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Free Vibration Analysis of Functionally Graded Beams

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Abstract—This work presents the highly accurate numerical calculation of the natural frequencies for functionally graded beams with simply supported boundary conditions. The Timoshenko first order shear deformation beam theory and the higher order shear deformation beam theory of Reddy have been applied to the functionally graded beams analysis. The material property gradient is assumed to be in the thickness direction. The Hamilton's principle is utilized to obtain the dynamic equations of functionally graded beams. The influences of the volume fraction index and thickness-to-length ratio on the fundamental frequencies are discussed. Comparison of the numerical results for the homogeneous beam with Euler-Bernoulli beam theory results show that the derived model is satisfactory.

Keywords—Functionally graded beam, Free vibration, Hamilton's principle.

I. INTRODUCTION

In recent years functionally graded materials (FGMs) have gained considerable importance as materials to be used in extremely high temperature environments such as nuclear reactors and high-speed spacecraft industries [1]. FGMs were first introduced by a group of scientists in Sendai Japan in 1984 [2]. FGMs are new inhomogeneous materials, in which the mechanical properties vary smoothly and continuously from one surface to the other. This is achieved by gradually varying the volume fraction of the constituent materials. This continuous change in composition results in the graded properties of FGMs [3]. This gradation in properties of the material reduces thermal stresses, residual stresses and stress concentration factors [4]. Typically these materials are made from a mixture of ceramic and metal or from a combination of different materials. The ceramic constituent of the material provides the high-temperature resistance due to its low thermal conductivity. The ductile metal constituent on the other hand, prevents fracture caused by stresses due to the high temperature gradient in a very short period of time. Furthermore a mixture of ceramic and metal with a continuously varying volume fraction can be easily manufactured [5]. The vibration of beams is important in many applications pertaining to mechanical, civil and aerospace engineering. Beams used in real practice may have appreciable thickness where the transverse shear and the rotary inertia are not negligible as assumed in the classical theories. As a result, the thick beam model based on the Timoshenko theory has gained more popularity. The static and free vibration analysis of sandwich beams were carried out to investigate the behavior of sandwich beams [6-8]. Shear deformation theories are those in which the transverse shear stresses are accounted for. The first order shear deformation theory (FSDT), commonly known as the Timoshenko beam theory, accounts for layerwise constant states of transverse shear stresses. In the first order shear deformation theory (FSDT) it is assumed that 1) the straight lines do not undergo axial deformation (i,e.,inextensible); 2) straight lines perpendicular to the midsurface (i.e., transverse normals) before deformation remain straight after deformation, whereas the third order shear deformation theory (TSDT) accounts for layerwise parabolic distribution of transverse shear stresses (i.e., assumption 2 is removed) [9].

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In the present paper, the Navier solution is developed to analyze the free vibration of simply supported functionally graded (FG) beams based on the first order and third order shear deformation theories (FSDT and TSDT). The objective is to study the frequency characteristics and the influence of graded material properties on the natural frequencies. A comparison of fundamental frequencies predicted by the two theories is presented and the results are validated with compression of the numerical results for homogeneous beam with Euler-Bernoulli beam theory results.

II. BASIC EQUATIONS

Consider a FG beam with rectangular cross section, length L, width b, and constant thickness h. The beam is assumed to be graded through the thickness direction. The constituent materials are assumed to be ceramic and metal. The volume fractions of the ceramic and metal corresponding to the power law are expressed as [10]

$$V_c = (\frac{2z+h}{2h})^k, \qquad V_m = 1 - V_c \tag{1}$$

where subscripts m and c refer to the metal and ceramic constituents, respectively. Also, z is the thickness coordinate $(-h/2 \preceq z \preceq h/2)$, and k is volume fraction index that takes values greater than or equal to zero. The variation of the composition of ceramics and metal is linear for $k = 1$.

The value of k equal to zero represents a fully ceramic beam. The mechanical properties of FGM are determined from the volume fraction of the material constituents. We assumed that the nonhomogeneous material properties, such as the modulus of elasticity E and and the density of material ρ change in the thickness direction z based on Voigt's rule over the whole range of the volume fraction [10]; while Poisson's ratio ν is assumed to be constant [11] as

$$\begin{aligned} &E = E(z) = E_c V_c + E_m(1 - V_c) \\ &\rho = \rho(z) = \rho_c V_c + \rho_m(1 - V_c) \\ &\nu(z) = \nu_0 \end{aligned} \tag{2}$$

