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Quadrilateral Decomposition by Two-Ear Property Resulting in CAD Segmentation

Maharavo Randrianarivony

***Abstract*—The objective is to split a simply connected polygon into a set of convex quadrilaterals without inserting new boundary nodes. The presented approach consists in repeatedly removing quadrilaterals from the polygon. Theoretical results pertaining to quadrangulation of simply connected polygons are derived from the usual 2-ear theorem. It produces a quadrangulation technique with $\mathcal{O}(n)$ number of quadrilaterals. The theoretical methodology is supplemented by practical results and CAD surface segmentation.**

***Keywords*—Quadrangulation, simply connected, two-ear theorem.**

I. Introduction

QUADRILATERAL decomposition is an important structure in computational geometry. It has several important applications in computer graphics, numerics and engineering [7], [6]. The main purpose of this paper is to propose a methodology for splitting a simply connected polygon into quadrilaterals. In fact, this is a completion of a previous work with Brunnett as described in [8], [9], [10]. The main contribution in this paper can be summarized as follows:

- Quadrilateral decomposition without prior triangulation,
- Introducing two operations for quadrilateral removal,
- Theoretical proofs supporting the algorithm,
- The number of quadrilaterals is $\mathcal{O}(n)$,
- Practical implementation and CAD application.

Related works are as follows, Ramaswami *et al.* [7] used the *percolation algorithm* to transform a triangular mesh into a quadrilateral one by using graph-based approach. Lee and Lo [4] have used a method which needs the *merging front* that is initialized to be the polygonal boundary of the domain. It is also an indirect method which needs a background triangular mesh. It recursively tries to merge a triangle which is incident upon the merging front and an adjacent one. The authors failed to give any theoretical proof of their merging algorithm. The Q-Morph (or quad morphing) algorithm [6] uses a similar approach as the above method where the author uses an advancing front consisting of a set of edges that must be updated every time new quadrilaterals are formed. In [1], the authors first fill the domain with circles which are tangent to one another. The gaps in the domain are then bounded by a few circles. The *circle packing* method amounts to generate edges whose endpoints are centers of those circles.

M. Randrianarivony is with the Faculty for Mathematics, University of Bonn, Germany, email: maharavo@informatik.tu-chemnitz.de.

The structure of this paper is as follows. First, it starts by formulating the problem accurately and by introducing several definitions. Section III introduces two operations with which a single quadrilateral can be removed from a polygon. In Section IV, those two operations are used to design an algorithm for quadrangulation. The purpose of Section V is twofold. First, it shows results from practical implementation of the theoretical methods. Second, it briefly describes how to apply the presented method to real CAD models.

II. Problem Setting and Definitions

Let P be a simply connected polygon with an even number of boundary vertices $\{\mathbf{x}_k\}$ $k=1,...,n$. The objective is to find a set of convex quadrilaterals $\{Q_i\}$ such that (see Fig. 1(a)):

(P0) $P=\bigcup_{i=1}^{m} Q_i$.
(P1) For $i,j=1,...,m$ $(i\neq j)$ the intersection $Q_i\cap Q_j$ is either empty or a single node or a complete edge.
(P2) Each vertex of a quadrilateral Q_i is either an element of $\{\mathbf{x}_k\}_{k=1}^{n}$ or it is strictly inside P. In other words, boundary Steiner points are not allowed.

It was required that the number of boundary nodes be even in order to guarantee [7] the solvability of this problem. To simplify the description of the approach, the following notations and definitions are introduced.
For two given points $\mathbf{a}$ and $\mathbf{b}$ in the plane, $[\mathbf{a},\mathbf{b}]$ and $]\mathbf{a},\mathbf{b}[$ will denote the closed and open line segments defined by

$$[\mathbf{a},\mathbf{b}] \;:=\; \{\lambda\mathbf{a}+(1-\lambda)\mathbf{b}, \quad \lambda\in[0,1]\subset\mathbf{R}\}, \quad (1)$$
$$]\mathbf{a},\mathbf{b}[\;:=\; \{\lambda\mathbf{a}+(1-\lambda)\mathbf{b}, \quad \lambda\in]0,1[\subset\mathbf{R}\}. \quad (2)$$

The line which passes through $\mathbf{a}$ and $\mathbf{b}$ splits the plane into two *half planes*:

$$(\mathbf{ab})^{+} \;:=\; \{\mathbf{z}\in\mathbf{R}^2: \quad \det(\vec{\mathbf{az}},\vec{\mathbf{ab}})>0\}, \quad (3)$$
$$(\mathbf{ab})^{-} \;:=\; \{\mathbf{z}\in\mathbf{R}^2: \quad \det(\vec{\mathbf{az}},\vec{\mathbf{ab}})<0\} \quad (4)$$

