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Stability of Interval Fractional-order systems with order $0 < \alpha < 1$

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Abstract—In this paper, some brief sufficient conditions for the stability of FO-LTI systems $\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t)$ with the fractional order are investigated when the matrix A and the fractional order α are uncertain or both α and A are uncertain, respectively. In addition, we also relate the stability of a fractional-order system with order $0 < \alpha \leq 1$ to the stability of its equivalent fractional-order system with order $1 \leq \beta < 2$, the relationship between α and β is presented. Finally, a numeric experiment is given to demonstrate the effectiveness of our results.

Keywords—Interval Fractional-order systems, Linear matrix inequality(LMI), Asymptotical stability

I. Introduction

RECENTLY, fractional-order systems have gained considerable importance mainly due to the following two facts. First, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes, such as dielectric [1], electrode-electrolyte polarization [2] and electromagnetic wave [3]. The advantages of the fractional-order systems are that we have more degrees of freedom in the model and that a memory is included in the model. Second, fractional-order controllers such as CRONE controller [4], TID controller [5] and fractional PID controller [6] have so far been implemented to enhance the robustness and the performance of the closed loop control system.

The problem of stability is a very essential and crucial issue for control systems certainly including fractional-order systems [10], [19]. Stability of a linear fractional-order system depends on the location of the system poles in the complex. For commensurate fractional-order systems, powerful criteria have been proposed. The most well known is the Matignon's stability theorem [7]. It permits us to check the system stability through the location in the complex plane of the dynamic matrix eigenvalues of the state space like system representation. Matignon's theorem is the starting point of several results in the field.

Recently, LMI approach [8], [9], [10], Lyapunov approach [11], [12] and Lambert W function approach [13], [14] have been used to investigate the stability of FO-LTI systems. Using LMI approach and Laplace transform, sufficient conditions are proposed in [9] for stability of FO-LTI system $\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t)$ with fractional order $0 < \alpha \leq 1$ and $1 \leq \alpha < 2$, respectively. According to [9], J.G. Lu and Y.Q. Chen ([10]) proposed sufficient conditions for robust stability and stabilization of interval FO-LTI systems by using the LMI approach. The paper [15] also investigated stability of FO-LTI systems by finding a linear time invariant system with integer order that has equivalently the same stability property as of the FO-LTI system. But, until now, only a few LMI stability conditions (one of which was given in [10]) were proposed for the stability of FO-LTI system with fractional order $0 < \alpha < 1$.

In this paper, the stability of system $\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t)$ is investigated. It is organized as follows. In Section 2, the problem formulation and some preliminaries are presented. The main results are derived in Section 3. We first presents a sufficient condition for the stability of FO-LTI system $\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t)$ with the fractional order when the matrix A is uncertain. Our condition is more briefer than the paper [10], and then some sufficient stability conditions for the same system are derived when the fractional order α is uncertain and when both α and A are uncertain, respectively. In Section 4, a numeric experiment are given to demonstrate the effectiveness of our results.

II. Problem Formulation and Preliminaries

The differ-integral operator, denoted by ${}_aD_t^\alpha$, is a combined differentiation and integration operator commonly used in fractional calculus which is defined by

$$ {}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}, & \alpha > 0 \\ 1, & \alpha = 0 \\ \int_a^t (d\tau)^{-\alpha}, & \alpha < 0. \end{cases} $$

There are different definitions for fractional derivatives [16]. The most commonly used definitions are the Grünwald-Letnikov, Riemann-Liouville and Caputo definitions. The Caputo definition is sometimes called smooth fractional derivative in literature because it is suitable to be treated by the Laplace transform technique, while the Riemann-Liouville definition is unsuitable.

In the rest of the paper, D^α is used to denote the Caputo fractional derivative of order α

$$ D^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha - m)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau \quad (1) $$

where m is an integer satisfying $m - 1 < \alpha \leq m$. This paper mainly focuses on the case that the fractional order is $0 < \alpha < 1$.

Next, consider the FO-LTI system described by the following form

$$ \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) \quad (2) $$

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where α is the fractional commensurate order, $x(t) \in \mathbb{R}^n$ denotes the state vector, $A \in \mathbb{R}^{n\times n}$ is the system matrix.

If the matrix A is uncertain, then the FO-LTI system (2) can be described by state space equation of the form

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) \tag{3}$$

where $A \in [A^m, A^M] = \{[a_{ij}] : a^m{}_{ij} \le a_{ij} \le a^M{}_{ij}, 1 \le i, j \le n\}$.

