideal of R. □

Definition 3.8. Let S and R be any two sets and $\varphi : S \to R$ be any function. A fuzzy subset f of S is called *φ-invariant* if $\varphi(x) = \varphi(y)$ implies $f(x) = f(y)$ where $x, y \in S$.

Lemma 3.13. Let S and R be ternary semirings and $\varphi : S \to R$ a homomorphism. Let f be a φ-invariant fuzzy ideal of S. If $x = \varphi(a)$, then $\varphi(f)(x) = f(a)$.

Proof. If $t \in \varphi^{-1}(x)$, then $\varphi(t) = x = \varphi(a)$. Since f is φ-invariant, $f(t) = f(a)$. This implies

$$\varphi(f)(x) = \sup_{t\in\varphi^{-1}(x)} f(t) = f(a).$$

Hence $\varphi(f)(x) = f(a)$. □

Theorem 3.14. Let S and R be ternary semirings and $\varphi : S \to R$ an onto homomorphism. If f is a φ-invariant fuzzy ideal of S, then $\varphi(f)$ is a fuzzy ideal of R.

Proof. Let $x, y, z \in R$. Then there exist $a, b, c \in S$ such that $\varphi(a) = x, \varphi(b) = y$ and $\varphi(c) = z$ and then $x + y = \varphi(a + b)$ and $xyz = \varphi(abc)$. Since f is φ-invariant, by Lemma 3.13, we have

$$\begin{aligned}\varphi(f)(x + y) &= f(a + b)\\ &\geq \min\{f(a), f(b)\}\\ &= \min\{\varphi(f)(x), \varphi(f)(y)\}\end{aligned}$$

and

$$\begin{aligned}\varphi(f)(xyz) &= f(abc)\\ &\geq \max\{f(a), f(b), f(c)\}\\ &= \max\{\varphi(f)(x), \varphi(f)(y), \varphi(f)(z)\}.\end{aligned}$$

Hence $\varphi(f)$ is a fuzzy ideal of R. □

Theorem 3.15. Let S and R be ternary semirings with zero 0 and $\varphi : S \to R$ an onto homomorphism. Let f be a φ-invariant fuzzy ideal of S. Then f is a k-fuzzy ideal S if and only if $\varphi(f)$ is a k-fuzzy ideal of R.

Proof. Suppose that f is a k-fuzzy ideal of S and let $x, y \in R$ such that $\varphi(f)(x + y) = \varphi(f)(0)$ and $\varphi(f)(y) = \varphi(f)(0)$. Since φ is onto, there exist $a, b \in S$ such that $\varphi(a) = x$ and $\varphi(b) = y$. By Lemma 3.13, $\varphi(f)(0) = f(0), \varphi(f)(x + y) = f(a + b)$ and $\varphi(f)(y) = f(b)$. Thus $f(a + b) = f(0)$ and $f(b) = f(0)$. Since f is a k-fuzzy ideal of S, $f(a) = f(0)$. By Lemma 3.13, $\varphi(f)(x) = f(a) = f(0) = \varphi(f)(0)$. Hence $\varphi(f)$ is a k-fuzzy ideal of R. Conversely, if $\varphi(f)$ is a k-fuzzy ideal of R, then for any $x \in S$,

$$\varphi^{-1}(\varphi(f))(x) = \varphi(f)(\varphi(x)) = f(x).$$

So $\varphi^{-1}(\varphi(f)) = f$. Since $\varphi(f)$ is a k-fuzzy ideal of R, by Theorem 3.9, $f = \varphi^{-1}(\varphi(f))$ is a k-fuzzy ideal of S. □

Next, we define fuzzy k-ideals of ternary semirings analogous to fuzzy k-ideals of semirings.

Definition 3.9. A fuzzy ideal f of a ternary semiring S is said to be a *fuzzy k-ideal* of S if

$$f(x) \geq \min\{f(x + y), f(y)\}$$

for all $x, y \in S$.

Example 3.7. Let f be a fuzzy subset of a ternary semiring $\mathbb{Z}_0^-$ under the usual addition and ternary multiplication defined by

$$f(x) = \begin{cases} 0 & \text{if } x = -1,\\ 0.5 & \text{otherwise.}\end{cases}$$

Then f is a fuzzy ideal of $\mathbb{Z}_0^-$ but not a fuzzy k-ideal of $\mathbb{Z}_0^-$ because set $x = -1$ and $y = -2$, we have $f(x) = 0 < 0.5 = \min\{f(x + y), f(y)\}$.

Example 3.8. let f be a fuzzy subset of a ternary semiring $\mathbb{Z}_0^-$ under usual addition and ternary multiplication defined by

$$f(x) = \begin{cases} 0.3 & \text{if } x \text{ is odd,}\\ 0.5 & \text{if } x \text{ is even.}\end{cases}$$

It is easy to show that f is a fuzzy k-ideal of $\mathbb{Z}_0^-$.

Lemma 3.16. Let S be a ternary semiring and f a fuzzy ideal of S. Then f is a fuzzy k-ideal of S if and only if for any $t \in [0, 1]$ such that $f_t \neq \emptyset, f_t$ is a k-ideal of S.