As in most papers in computational geometry, the vertices of a polygon is given in counter-clockwise orientation. Let $\mathbf{x}_{i-1}$, $\mathbf{x}_i$, $\mathbf{x}_{i+1}$ be three consecutive vertices of a polygon P. The next region is called the *wedge* of $\mathbf{x}_i$

$$\mathcal{W}(\mathbf{x}_i):=(\mathbf{x}_{i-1}\mathbf{x}_i)^{-}\cap(\mathbf{x}_{i+1}\mathbf{x}_i)^{+}. \quad (5)$$

The vertex $\mathbf{x}_i$ is called a *reflex* vertex if $\det(\mathbf{x}_i-\mathbf{x}_{i-1},\mathbf{x}_{i+1}-\mathbf{x}_i)<0$. A point $\mathbf{a}$ is *visible* from a vertex $\mathbf{x}_k\in P$ if $]\mathbf{a},\mathbf{x}_k[$ does not intersect any edge of P. The *kernel* $\ker(P)$ of P is

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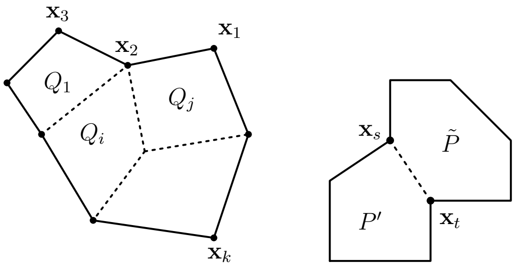
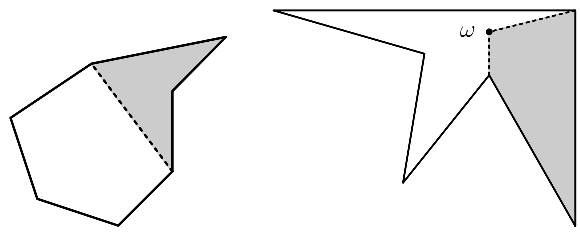


(a) (b)

Fig. 1. (a) Quadrangulation (b) Chopping off a polygon from an initial one.



(a) (b)

Fig. 2. (a) Chop off a quadrilateral with one cut (b) Remove a quadrilateral with two cuts and one internal node.

the set of points inside P which are visible from all vertices of P
A *cut* within P is a line segment $[\mathbf{x}_p, \mathbf{x}_q]$ which is formed by two non-consecutive vertices of P and which is inside P such that $]\mathbf{x}_p, \mathbf{x}_q[$ does not intersect any edge of P. Chopping off a subpolygon P' from P means introducing a cut e that splits P into P' and the remaining polygon $\tilde{P}$, i.e.

$$P = P' \cup \tilde{P}, \quad \text{and} \quad P' \cap \tilde{P} = e. \tag{6}$$

An *ear* T of a polygon P is a triangle formed by three consecutive vertices of P such that one edge of T is a cut. Since the presented method is based on the 2-ear theorem [5], the next theorem is recalled.

Theorem 1 (Meister, 1975): Every simply connected polygon having at least four vertices has two nonoverlapping ears.

III. Two Operations for Quadrilateral Removal

The next theorem will be used to split a simply connected polygon into quadrilaterals without additional boundary vertices.

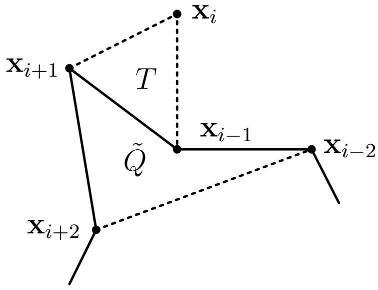
Theorem 2: Let P be a simply connected polygon having at least five vertices. Then, one of the next two operations can be applied:

(Op1) One can remove a quadrilateral which is not necessarily convex by inserting a single cut as in Fig. 2(a).

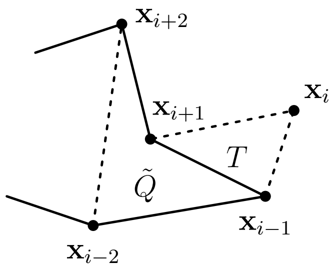
(Op2) There exists a point ω in the interior of P such that one can remove a *convex* quadrilateral by inserting two line segments emanating from ω to two vertices of P (Fig. 2(b)).