If the fractional commensurate order α is uncertain, then the FO-LTI system (2) can be described by the state space equation of the form

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) \tag{4}$$

where $\alpha \in [\alpha_1, \alpha_2], \alpha_1, \alpha_2 \in \mathbb{R}$.

If there are some coupling relationships in FO-LTI interval system, the perturbation of model parameters A can be considered as a function of variable α. Therefore, the FO-LTI interval system can be described by state space equation of the form ([17])

$$\frac{d^{\alpha_0+\Delta\alpha}x(t)}{dt^{\alpha_0+\Delta\alpha}} = [A_0 + k(\alpha_0 + \Delta\alpha)\Delta A]x(t) \tag{5}$$

where $A_0 = \frac{A+\overline{A}}{2}, \alpha \in [\alpha_1, \alpha_2], \alpha_1, \alpha_2 \in \mathbb{R}, \alpha_0 = \frac{(\alpha_1+\alpha_2)}{2}, \Delta\alpha = \frac{(\alpha_2-\alpha_1)}{2}$.

Next, to prove the main results in the next section, we need the following lemmas.

Lemma 2.1 ([7]). Autonomous system

$$D^\alpha x(t) = Ax(t) \tag{6}$$

with $x(t_0) = x_0$ and $0 < \alpha < 2$ is asymptotically stable if and only if $|arg(spec(A))| > \alpha\frac{\pi}{2}$, where $spec(A)$ is the spectrum (the set of all eigenvalues)of A. Also the state vector $x(t)$ decays towards 0 and meets the following condition: $||x(t)|| < Nt^{-\alpha}, t > 0, \alpha > 0$.

Lemma 2.2 ([10]). For any matrix X and Y with appropriate dimensions, we have
$X^TY + Y^TX \le \varepsilon X^TX + \frac{1}{\varepsilon}Y^TY$, for any $\varepsilon > 0$.

Lemma 2.3 ([10]). Let X, Y, F be real matrices of suitable dimensions. Then, for any $x \in \mathbb{R}^n$

$$max\{(x^TXFYx)^2 : F^TF \le I\} = (x^TXX^Tx)(x^TY^TYx).$$

Lemma 2.4 ([18]). The FO-LTI systems (2) with order $0 < \alpha \le 1$ is asymptotically stable if there exist a matrix $X = X^H \in \mathbb{C}^{n\times n} > 0$, such that

$$\Psi \triangleq (rX + \overline{r}\overline{X})^T A^T + A(rX + \overline{r}\overline{X}) < 0, \tag{7}$$

where $r = e^{j(1-\alpha)\frac{\pi}{2}}$.

III. Main Results

A. Stability analysis of systems Eq.(2)-(5) with fractional order $0 < \alpha < 1$

LMI have played an important role in control theory since the early 1960s due to this particular form. The main issue when dealing with LMI is the convexity of the optimization set. As the stability domain of a fractional system with order $1 \le \alpha < 2$ ia a convex set, various LMI methods for defining such a region have already been developed.

In this section, a new LMI-based sufficient condition for stability of systems (3) with order $0 < \alpha < 1$ and some sufficient conditions of system (4)-(5) with order $0 < \alpha < 1$ are presented. Both our result and the condition in paper [10] are sufficient condition, but the result in this paper is more brief than that of [10].

First, for convenience, let us rewrite Lemma 2.5 as follows.

Lemma 3.1. The FO-LTI system (2) with order $0 < \alpha < 1$ is asymptotically stable if there exists a real symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a real skew-symmetric matrix $Q \in \mathbb{R}^{n\times n}$ such that

1. $\begin{bmatrix} P & Q \\ -Q & P \end{bmatrix} > 0,$
2. $\Phi = \sin\frac{\alpha\pi}{2}(PA^T + AP) + \cos\frac{\alpha\pi}{2}(QA^T - AQ) < 0.$ (8)

Proof: Using Lemma 2.4, we have the system (2) with order $0 < \alpha < 1$ is asymptotically stable if and only if

$$\Psi = (rX + \overline{r}\overline{X})^T A^T + A(rX + \overline{r}\overline{X}) < 0. \tag{9}$$