Substituting Eqs.(1) into Eqs.(2), material properties of the FG beam are determined, which are the same as the equations proposed by Praveen and Reddy [14]

$$E = E(z) = E_m + (E_{cm})(\frac{2z+h}{2h})^k$$
$$\rho = \rho(z) = \rho_m + (\rho_{cm})(\frac{2z+h}{2h})^k$$
$$\nu(z) = \nu_0 \quad (3)$$

where

$$E_{cm} = E_c - E_m,$$
$$G_{cm} = G_c - G_m, \quad (4)$$
$$\rho_{cm} = \rho_c - \rho_m$$

The third order shear deformation theory (TSDT) of Reddy used in the present study is based on the following displacement field [13]

$$u(x,z,t) = u_0 + z[\phi_0 - \alpha z^2(\phi_0 + \frac{\partial w}{\partial x})]$$
$$w(x,z,t) = w_0 \quad (5)$$

where $\alpha = 4/(3h^2)$, and u and w denote the displacement components in the x and z directions, respectively; and ϕ_0 is the rotation of the transverse normal about y axis. All of the generalized displacements (u_0, w_0, ϕ_0) are functions of x and t. Note that the displacement field of the first order shear deformation theory (FSDT) can be deduced from Eq. (5) by setting $\alpha = 0$. The linear strain-displacement relations for plane-strain condition are

$$\epsilon_{xx} = \epsilon_{xx}^{(0)} + z\epsilon_{xx}^{(1)} + z^3\epsilon_{xx}^{(2)},$$
$$\gamma_{xz} = \gamma_{xz}^{(0)} + z^2\gamma_{xz}^{(1)} \quad (6)$$

where

$$\epsilon_{xx}^{(0)} = \frac{\partial u_0}{\partial x}, \qquad \epsilon_{xx}^{(1)} = \frac{\partial \phi_0}{\partial x},$$
$$\epsilon_{xx}^{(2)} = -\alpha(\frac{\partial \phi_0}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}) \quad (7)$$
$$\gamma_{xz}^{(0)} = \phi_0 + \frac{\partial w_0}{\partial x}, \ \ \gamma_{xz}^{(1)} = -3\alpha(\phi_0 + \frac{\partial w_0}{\partial x})$$

where ϵ_{xx} and γ_{xz} are the normal and shear strains, respectively. Hook's law for a FG beam is defined as

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{xz} \end{pmatrix} = \begin{pmatrix} Q_{11} & 0 \\ 0 & Q_{55} \end{pmatrix} \begin{pmatrix} \epsilon_{xx} \\ \gamma_{xz} \end{pmatrix} \quad (8)$$

The normal stresses σ_{zz} are assumed to be zero. Also, Q_{ij} are defined by

$$Q_{11} = \frac{E(z)}{1-\nu_0^2}, \qquad Q_{55} = \frac{E(z)}{2(1+\nu_0)} \quad (9)$$

The equations of motion appropriate for the displacement field, Eq. (5), can be derived using the Hamilton's principle as

$$\frac{\partial N_{xx}}{\partial x} = I_0\ddot{u}_0 + \bar{I}_1\ddot{\phi}_1 - \alpha I_3\frac{\partial \ddot{w}_0}{\partial x}$$
$$\frac{\partial M_{xx}}{\partial x} - \bar{Q}_x - \alpha\frac{\partial P_{xx}}{\partial x} = \bar{I}_1\ddot{u}_0 + \bar{I}_2\ddot{\phi}_1 - \alpha\bar{I}_4\frac{\partial \ddot{w}_0}{\partial x}$$
$$\frac{\partial \bar{Q}_x}{\partial x} + \alpha\frac{\partial^2 P_{xx}}{\partial x^2} + q = I_0\ddot{w}_0 + \alpha I_3\frac{\partial \ddot{u}_0}{\partial x}$$
$$+\alpha\bar{I}_4\frac{\partial \ddot{\phi}_1}{\partial x} - \alpha^2 I_6\frac{\partial^2 \ddot{w}_0}{\partial x^2} \quad (10)$$

where

$$\bar{I}_1 = I_1 - \alpha I_3, \qquad \bar{I}_2 = I_2 - 2\alpha I_4 + \alpha^2 I_6$$
$$\bar{I}_4 = I_4 - \alpha I_6, \qquad \bar{Q}_x = Q_x - 3\alpha R_x \quad (11)$$

Here the applied in-plane forces are assumed to be zero. The superposed dot denotes differentiation with respect to time, q is the distributed transverse load, and (N_{xx}, M_{xx}, P_{xx}) and (Q_{xx}, R_{xx}) are the stress and moment resultants, respectively, and defined as

$$\begin{pmatrix} N_{xx} \\ M_{xx} \\ P_{xx} \end{pmatrix} = \int_{-h/2}^{h/2} \sigma_{xx} \begin{pmatrix} 1 \\ z \\ z^3 \end{pmatrix} dz$$
$$\begin{pmatrix} Q_x \\ R_x \end{pmatrix} = \int_{-h/2}^{h/2} \sigma_{xz} \begin{pmatrix} 1 \\ z^2 \end{pmatrix} dz \quad (12)$$