Proof: It is proved by induction with respect to the number n of vertices of P. For $n = 5$, use the theorem of Meister to chop off one triangle from P and the remaining four vertices form the quadrilateral which can be discarded from P by using operation (Op1). Suppose that the claim holds for every polygon having n vertices. Now, consider a polygon P with $n+1$ vertices. A quadrilateral Q which can be discarded from P is now sought.
First, apply the 2-ear theorem to chop off a triangle $T := [\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}]$ from P and denote by $\tilde{P}$ the remaining polygon which must have n vertices. That is,

$$\tilde{P} := [\mathbf{x}_0, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_n]. \tag{7}$$



(a)



(b)

Fig. 3. The ear $[\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}]$ and the quadrilateral $\tilde{Q}$ are adjacent.

After applying the hypothesis of induction to the new polygon $\tilde{P}$, a quadrilateral $\tilde{Q}$ is obtained. Three different cases have to be distinguished.
Case 1: If the quadrilateral $\tilde{Q}$ is not incident upon the edge $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$ of T, then one simply defines $Q := \tilde{Q}$.
Case 2: Suppose that $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$ is an edge of $\tilde{Q}$ and all vertices of $\tilde{Q}$ are elements of $\tilde{P}$ (Fig. 3 and Fig. 4). Three subcases are investigated.
Case 2.a: Assume that $\tilde{Q} = [\mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}]$ as in Fig. 3. Observe that with respect to the quadrilateral $\tilde{Q}$, $\mathbf{x}_{i-1}$ is visible from $\mathbf{x}_{i+2}$ (Fig. 3(a)) or $\mathbf{x}_{i+1}$ is visible from $\mathbf{x}_{i-2}$ (Fig. 3(b)) (In fact, you can apply the 2-ear theorem to $\tilde{Q}$). In the first situation, define $Q := [\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{x}_{i-1}]$. In the other case, define $Q := [\mathbf{x}_{i+1}, \mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_i]$. In other words, operation (Op1) was just applied to P in case 2.a.
Case 2.b: Assume now that $\tilde{Q} = [\mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{x}_{i+3}]$ as in Fig. 4. If the quadrilateral $\tilde{Q}$ is convex, then (Op1) is applied by defining

$$Q := [\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}]. \tag{8}$$

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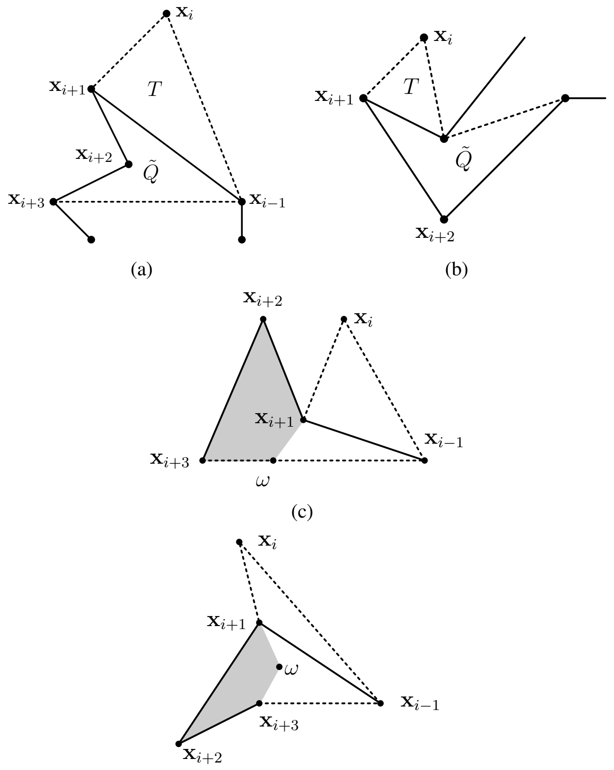

Fig. 4. (a)$\mathbf{x}_{i+2}$ is reflex in $\tilde{Q}$ (b)$\mathbf{x}_{i-1}$ is reflex in $\tilde{Q}$ (c)$\mathbf{x}_{i+1}$ is reflex (d)$\mathbf{x}_{i+3}$ is the reflex vertex in $\tilde{Q}$.

In the situation that $\tilde{Q}$ is nonconvex, four configurations with respect to the position of the reflex vertex within $\tilde{Q}$ are distinguished:

(i) If the vertex $\mathbf{x}_{i+2}$ is the reflex vertex as depicted in Fig. 4(a), define $Q := [\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}]$.

(ii) If $\mathbf{x}_{i-1}$ is reflex (Fig. 4(b)), define $Q := [\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}]$.