Let $X = P + jQ, P \in \mathbb{R}^{n\times n}, Q \in \mathbb{R}^{n\times n}$, $X > 0$ is equivalent to the condition 1 in Theorem 3.1. Substitute the positive matrix X and $r = e^{j(1-\alpha)\frac{\pi}{2}} = \sin\frac{\alpha\pi}{2} + j\cos\frac{\alpha\pi}{2}$ in the left side of (9), we obtain

$$\begin{aligned}\Psi &= [(\sin\tfrac{\alpha\pi}{2} + j\cos\tfrac{\alpha\pi}{2})(P + jQ)]^T \\ &\quad + [(\sin\tfrac{\alpha\pi}{2} - j\cos\tfrac{\alpha\pi}{2})(P - jQ)]^T A^T \\ &\quad + A[(\sin\tfrac{\alpha\pi}{2} + j\cos\tfrac{\alpha\pi}{2})(P + jQ)] \\ &\quad + A[(\sin\tfrac{\alpha\pi}{2} - j\cos\tfrac{\alpha\pi}{2})(P - jQ)] \\ &= 2(\sin\tfrac{\alpha\pi}{2}P + \cos\tfrac{\alpha\pi}{2}Q)A^T + 2A(\sin\tfrac{\alpha\pi}{2}P - \cos\tfrac{\alpha\pi}{2}Q) \\ &= 2[\sin\tfrac{\alpha\pi}{2}(PA^T + AP) + \cos\tfrac{\alpha\pi}{2}(QA^T - AQ)].\end{aligned} \tag{10}$$

So, $\Psi < 0$ if and only if the condition 2 in Theorem 3.1 holds. This completes the proof. □

To deal with the uncertainty interval, we introduce the following lemma.

Lemma 3.2([19]). The interval matrix A in the system (3) is equal to

$$A = \{A_0 + D_AF_AE_A | F_A{}^TF_A \le I\}, \tag{11}$$

where

$$\begin{aligned} A_0 &= \tfrac{1}{2}(A^m + A^M), \Delta A = \tfrac{1}{2}(A^M - A^m) = \{\gamma_{ij}\}_{n\times n}, \\ D_A &= [\sqrt{\gamma_{11}}e_1^n, \cdots, \sqrt{\gamma_{1n}}e_1^n, \cdots, \sqrt{\gamma_{n1}}e_n^n, \cdots, \sqrt{\gamma_{nn}}e_n^n]_{n\times n^2}, \\ E_A &= [\sqrt{\gamma_{11}}e_1^n, \cdots, \sqrt{\gamma_{1n}}e_n^n, \cdots, \sqrt{\gamma_{n1}}e_1^n, \cdots, \sqrt{\gamma_{nn}}e_n^n]^T_{n^2\times n}, \end{aligned} \tag{12}$$

and $e_k^n \in \mathbb{R}^n (k = 1, \cdots, n)$ denote the column vectors with the kth element being 1 and all the others being 0.

Next, let us establish a stable result of FO-LTI interval system (3).

Theorem 3.1. Let $A \in \mathbb{R}^{n\times n}$ and $0 < \alpha < 1$. The fractional-order system (3) is asymptotically stable if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a skew-symmetric matrix $Q \in \mathbb{R}^{n\times n}$, and two scalar constants

$\varepsilon_i > 0 (i = 1, 2)$ such that

$$1.\ \begin{bmatrix} P & Q \\ -Q & P \end{bmatrix} > 0,$$
$$2.\ \begin{bmatrix} M_1 & \sin\frac{\alpha\pi}{2} PE_A^T & \cos\frac{\alpha\pi}{2} QE_A^T \\ \sin\frac{\alpha\pi}{2} E_A P & -\varepsilon_1 I & 0 \\ \cos\frac{\alpha\pi}{2}(-E_A Q) & 0 & -\varepsilon_2 I \end{bmatrix} < 0, \quad (13)$$

where

$$M_1 = \sin\frac{\alpha\pi}{2}(PA_0{}^T + A_0 P) + \cos\frac{\alpha\pi}{2}(QA_0{}^T - A_0 Q) + (\varepsilon_1 + \varepsilon_2) D_A D_A^T.$$

Proof:Suppose that (11) holds, it follows from Lemma 3.1 that

$$\begin{aligned} \Phi &= \sin\frac{\alpha\pi}{2}(P(A_0 + D_A F_A E_A)^T + (A_0 + D_A F_A E_A)P) \\ &\quad + \cos\frac{\alpha\pi}{2}(Q(A_0 + D_A F_A E_A)^T - (A_0 + D_A F_A E_A)Q) \\ &= \sin\frac{\alpha\pi}{2}(PA_0^T + A_0 P) + \cos\frac{\alpha\pi}{2}(QA_0^T - A_0 Q) \\ &\quad + \sin\frac{\alpha\pi}{2}(PE_A^T F_A^T D_A^T + D_A F_A E_A P) \\ &\quad + \cos\frac{\alpha\pi}{2}(QE_A^T F_A^T D_A^T - D_A F_A E_A Q). \end{aligned} \quad (14)$$