Proof. By Theorem 3.5, f_t is an ideal of S. Let $x, y \in f_t$ and assume $x + y, y \in f_t$. Then $f(x + y), f(y) \geq t$. Since f is a fuzzy k-ideal of S, $f(x) \geq \min\{f(x + y), f(y)\} \geq t$. So $x \in f_t$. Therefore f_t is a k-ideal of S. Conversely, assume for any $t \in [0, 1]$ such that $f_t \neq \emptyset, f_t$ is a k-ideal of S. By Theorem 3.5, f is a fuzzy ideal of S. Next, let $x, y \in S$. Set $t = \min\{f(x + y), f(y)\}$ Then $f(x + y), f(y) \geq t$. So $x + y, y \in f_t$. By assumption, f_t is a k-ideal of S, this implies $x \in f_t$. Hence $f(x) \geq t = \min\{f(x + y), f(y)\}$. Therefore f is a fuzzy k-ideal of S. □

Theorem 3.17. Let S be a ternary semiring with zero 0 and f a fuzzy ideal of S. If f is a fuzzy k-ideal of S, then f is a k-fuzzy ideal of S.

Proof. Let $x, y \in S$ such that $f(x + y) = f(0)$ and $f(y) = f(0)$. Set $t = f(0)$. So $x + y, y \in f_t$. By Lemma 3.16, the level subset f_t is a k-ideal of S. So $x \in f_t$. This implies $f(x) \geq t = f(0)$. By Lemma 3.3, $f(x) = f(0)$. □

However, the converse of Theorem 3.17 does not hold. We can see this example.

Example 3.9. Consider the ternary semiring $\mathbb{Z}_0^-$ under usual addition and ternary multiplication. Define a fuzzy subset f on $\mathbb{Z}_0^-$ by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is even,}\\ 0 & \text{if } x = -1,\\ 0.5 & \text{otherwise.}\end{cases}$$

By Example 3.6, we known that f is a k-fuzzy ideal of $\mathbb{Z}_0^-$. However, f is not a fuzzy k-ideal of $\mathbb{Z}_0^-$ because $f((-1) + (-2)) = f(-3) = 0.5$ and $f(-2) = 1$ but $f(-1) = 0 < 0.5 = \min\{f((-1) + (-2)), f(-2)\}$.

Definition 3.10. Let f and g be fuzzy subset of a non-empty subset S. A fuzzy subset $f\cap g$ of S is defined by $(f\cap g)(x) = \min\{f(x), g(x)\}$ for all $x \in S$.

Lemma 3.18. Let f and g be fuzzy subset of a ternary semiring S. If f and g are fuzzy ideals of S, then $f\cap g$ is a fuzzy ideal of S.

Proof. Let $x, y, z \in S$. We have

$$\begin{aligned}(f \cap g)(x+y) &= \min\{f(x+y), g(x+y)\} \\ &\geq \min\{f(x), f(y), g(x), g(y)\} \\ &= \min\{(f\cap g)(x), (f \cap g)(y)\}\end{aligned}$$

and

$$\begin{aligned}(f \cap g)(xyz) &= \min\{f(xyz), g(xyz)\} \\ &\geq \min\{\max\{f(x), f(y), f(z)\}, \max\{g(x), g(y), g(z)\}\} \\ &\geq \max\{(f\cap g)(x), (f\cap g)(y), (f \cap g)(z)\}.\end{aligned}$$

Hence $f \cap g$ is a fuzzy ideal of S. □

Theorem 3.19. Let f and g be fuzzy subset of a ternary semiring S. If f and g are fuzzy k-ideals of S, then $f \cap g$ is a fuzzy k-ideal of S.

Proof. By Lemma 3.18, $f\cap g$ is a fuzzy ideal of S. Let $x, y \in S$. We have

$$\begin{aligned}(f \cap g)(x) &\geq \min\{f(x), g(x)\} \\ &\geq \min\{f(x+y), f(y), g(x+y), g(y)\} \\ &= \min\{(f\cap g)(x+y), (f \cap g)(y)\}.\end{aligned}$$

Hence $f\cap g$ is a fuzzy k-ideal of S. □

Let f and g be k-fuzzy ideals of a ternary semiring S. In general, a fuzzy ideal $f\cap g$ need not be a k-fuzzy ideal of S. See this example.

Example 3.10. Consider the ternary semiring $\mathbb{Z}_0^-$ under usual addition and ternary multiplication. Let f and g be fuzzy subsets on $\mathbb{Z}_0^-$ by

$$f(x) = \begin{cases} 0.3 & \text{if } x = 0, \\ 0.1 & \text{if } x = -1, \\ 0.2 & \text{otherwise} \end{cases}$$

and $g(x) = 0.2$ for all $x \in \mathbb{Z}_0^-$. It is easy to verify that f and g are k-fuzzy ideal of $\mathbb{Z}_0^-$. We have

$$(f \cap g)(x) = \begin{cases} 0.1 & \text{if } x = -1, \\ 0.2 & \text{otherwise.} \end{cases}$$

Set $x = -1$ and $y = -2$. We have $(f \cap g)(x + y) = 0.2 = (f\cap g)(0)$ and $(f\cap g)(y) = 0.2 = (f\cap g)(0)$ but $(f\cap g)(x) = 0.1 \neq 0.2 = (f \cap g)(0)$. Thus $f \cap g$ is not k-fuzzy ideal of $\mathbb{Z}_0^-$.

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