(iii) In the case that $\mathbf{x}_{i+1}$ is the reflex vertex of $\tilde{Q}$, take any node ω on the open segment $]\mathbf{x}_{i+3}, \mathbf{x}_{i-1}[$ in the wedge of $\mathbf{x}_{i+2}$ as illustrated in Fig. 4(c) and apply (Op2) by defining

$$Q := [\mathbf{x}_{i+2}, \mathbf{x}_{i+3}, \omega, \mathbf{x}_{i+1}]. \quad (9)$$

Observe that in this case Q is convex because ω is visible from $\mathbf{x}_{i+1}$, $\mathbf{x}_{i+2}$, $\mathbf{x}_{i+3}$.

(iv) If $\mathbf{x}_{i+3}$ is the reflex vertex of $\tilde{Q}$, then Q is defined as in (9) but the internal Steiner point ω is chosen within the interior of the triangle (Fig. 4(d)) $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \mathbf{x}_{i+3}]$ and within the wedge of $\mathbf{x}_{i+2}$.

Case 2.c: If $\tilde{Q} = [\mathbf{x}_{i-3}, \mathbf{x}_{i-2}, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$, then proceed analogously to case 2.b.

Case 3: This case considers the situation that the segment $[\mathbf{x}_{i-1}, \mathbf{x}_{i+1}]$ is an edge of $\tilde{Q}$ which has a vertex $\tilde{\omega}$ that is not a vertex of $\tilde{P}$ (Fig. 5(a)). Then $\tilde{Q}$ is convex and any point ω is chosen within the interior of $\tilde{Q}$ and within the wedge of $\mathbf{x}_i$. Due to the convexity of $\tilde{Q}$, both $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ must be visible from the node ω. Therefore, define

$$Q := [\mathbf{x}_i, \mathbf{x}_{i+1}, \omega, \mathbf{x}_{i-1}] \quad (10)$$

as a quadrilateral which can be removed from P by using operation (Op2). ■

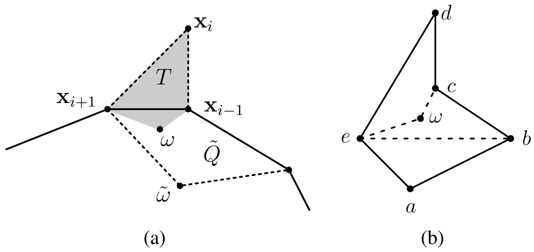

Fig. 5. (a)Introducing a Steiner point ω in $Q \cap \mathcal{W}(\mathbf{x}_i)$ (b) Case of a pentagon.

IV. Quadrilateral Decomposition

After applying operation (Op1) to a polygon having n vertices, the number of vertices of the remaining polygon is reduced to $(n-2)$. However, applying operation (Op2) as illustrated in Fig. 2(b) does not reduce the number of vertices. Thus, it is not obvious that the recursive application of the above theorem splits a polygon into a set of quadrilaterals.

Theorem 3: The internal node ω can be chosen in such a way that if a quadrilateral Q_1 has been removed from P via operation (Op2), then there is a quadrilateral Q_2 adjacent to Q_1 that can be removed from $P \setminus Q_1$ via operation (Op1). Thus, there are four cases which are illustrated in Fig. 6:

$$Q_1 = [\omega, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}] \quad Q_2 = [\omega, \mathbf{x}_{i-3}, \mathbf{x}_{i-2}, \mathbf{x}_{i-1}], \quad (11)$$

$$Q_1 = [\omega, \mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{x}_{i+1}] \quad Q_2 = [\omega, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \mathbf{x}_{i+3}], \quad (12)$$

$$Q_1 = [\mathbf{x}_i, \mathbf{x}_{i+1}, \omega, \mathbf{x}_{i-1}] \quad Q_2 = [\mathbf{x}_{i-1}, \omega, \mathbf{x}_{i+1}, \mathbf{x}_{i+2}], \quad (13)$$

$$Q_1 = [\mathbf{x}_{i+1}, \mathbf{x}_{i+2}, \omega, \mathbf{x}_i] \quad Q_2 = [\mathbf{x}_i, \omega, \mathbf{x}_{i+2}, \mathbf{x}_{i-1}]. \quad (14)$$

In cases (11) and (12), the union $Q_1 \cup Q_2$ is a hexagon, while it is a quadrilateral in (13) and (14).

Proof: This is proved by using induction in a very similar manner as in the preceding theorem where one ear T is removed to obtain an auxiliary polygon with n vertices from a polygon with $(n+1)$ vertices. For the case $n = 5$, suppose that the vertices are $[a, b, c, d, e]$ as Fig. 5(b). Discard an ear which is supposedly $[a, b, e]$. For the remaining quadrilateral, choose a point $\omega \in \mathcal{W}(d) \cap (ec)^+$. Therefore, $Q_1 := [e, \omega, c, d]$ and $Q_2 := [c, \omega, e, b]$.