Note that $F_A^T F_A \leq I$,it follows from Lemma 2.2 that for any real scalars $\varepsilon_i > 0 (i = 1, 2)$

$$\begin{aligned} &\sin\frac{\alpha\pi}{2}(PE_A^T F_A^T D_A^T + D_A F_A E_A P) \\ &\quad \leq \varepsilon_1 D_A D_A^T + \frac{1}{\varepsilon_1}\sin^2\frac{\alpha\pi}{2} PE_A^T E_A P, \\ &\cos\frac{\alpha\pi}{2}(QE_A^T F_A^T D_A^T - D_A F_A E_A Q) \\ &\quad \leq \varepsilon_2 D_A D_A^T + \frac{1}{\varepsilon_2}\cos^2\frac{\alpha\pi}{2} QE_A^T E_A(-Q). \end{aligned} \quad (15)$$

Substituting (15) into (14), one has

$$\begin{aligned} \Phi &\leq \sin\frac{\alpha\pi}{2}(PA_0^T + A_0 P) + \cos\frac{\alpha\pi}{2}(QA_0^T - A_0 Q) \\ &\quad + (\varepsilon_1 + \varepsilon_2) D_A D_A^T + \frac{1}{\varepsilon_1}\sin^2\frac{\alpha\pi}{2} PE_A^T E_A P \\ &\quad + \frac{1}{\varepsilon_2}\cos^2\frac{\alpha\pi}{2} QE_A^T E_A(-Q) \\ &= M_1 - (-\frac{1}{\varepsilon_1}\sin^2\frac{\alpha\pi}{2} PE_A^T E_A P) \\ &\quad - (-\frac{1}{\varepsilon_2}\cos^2\frac{\alpha\pi}{2} QE_A^T E_A(-Q). \end{aligned} \quad (16)$$

Using the Schur complement of (16),we have $\Phi < 0$ if the condition 2 in Theorem 3.2 holds. so the inequality (13) is the sufficient stable condition of system (3). The proof is completed. □.

Now, let us consider the FO-LTI interval system (4).

Theorem 3.2. Let $A \in \mathbb{R}^{n\times n}$ and $0 < \alpha < 1$. The fractional-order interval system (4) is asymptotically stable if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a skew-symmetric matrix $Q \in \mathbb{R}^{n\times n}$, such that

$$\begin{aligned} &\begin{bmatrix} P & Q \\ -Q & P \end{bmatrix} > 0, \\ &X(\alpha_0) = sin\frac{\alpha_0\pi}{2}(PA^T + AP) + cos\frac{\alpha_0\pi}{2}(QA^T - AQ) < 0, \\ &X(1) = PA^T + AP < 0, \\ &X(\alpha_0)\frac{\cos(\frac{\alpha_0\pi}{2} + \frac{\alpha_M\pi}{2})}{\cos\frac{\alpha_0\pi}{2}} - \frac{X(1)}{\cos\frac{\alpha_0\pi}{2}}\sin\frac{\alpha_M\pi}{2} < 0, \end{aligned} \quad (17)$$

where

$$\Delta\alpha = [\frac{\alpha_1 - \alpha_2}{2}, \frac{\alpha_2 - \alpha_1}{2}] = [-\alpha_M, \alpha_M].$$

Proof: Define

$$X(\alpha) = (sin\frac{\alpha\pi}{2}P + cos\frac{\alpha\pi}{2}Q)A^T + A(sin\frac{\alpha\pi}{2}P - cos\frac{\alpha\pi}{2}Q).$$

Substitute $\alpha = \alpha_0 + \Delta\alpha$ into $X(\alpha)$, and then it yields

$$\begin{aligned} X(\alpha) &= (\sin\frac{(\alpha_0+\Delta\alpha)\pi}{2}P + \cos\frac{(\alpha_0+\Delta\alpha)\pi}{2}Q)A^T \\ &\quad + A(\sin\frac{(\alpha_0+\Delta\alpha)\pi}{2}P - \cos\frac{(\alpha_0+\Delta\alpha)\pi}{2}Q) \\ &= (\sin\frac{\alpha_0\pi}{2}(PA^T + AP) + \cos\frac{\alpha_0\pi}{2}(QA^T - AQ))\cos\frac{\Delta\alpha\pi}{2} \\ &\quad + (\cos\frac{\alpha_0\pi}{2}(PA^T + AP) - \sin\frac{\alpha_0\pi}{2}(QA^T - AQ))\sin\frac{\Delta\alpha\pi}{2} \\ &= X(\alpha_0)\cos\frac{\Delta\alpha\pi}{2} + X(\alpha_0 + 1)\sin\frac{\Delta\alpha\pi}{2}. \end{aligned} \quad (18)$$