Also, the inertias I_i $(i = 0,1,2,3,4,6)$ are defined by

$$I_i = \int_{-h/2}^{h/2} \rho\,(z)^i\, dz \quad (13)$$

The force and moment resultants can be expressed in terms of the strains as

$$N_{xx} = A_{11}\epsilon_{xx}^{(0)} + B_{11}\epsilon_{xx}^{(1)} + D_{11}\epsilon_{xx}^{(2)}$$
$$M_{xx} = B_{11}\epsilon_{xx}^{(0)} + E_{11}\epsilon_{xx}^{(1)} + F_{11}\epsilon_{xx}^{(2)}$$
$$P_{xx} = D_{11}\epsilon_{xx}^{(0)} + F_{11}\epsilon_{xx}^{(1)} + H_{11}\epsilon_{xx}^{(2)}$$
$$Q_x = A_{55}\gamma_{xz}^{(0)} + D_{55}\gamma_{xz}^{(1)}$$
$$R_x = D_{55}\gamma_{xz}^{(0)} + F_{55}\gamma_{xz}^{(1)} \quad (14)$$

where

$$(A_{11}, B_{11}, D_{11}, , E_{11}, F_{11}, H_{11})$$
$$= \int_{-h/2}^{h/2} (1, z, z^2, z^3, z^4, z^6)\, Q_{11}\, dz$$
$$(A_{55}, D_{55}, F_{55}) = \int_{-h/2}^{h/2} (1, z^2, z^4)\, Q_{55}\, dz \quad (15)$$

where A_{ij}, B_{ij}, and D_{ij} are extensional, bending extensional coupling, and bending stiffnessess, respectively. Also, E_{ij}, F_{ij}, and H_{ij} are high order stiffnessess.

III. The Navier Solution

Navier solution is used to analyze the free vibration problem of simply supported FG beams. We express the generalized displacements as products of undetermined functions and known trigonometric functions so as to satisfy identically the simply supported boundary conditions at $x = 0, L$

For TSDT:

$$w = N_{xx} = M_{xx} = P_{xx} = 0 \quad (16)$$

For FSDT:

$$w = N_{xx} = M_{xx} = 0 \quad (17)$$

For the free vibration case, we set the load term q to zero, and represent the displacement quantities as

$$\begin{pmatrix} \mathrm{u}_0(\mathrm{x,z,t}) \\ \mathrm{w}_0(\mathrm{x,z,t}) \\ \phi_1(\mathrm{x,z,t}) \end{pmatrix} = \begin{pmatrix} \mathrm{U}\cos\beta x \\ \mathrm{W}\sin\beta x \\ \mathrm{A}\cos\beta x \end{pmatrix} e^{-i\omega t} \quad (18)$$

where $\beta = m\eta/L$, and ω denotes the natural frequency. The representation (18) is valid for TSDT and FSDT. By substituting Eqs. (18) into Eqs. (10), three differential equations can be obtain as

$$(\mathbf{C} - \omega^2\mathbf{M})\begin{pmatrix} \mathrm{U} \\ \mathrm{W} \\ \mathrm{A} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (19)$$

where the matrices $\mathbf{C}$ and $\mathbf{M}$ are symmetric matrices and defined for TSDT and FSDT as (3×3) matrices by

$$\begin{aligned} c_{11} &= -A_{11}\beta^2 \\ c_{12} &= -(B_{11} - \alpha E_{11})\beta^2 \\ c_{13} &= \alpha E_{11}\beta^3 \\ c_{22} &= -(D_{11} - 2\alpha F_{11} + \alpha^2 K_{11})\beta^2 \\ &\qquad +6\alpha D_{55} - A_{55} - 9\alpha^2 F_{55} \\ c_{23} &= -(\alpha^2 K_{11} - \alpha F_{11})\beta^3 + (6\alpha D_{55} - A_{55} - 9\alpha^2 F_{55})\beta \\ c_{33} &= (6\alpha D_{55} - A_{55} - 9\alpha^2 F_{55})\beta^2 - \alpha^2 H_{11}\beta^4 \end{aligned} \quad (20)$$

$$\begin{aligned} m_{11} &= -I_0 \\ m_{12} &= -(I_1 - \alpha I_3) \\ m_{13} &= \alpha I_3 \beta \\ m_{22} &= -(I_2 - 2\alpha I_4 + \alpha^2 I_6) \\ m_{23} &= -\alpha\beta(\alpha - I_4) \\ m_{33} &= -(I_0 + \alpha^2 I_6 \beta^2) \end{aligned} \quad (21)$$

To obtain the nontrivial solution, the determinant should be zero

$$|\ c_{ij} - m_{ij}\omega^2\ | = 0 \quad (22)$$

By solving the achieved equation for ω, the values of natural frequencies of simply supported FG beams will be derived.