Suppose in the hypothesis of induction that there are two quadrilaterals $\tilde{Q}_1$ and $\tilde{Q}_2$ and an internal node $\tilde{\omega}$ fulfilling relation (11) or (12) or (13) or (14). Let us prove that after removing an ear T, two quadrilaterals Q_1 and Q_2 and a point ω satisfying those relations can be found. The trivial case consists of an ear which is neither incident upon $\tilde{Q}_1$ nor

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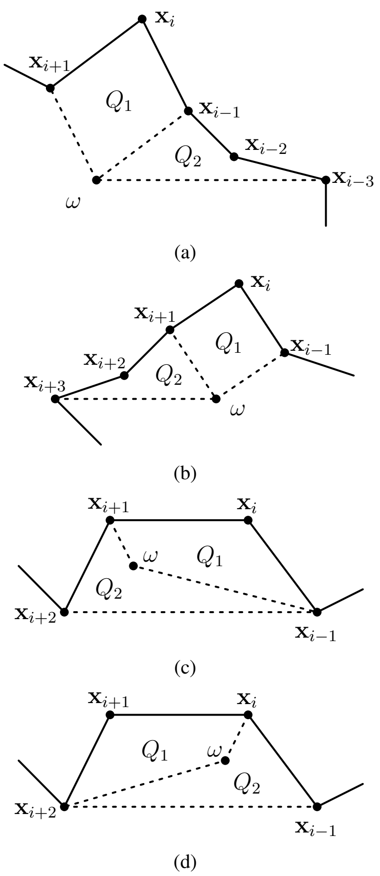


Fig. 6. Q_1 and Q_2 which are discarded by using operations (Op1) and (Op2) are adjacent.

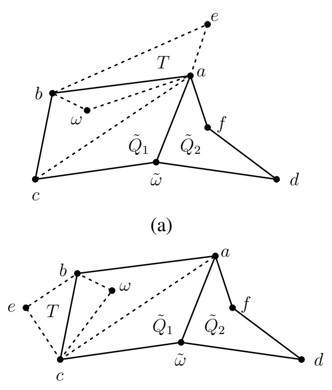


Fig. 7. (a)Ear T incident upon edge $[a, b]$ of $\tilde{Q}_1$, (b)Ear T incident upon edge $[b, c]$ of $\tilde{Q}_1$.

upon $\tilde{Q}_2$. In such a case, $Q_1 := \tilde{Q}_1$, $Q_2 := \tilde{Q}_2$ and $\omega := \tilde{\omega}$. Two nontrivial cases are distinguished according to whether $\tilde{Q}_1 \cup \tilde{Q}_2$ is a hexagon or a quadrilateral.

Case A: Suppose $\tilde{Q}_1 \cup \tilde{Q}_2$ is a (non-convex) hexagon. Without loss of generality, it is assumed that $\tilde{Q}_1$ and $\tilde{Q}_2$ satisfy (11). Let the vertices of $\tilde{Q}_1$ be denoted by $[a, b, c, \tilde{\omega}]$,

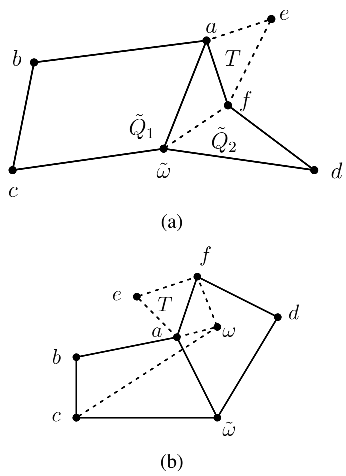


Fig. 8. Case where an ear T is incident upon the edge $[a, f]$ of $\tilde{Q}_2$.

and those of $\tilde{Q}_2$ by $[d, f, a, \tilde{\omega}]$. Let e be the vertex of T which does not belong to $\tilde{Q}_1 \cup \tilde{Q}_2$ as illustrated in Fig. 7. Four subcases are considered with respect to the position of the ear T.

Case A.1: If the ear T is incident upon the edge $[a, b]$ of $\tilde{Q}_1$ as illustrated in Fig. 7(a), consider the diagonal $[c, a]$ which must be inside the quadrilateral $\tilde{Q}_1$ because $\tilde{Q}_1$ is convex. Choose then a point $\omega \in \tilde{Q}_1$ which is in $(ca)^-$ and which is visible from the vertex e. Define $Q_1 := [\omega, a, e, b]$ and $Q_2 := [c, a, \omega, b]$ which are adjacent and which form a quadrilateral union.

Case A.2: If the ear T is incident upon the edge $[b, c]$ of $\tilde{Q}_1$ as in Fig. 7(b), proceed as in case A.1 but define now $Q_1 := [e, c, \omega, b]$ and $Q_2 := [b, \omega, c, a]$.