Noting that

$$X(\alpha_0)sin\frac{\alpha_0\pi}{2} + X(\alpha_0 + 1)cos\frac{\alpha_0\pi}{2} = X(1),$$

we have that

$$X(\alpha_0 + 1) = \frac{X(1) - X(\alpha_0)\sin\frac{\alpha_0\pi}{2}}{\cos\frac{\alpha_0\pi}{2}}. \quad (19)$$

Substitute (19) into (18), and then it yields

$$\begin{aligned} X(\alpha) &= X(\alpha_0)\cos\frac{\Delta\alpha\pi}{2} + \frac{X(1) - X(\alpha_0)\sin\frac{\alpha_0\pi}{2}}{\cos\frac{\alpha_0\pi}{2}}\sin\frac{\Delta\alpha\pi}{2} \\ &= X(\alpha_0)\frac{\cos(\frac{\alpha_0\pi}{2} + \frac{\Delta\alpha\pi}{2})}{\cos\frac{\alpha_0\pi}{2}} + \frac{X(1)}{\cos\frac{\alpha_0\pi}{2}}\sin\frac{\Delta\alpha\pi}{2}. \end{aligned} \quad (20)$$

We have that $0 < \alpha_0 < 1, 0 < \alpha_0 + \Delta\alpha < 1$, and $\sin\frac{\alpha_0\pi}{2} > 0, \cos\frac{\alpha_0\pi}{2} > 0, -\sin\alpha_M \leq \sin\Delta\alpha \leq \sin\alpha_M$, $\cos(\alpha_0 + \alpha_M) \leq \cos(\frac{\alpha_0\pi}{2} + \frac{\Delta\alpha\pi}{2}) \leq \cos(\alpha_0 - \alpha_M)$, such that

$$\begin{aligned} X(\alpha) &= X(\alpha_0)\frac{\cos(\frac{\alpha_0\pi}{2} + \frac{\Delta\alpha\pi}{2})}{\cos\frac{\alpha_0\pi}{2}} + \frac{X(1)}{\cos\frac{\alpha_0\pi}{2}}\sin\frac{\Delta\alpha\pi}{2} \\ &\leq X(\alpha_0)\frac{\cos(\frac{\alpha_0\pi}{2} + \frac{\alpha_M\pi}{2})}{\cos\frac{\alpha_0\pi}{2}} - \frac{X(1)}{\cos\frac{\alpha_0\pi}{2}}\sin\frac{\alpha_M\pi}{2} < 0. \end{aligned} \quad (21)$$

That is, the system (4) is asymptotically stable. □

Similarly, a stable result of FO-LTI interval system (5) can be also established.

Theorem 3.3. Let $A \in \mathbb{R}^{n\times n}$ and $0 < \alpha < 1$. The fractional-order interval system (5) is asymptotically stable if there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, a skew-symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and four real scalar constant $\varepsilon_i > 0, i = 1, 2, 3, 4$, such that

$$\begin{aligned} &\begin{bmatrix} P & Q \\ -Q & P \end{bmatrix} > 0, \\ &\begin{bmatrix} M_2 & M_3 & M_4 & M_5 & M_6 \\ M_3^T & -\varepsilon_1 I & 0 & 0 & 0 \\ M_4^T & 0 & -\varepsilon_2 I & 0 & 0 \\ M_5^T & 0 & 0 & -\varepsilon_3 I & 0 \\ M_6^T & 0 & 0 & 0 & -\varepsilon_4 I \end{bmatrix} < 0, \end{aligned} \quad (22)$$

where

$$\begin{aligned} M_2 &= Y(A_0, \alpha_0)\frac{\cos(\frac{\alpha_0\pi}{2} + \frac{\alpha_M\pi}{2})}{\cos\frac{\alpha_0\pi}{2}} + Y(A_0, 1)\frac{\sin\frac{\alpha_M\pi}{2}}{\cos\frac{\alpha_0\pi}{2}} \\ &\quad + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4) D_A D_A^T, \\ M_3 &= k\alpha_0 PE_A^T, \quad M_4 = k\alpha_0 QE_A^T, \\ M_5 &= k\frac{\alpha_2 - \alpha_1}{2} PE_A^T, \quad M_6 = k\frac{\alpha_2 - \alpha_1}{2} QE_A^T. \end{aligned}$$