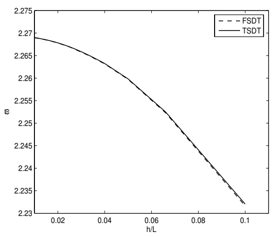
IV. Results and Discussion

The Navier solution procedure developed in the previous section is used to evaluate the natural frequencies of a FG beam. Too illustrate the behavior of the derived vibration equations, an FGM beam of aluminum and alumina is considered. Young's modulus, Poisson's ratio, and density for aluminum are 70 Gpa, 0.3 and 2707 kg/m^3. For alumina they are 380 Gpa, 0.3 and 3800 kg/m^3, respectively. Note that Poisson's ratio is selected to be constant and equal to 0.3. The effects of thickness-to-length ratio h/L and the volume fraction index k on the natural frequency of simply supported FG beam is investigated, and the non-dimensional natural frequencies obtained using the first order and third order shear deformation theories (FSDT and TSDT) for homogenous beam (k=0) are compared with Euler-Bernoulli beam theory results [12] in Table 1. As can be seen the results of two shear deformation theories are in good agreement with the Euler-Bernoulli beam theory results. Also, the frequencies predicted by the two shear deformation theories are very close to each other.

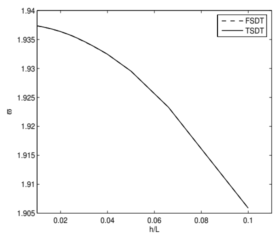
The values of the non-dimensional natural frequency of FG beams for various values of k based on two shear deformation theories are shown in Fig. 1-6. The results of the two shear deformation theories are converged to each other at $k = 1$ that shown in fig. 2. The natural frequencies decrease with increasing the thickness-to-length ratio h/L and volume fraction index k. Also, by decreasing of the thickness h the results of FSDT and TSDT are completely synchronize and converged to the constant values, because, the thickness h in TSDT effects on transvers shear stresses as a coefficient, but in FSDT the transverse shear stresses are constant along the thickness and are independent of the thickness h. In other word, with increasing the thickness of beam, the difference between FSDT and TSDT will increases.

Table 1. Non-dimensional natural frequencies of simply supported homogenous beam versus thickness-to-length ratio (k=0): $\bar{\omega} = (\omega L\ /h)(\rho_c/E_c)$.

h/L	Euler-Bernoulli [12]	FSDT	TSDT
0.01	2.985526	2.986137	2.9861380
0.0125	2.985232	2.985827	2.9858280
0.0142	2.984340	2.985556	2.9855680
0.0166	2.984865	2.985155	2.9851680
0.02	2.983701	2.984505	2.9845054
0.025	2.982588	2.983285	2.9832858
0.033	2.979668	2.980657	2.9806572
0.04	2.976570	2.978020	2.9780220
0.05	2.971688	2.973193	2.9731941
0.066	2.961235	2.962858	2.9628610
0.1	2.931568	2.934044	2.9340570



(a)



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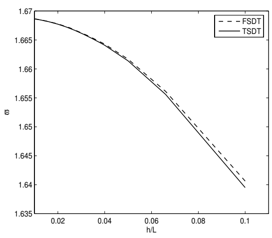

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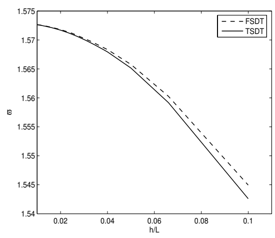

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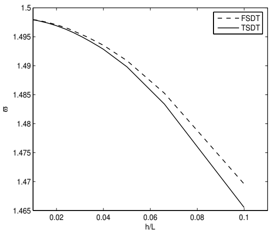

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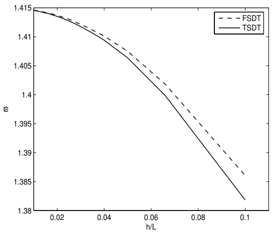

(f)

V. Conclusions

An exact analytical solution is developed for free vibration of simply supported FG beams based on the first order and third order shear deformation theories (FSDT and TSDT). The effects of volume fraction ratio and thickness-to-length ratio on fundamental frequencies are investigated. The first order and third order shear deformation theories (FSDT and TSDT) can be used replace by another one for thin beams with high accuracy, but TSDT has higher accuracy for thick beams, therefore it is better to use this theory for thick beams.

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