Case A.3: Suppose now that the ear T is incident upon the edge $[a, f]$ as in Fig. 8. If $\tilde{\omega} \in \mathcal{W}(e)$ (see Fig. 8(a)), define $\omega := \tilde{\omega}$, $Q_2 := \tilde{Q}_1$ and $Q_1 := [a, \omega, f, e]$ which is a convex quadrilateral. Else, choose $w \in \mathcal{W}(e)$ such that ω is visible from c (see Fig. 8(b)). Define afterwards $Q_1 := [\omega, f, e, a]$ and $Q_2 := [b, c, \omega, a]$.

Case A.4: Suppose that T is incident upon the edge $[f, d]$.

(i) If c and f are mutually visible in $\tilde{Q}_1 \cup \tilde{Q}_2$ as in Fig.9(a), choose $\omega \in \tilde{Q}_1$ such that $\omega \in (ca)^+ \cap (cf)^-$ and define $Q_1 := [\omega, a, b, c]$ and $Q_2 := [f, a, \omega, c]$.

(ii) Suppose that a and d are mutually visible in $\tilde{Q}_2$ (Fig.9(b)). That means, f must be in $(da)^+$. Choose $\omega \in \tilde{Q}_2$ such that $\omega \in \mathcal{W}(e)$ and define $Q_1 := [e, f, \omega, d]$ and $Q_2 := [a, d, \omega, f]$.

(iii) Suppose that none of (i) and (ii) occurs (Fig.9(c)). Since a is not visible from d in $\tilde{Q}_2$, f must be in $(da)^-$. Define $\omega := \tilde{\omega}$, $Q_1 := [\omega, f, a, b]$ and $Q_2 := [f, \omega, d, e]$.

Case B: Suppose $\tilde{Q}_1 \cup \tilde{Q}_2$ is a quadrilateral. Without loss of generality, let $\tilde{Q}_1$ and $\tilde{Q}_2$ satisfy (13). Let $[a, b, c, d]$ denote that union as depicted in Fig. 10. Three subcases are considered according to the incidence of the ear T on $[a, b, c, d]$

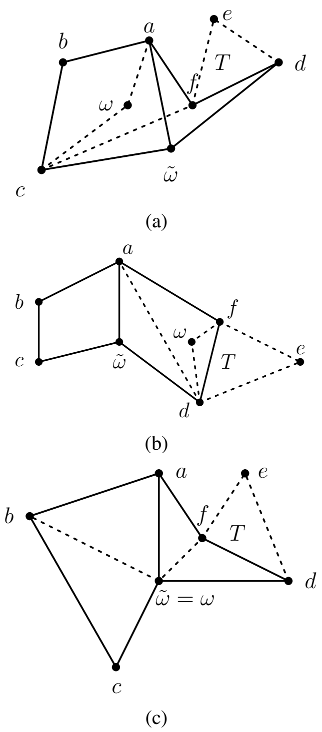


Fig. 9. Case where the ear T is incident upon the edge $[f, d]$ of $\tilde{Q}_2$.

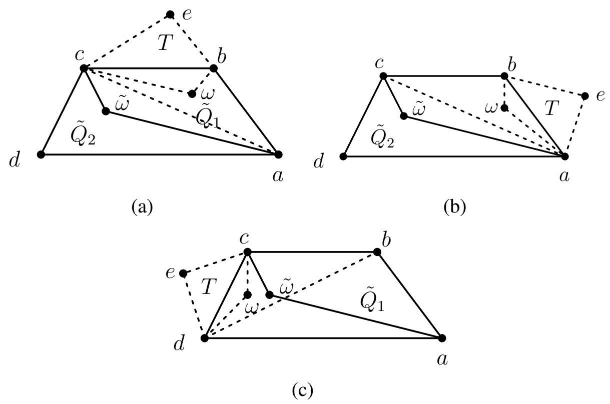


Fig. 10. Situation where the union $\tilde{Q}_1 \cup \tilde{Q}_2$ forms a quadrilateral.

Case B.1: Suppose that the ear T is incident upon the edge $[b, c]$ as shown in Fig. 10(a). The diagonal $[a, c]$ of the union $[a, b, c, d]$ is considered and $\omega \in \tilde{Q}_1$ is chosen such that $\omega \in (ac)^+ \cap \mathcal{W}(e)$. Then, $Q_1 := [b, e, c, \omega]$ and $Q_2 := [a, b, \omega, c]$ are defined.