Proof. Let $Y(A, \alpha) = \sin\frac{\alpha\pi}{2}(PA^T + AP) + \cos\frac{\alpha\pi}{2}(QA^T - AQ)$. Substitute $A = A_0 + k(\alpha_0 + \Delta\alpha)\Delta A = A_0 + k(\alpha_0 +$

$\Delta\alpha)D_AF_AE_A$ into $Y(A,\alpha)$, we obtain that

$$\begin{array}{l} Y(A,\alpha) \\ = \sin\frac{\alpha\pi}{2}(P(A_0+k\alpha D_AF_AE_A)^T + (A_0+k\alpha D_AF_AE_A)P) \\ \quad + \cos\frac{\alpha\pi}{2}(Q(A_0+k\alpha D_AF_AE_A)^T - (A_0+k\alpha D_AF_AE_A)Q) \\ = \sin\frac{\alpha\pi}{2}(PA_0^T + A_0P) + \cos\frac{\alpha\pi}{2}(QA_0^T - A_0Q) \\ \quad + \sin\frac{\alpha\pi}{2}(k\alpha PE_A^TF_A^TD_A^T + k\alpha D_AF_AE_AP) \\ \quad + \cos\frac{\alpha\pi}{2}(k\alpha QE_A^TF_A^TD_A^T - k\alpha D_AF_AE_AQ). \end{array} \tag{23}$$

Set

$$\begin{array}{l} Y(A_0,\alpha) = \sin\frac{\alpha\pi}{2}(PA_0^T + A_0P) + \cos\frac{\alpha\pi}{2}(QA_0^T - A_0Q), \\ Y(\Delta A,\alpha) = \sin\frac{\alpha\pi}{2}(k\alpha PE_A^TF_A^TD_A^T + k\alpha D_AF_AE_AP) \\ \qquad\qquad + \cos\frac{\alpha\pi}{2}(k\alpha QE_A^TF_A^TD_A^T - k\alpha D_AF_AE_AQ), \end{array}$$

then

$$Y(A,\alpha) = Y(A_0,\alpha) + Y(\Delta A,\alpha). \tag{24}$$

For $Y(A_0,\alpha)$, substitute $\alpha = \alpha_0 + \Delta\alpha$ and according to Theorem 3.3, we obtain

$$Y(A_0,\alpha) \le Y(A_0,\alpha_0)\frac{\cos(\frac{\alpha_0\pi}{2} + \frac{\alpha_M\pi}{2})}{\cos\frac{\alpha_0\pi}{2}} + Y(A_0,1)\frac{\sin\frac{\alpha_M\pi}{2}}{\cos\frac{\alpha_0\pi}{2}}. \tag{25}$$

For $Y(\Delta A,\alpha)$, substitute $\alpha = \alpha_0 + \Delta\alpha$, we get

$$\begin{array}{l} Y(\Delta A,\alpha) \\ = \sin\frac{\alpha\pi}{2}(k\alpha PE_A^TF_A^TD_A^T + k\alpha D_AF_AE_AP) \\ \quad + \cos\frac{\alpha\pi}{2}(k\alpha QE_A^TF_A^TD_A^T - k\alpha D_AF_AE_AQ) \\ = \sin\frac{(\alpha_0+\Delta\alpha)\pi}{2}k\alpha_0(PE_A^TF_A^TD_A^T + D_AF_AE_AP) \\ \quad + \sin\frac{(\alpha_0+\Delta\alpha)\pi}{2}k\Delta\alpha(PE_A^TF_A^TD_A^T + D_AF_AE_AP) \\ \quad + \cos\frac{(\alpha_0+\Delta\alpha)\pi}{2}k\alpha_0(QE_A^TF_A^TD_A^T - D_AF_AE_AQ) \\ \quad + \cos\frac{(\alpha_0+\Delta\alpha)\pi}{2}k\Delta\alpha(QE_A^TF_A^TD_A^T - D_AF_AE_AQ). \end{array} \tag{26}$$