Case B.2: Suppose that the ear T is incident upon the edge $[a, b]$ as in Fig. 10(b). Generate $\omega \in \tilde{Q}_1$ such that $\omega \in (ac)^+ \cap \mathcal{W}(e)$ and define $Q_1 := [e, b, \omega, a]$ and $Q_2 := [a, \omega, b, c]$.

Case B.3: If the ear T is incident upon the edge $[c, d]$ like in Fig. 10(c), proceed as in case B.1 but define now $Q_1 := [e, d, \omega, c]$ and $Q_2 := [c, \omega, d, b]$. ■

As a consequence of those statements, the next algorithm terminates because in at most two iterations the number of vertices decrements by two. The number of resulting quadrilaterals is of order $\mathcal{O}(n)$ for a polygon having n vertices.

Algorithm: Quadrangulation of P

While (the number of vertices of $P > 4$)
    Use (Op1) to chop off a quad Q if possible.
    Else use (Op2) to chop off a quad Q.
    $P := P \setminus Q$.
    Output(Q).
End While
Output(P).

The conversion of a quadrangulation which has nonconvex quadrilaterals into another one which contains only convex quadrilaterals is now described. Note that two adjacent quadrilaterals q and p form either a single quadrilateral or a hexagon. In the first case, the quadrilaterals q and p share two edges and it is possible that the union $q \cup p$ is a nonconvex or a convex quadrilateral. In the second case, only one edge is shared by q and p. Now, the following result about hexagon quadrangulations is recalled.

Theorem 4 (Bremner, 2001): Every hexagon (which may include reflex vertices) can be decomposed into a set of convex quadrilaterals by using at most three internal Steiner points.

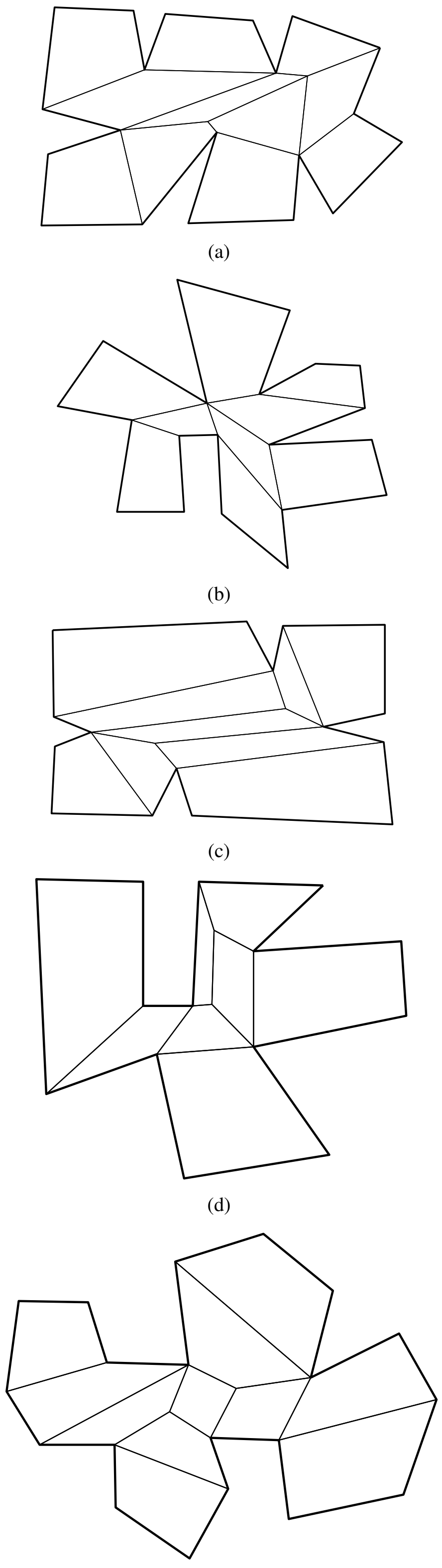
Bremner [2] proved this theorem but he did not specify the way of exactly choosing the internal Steiner points. That specification is found in [8]. Based on those facts, the next two steps perform the conversion into convex quadrangulation:

Step1: For every nonconvex quadrilateral p having a neighboring quadrilateral q such that $p \cup q$ is a quadrilateral, replace p by $p \cup q$ and remove the quadrilateral q from the quadrangulation. This step is repeated until such a union does not exist any more. After this step, there can only exist nonconvex quadrilaterals whose union with a neighboring quadrilateral forms a hexagon.

Step2: A nonconvex quadrilateral q is merged with a neighboring quadrilateral p in order to have a hexagon $q \cup p$. If there is a choice then a nonconvex neighbor p is selected. Then, the resulting hexagon is re-quadrilated by using the hexagon quadrangulation method from the above theorem in order to obtain a local *convex* quadrangulation $\mathcal{Q}_{\mathrm{loc}}$. Afterwards, the union $q \cup p$ in the quadrangulation is substituted by $\mathcal{Q}_{\mathrm{loc}}$.

V. Practical Results and Application to CAD

In this section, practical applications of the former theory are described. First, quadrangulations of simply connected polygons are considered. Then, its application to CAD models is briefly described. In fact, the former theory was implemented in C/C++ in order to see its practical behavior. In Fig. 11(a)–11(e), some quadrangulations of a few polygons are displayed. No new boundary nodes at all are used as already forecast in the theory.



(a) (b) (c) (d) (e)

Fig. 11. Decomposition of a few simply connected polygons.

There are many possible applications of the previous quadrangulation technique but it is used here in CAD segmentation. The goal is to split the surface boundary of a CAD model into four-sided patches Γ_k (Fig. 12). The main steps of the segmentation is summarized below and their details are found in [8], [10]. The CAD surfaces are collection of trimmed surfaces [3] which are images of $\mathcal{D}_i \subset \mathbf{R}^2$ by bivariate parametrizations ψ_i. The first step consists in approximating the parameter domains $\mathcal{D}_i$ by polygons $P^{(i)}$. Note that if too few vertices are taken in the polygonal approximation, the resulting polygon may have imperfections such as different edges which intersect. But if the polygonal approximation is too fine, then it ends up with too many four-sided patches. Adaptive method has been used to solve that. Afterwards, each polygon $P^{(i)}$ is decomposed into convex quadrilaterals $q_{k,i}$. For simply connected polygons, the quadrangulation of the previous sections is used. For multiply connected ones, internal cuts are inserted and the previous results are generalized to more complicated polygons [8]. Four-sided domains $Q_{k,i}$ are obtained from $q_{k,i}$ by replacing each straight boundary edge of $q_{k,i}$ by the corresponding curve portion of $\mathcal{D}_i$ such as $\mathcal{D}_i = \bigcup_k Q_{k,i}$. The final four-sided patches Γ_k are therefore the images by ψ_i of the 2D domains $Q_{k,i}$. Careful operations must be done to remove boundary intersection caused by the curve replacement as detailed in [8]. As a result, two CAD surface segmentations can be observed in Fig. 12.

VI. Conclusion

Two operations were presented for discarding a quadrilateral from a simply connected polygon. It was proved that one of those operations can always be applied. From that fact, an algorithm was designed for generating a quadrangulation from a polygon. The algorithm was then implemented in order to obtain interesting practical results.

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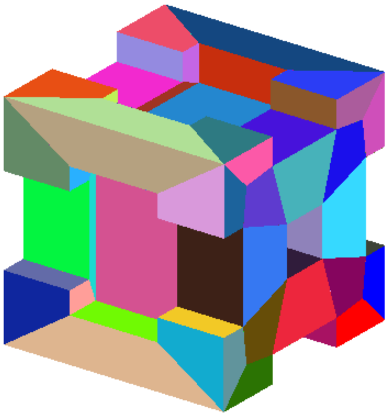
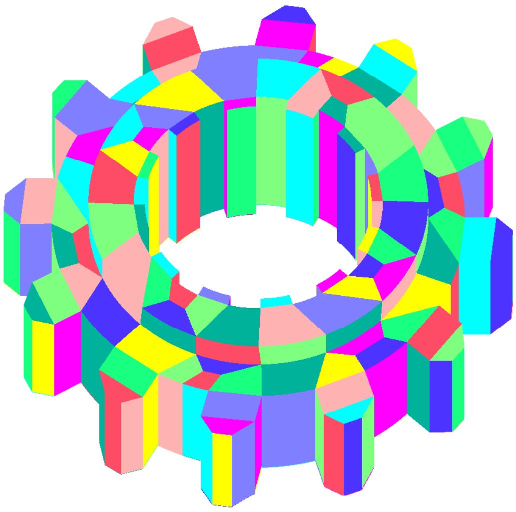


(a)



(b)

Fig. 12. Practical segmentation of two CAD surfaces into four-sided patches.

Maharavo Randrianarivony obtained his Master of Science from the Mathematics Department of the University of Chemnitz, Germany in 2001. He was mainly interested in numerical mathematics and computer science. He obtained his Ph.D. from the Department of Computer Science of the same University in 2006. The topic of his Ph.D. was in computer aided geometric design and CAD.

Right after his studies, he has done a PostDoc research for one year at the Institute of Computer Science in Kiel, Germany. He has treated there various scientific topics related to Geometric Modeling and CAD. Now, he holds a position as an Advanced Researcher at the Faculty of Mathematics of the University of Bonn, Germany.