Using Lemma 2.3, we have

$$\begin{array}{l} \sin\frac{(\alpha_0+\Delta\alpha)\pi}{2}k\alpha_0(PE_A^TF_A^TD_A^T + D_AF_AE_AP) \\ \quad \le \varepsilon_1D_AF_AF_A^TD_A^T + \varepsilon_1^{-1}\sin^2\frac{(\alpha_0+\Delta\alpha)\pi}{2}(k\alpha_0)^2PE_A^TE_AP \\ \quad \le \varepsilon_1D_AD_A^T + \varepsilon_1^{-1}(k\alpha_0)^2PE_A^TE_AP, \\ \sin\frac{(\alpha_0+\Delta\alpha)\pi}{2}k\Delta\alpha(PE_A^TF_A^TD_A^T + D_AF_AE_AP) \\ \quad \le \varepsilon_3D_AF_AF_A^TD_A^T + \varepsilon_3^{-1}\sin^2\frac{(\alpha_0+\Delta\alpha)\pi}{2}(k\Delta\alpha)^2PE_A^TE_AP \\ \quad \le \varepsilon_3D_AD_A^T + \varepsilon_3^{-1}k^2\frac{(\alpha_2-\alpha_1)}{4}^2PE_A^TE_AP, \\ \cos\frac{(\alpha_0+\Delta\alpha)\pi}{2}k\alpha_0(QE_A^TF_A^TD_A^T - D_AF_AE_AQ) \\ \quad \le \varepsilon_2D_AF_AF_A^TD_A^T + \varepsilon_2^{-1}\cos^2\frac{(\alpha_0+\Delta\alpha)\pi}{2}(k\alpha_0)^2QE_A^TE_A(-Q) \\ \quad \le \varepsilon_2D_AD_A^T + \varepsilon_2^{-1}(k\alpha_0)^2QE_A^TE_A(-Q), \\ \cos\frac{(\alpha_0+\Delta\alpha)\pi}{2}k\Delta\alpha(QE_A^TF_A^TD_A^T - D_AF_AE_AQ) \\ \quad \le \varepsilon_4D_AF_AF_A^TD_A^T \\ \quad\; +\varepsilon_4^{-1}\cos^2\frac{(\alpha_0+\Delta\alpha)\pi}{2}(k\Delta\alpha)^2QE_A^TE_A(-Q) \\ \quad \le \varepsilon_4D_AD_A^T + \varepsilon_4^{-1}k^2\frac{(\alpha_2-\alpha_1)}{4}^2QE_A^TE_A(-Q). \end{array} \tag{27}$$

Substituting (27) into (24), we obtain

$$\begin{array}{l} Y(A,\alpha) = Y(A_0,\alpha) + Y(A\Delta,\alpha) \\ \le Y(A_0,\alpha_0)\frac{\cos(\frac{\alpha_0\pi}{2}+\frac{\alpha_M\pi}{2})}{\cos\frac{\alpha_0\pi}{2}} + Y(A_0,1)\frac{\sin\frac{\alpha_M\pi}{2}}{\cos\frac{\alpha_0\pi}{2}} \\ \quad +\varepsilon_1D_AD_A^T + \varepsilon_2D_AD_A^T + \varepsilon_3D_AD_A^T + \varepsilon_4D_AD_A^T \\ \quad +\varepsilon_1^{-1}(k\alpha_0)^2PE_A^TE_AP + \varepsilon_2^{-1}(k\alpha_0)^2QE_A^TE_A(-Q) \\ \quad +\varepsilon_3^{-1}k^2\frac{(\alpha_2-\alpha_1)}{4}^2PE_A^TE_AP \\ \quad +\varepsilon_4^{-1}k^2\frac{(\alpha_2-\alpha_1)}{4}^2QE_A^TE_A(-Q) \\ = Y(A_0,\alpha_0)\frac{\cos(\frac{\alpha_0\pi}{2}+\frac{\alpha_M\pi}{2})}{\cos\frac{\alpha_0\pi}{2}} + Y(A_0,1)\frac{\sin\frac{\alpha_M\pi}{2}}{\cos\frac{\alpha_0\pi}{2}} \\ \quad +(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)D_AD_A^T \\ \quad -(-\varepsilon_1^{-1})(k\alpha_0)^2PE_A^TE_AP \\ \quad -(-\varepsilon_2^{-1})(k\alpha_0)^2QE_A^TE_A(-Q) \\ \quad -(-\varepsilon_3^{-1})k^2\frac{(\alpha_2-\alpha_1)}{4}^2PE_A^TE_AP \\ \quad -(-\varepsilon_4^{-1})k^2\frac{(\alpha_2-\alpha_1)}{4}^2QE_A^TE_A(-Q). \end{array} \tag{28}$$

Using the Schur complement of (28), one obtains that the FO-LTI interval system (5) with order $0 < \alpha < 1$ is asymptotically stable. □

B. Equivalent 0f fractional order systems with order $0 < \alpha \le 1$ and with order $1 \le \beta < 2$

In this section, stability relations of a fractional-order system with order $0 < \alpha \le 1$ and its equivalent fractional-order system with order $1 \le \beta < 2$ are given, and the relationship between α and β is also presented.

Theorem 3.4. All eigenvalues of the FO-LTI system (2) with order $0 < \alpha \le 1$ and output $u(t) = 0$ settle in the unstable region if and only if the fractional-order system

$$\frac{d^\beta x(t)}{dt^\beta} = -Ax(t), \quad 1 \le \beta = 2-\alpha < 2 \tag{29}$$

is asymptotically stable, see Figure 1.

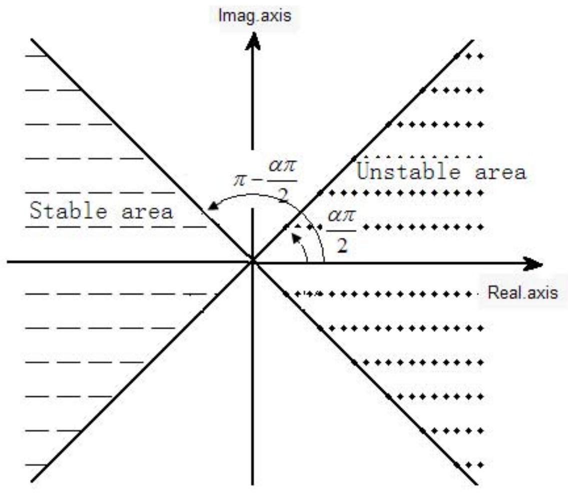

Fig. 1: Relation of stability domain and unstability domain.

Proof. Since all eigenvalues of the FO-LTI system (2) with order $0 < \alpha \le 1$ and output $u(t) = 0$ lie on the unstable region $\Omega_1 = \left\{\lambda : |arg(\lambda) < \alpha\frac{\pi}{2}\right\}$, then the eigenvalues of matrix $-A$ settle in the region

$$\Omega_2 = \left\{\lambda : |arg(\lambda) > \pi - \alpha\frac{\pi}{2}\right\} = \left\{\lambda : |arg(\lambda) > (2-\alpha)\frac{\pi}{2}\right\}.$$

Set $\beta = 2 - \alpha$, then $1 \leq \beta < 2$. One can rewrite the set Ω_2 as

$$\Omega_2 = \left\{\lambda : |arg(\lambda) > \beta\frac{\pi}{2}\right\}.$$

Therefore, according to Lemma 2.1, the fractional-order system (**??**) is asymptotically. The proof of the inverse is similar and it is not given here. □

Remark 3.1: Theorem 3.4 relates the stability of a fractional-order system with order $0 < \alpha \leq 1$ to the stability of its equivalent fractional-order system with order $1 \leq \beta < 2$. Thus, one can obtain some other analogical conclusions on the order $0 < \alpha \leq 1$ systems, according to the corresponding ones on the order $1 \leq \beta < 2$ systems.

IV. Numerical Examples

Example 4.1. Consider the stability of the following interval FO-LTI system

$$\frac{d^\alpha x(t)}{dt^\alpha} = Ax(t)$$

where $\alpha = 0.5$, and $A \in [A^m, A^M]$ with

$$A^m = \begin{bmatrix} -1.95 & 0.35 & 0.7 \\ -1.3 & -3.9 & 0.7 \\ -0.65 & -1.95 & -3.25 \end{bmatrix},$$

$$A^M = \begin{bmatrix} -1.05 & 0.65 & 1.3 \\ -0.7 & -2.1 & 1.3 \\ -0.35 & -1.05 & -1.75 \end{bmatrix}.$$

Using the Matlab LMI toolbox, it is found that the linear matrix inequalities (13) in Theorem 3.2 are easily feasible.

V. Conclusions

In summary, this paper presents some brief sufficient conditions for the stability of a class of FO-LTI system with uncertain parameters, which may be easily feasible by the Matlab LMI toolbox. In addition, we also relate the stability of a fractional-order system with order $0 < \alpha \leq 1$ to the stability of its equivalent fractional-order system with order $1 \leq \beta < 2$, and the relationship between α and β is also presented.

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