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# Fuzzy T-Neighborhood Groups Acting on Sets

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***Abstract*—**In this paper, The T-G-action topology on a set acted on by a fuzzy T-neighborhood (T-neighborhood, for short) group is defined as a final T-neighborhood topology with respect to a set of maps. We mainly prove that this topology is a T-regular T-neighborhood topology.

***Keywords*—**Fuzzy set, Fuzzy topology, Triangular norm, Separation axioms.

## I. Introduction

A T-neighborhood topology on a set can be defined by several method e.g., via closures, interiors, filters, etc. Sometimes a T-neighborhood topology constructed out of given T-neighborhood topologies may be useful. In the classical theory of topological groups, when a topological group $G$ acts on a set $X$, it confers a topology on $X$, called the G-action topology on $X$. In this paper we develop a fuzzy extension of that notion, in the case $G$ is a T-neighborhood group. Varity of useful characterizations of this T-neighborhood topology are considered. We show that the T-G-action topology $\tau_X^{T-G}$ coincides with the final T-neighborhood topology $\tau_f$ introduced on $X$ by a set of functions $\left\{\hat{g}\right\}$;

$$\hat{g} : G \rightarrow X .$$

## II. Definition and Preliminaries

**Definition 2.1. [8]** A topological group $G$ acts on a non-empty set $X$, if to each $g \in G$ and each $x \in X$ there corresponds a unique element $gx$ such that

$g_2(g_1x) = (g_2g_1)x \quad \forall\ x \in X$ and $g_1, g_2 \in G$

$ex = x$.

When $G$ acts on a set $X$, two families of functions can be defined as follows:

To each $g \in G$, we define $\hat{g} : X \rightarrow X$,

$\hat{g}(x) = gx$.

To each $x \in X$, we define $\hat{x} : G \rightarrow X$,

$\hat{x}(g) = gx$.

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We will use two important theorems which introduced in [7]. The first gives necessary and sufficient conditions for a group structure and T-neighborhood system to be compatible, and the second gives necessary and sufficient conditions for a filter to be the T-neighborhood filter of $e$ in a T-neighbourhood group.

**Theorem 2.1** [7] Let $(G, .)$ be a group and $\beta$ a T-neighborhood base on $G$. Then $(G, .,t(\beta))$ is a T-neighborhood group if and only if the following are fulfilled:

(a) For every $a \in G$ we have

$$\beta(a)= \{\zeta_a(\mu) \mid \mu \in \beta(e)\}$$

$$(\text{res. } \beta(a)= \{R_a(\mu) \mid \mu \in \beta(e)\} \text{ and}$$

$\beta(a)= \{\zeta_a(\mu) \mid \mu \in \beta(e)\}$ is a T-neighborhood base at $a$.

(b) For all $\mu \in \beta(e)$ and for all $\varepsilon \in I_0$ there exists $\nu \in \beta(e)$ such that $\nu - \varepsilon \leq \mu^{-1}$, i.e., $r$ is continuous at $e$.

(c) For all $\mu \in \beta(e)$ and for all $\varepsilon \in I_0$ there exists $\nu \in \beta(e)$ such that $\nu.\nu - \varepsilon \leq \mu$, i.e., $m$ is continuous at $(e, e)$.

(d) For all $\mu \in \beta(e)$, for all $\varepsilon \in I_0$ and for all $x \in G$ there exist $\nu \in \beta(e)$ such that $1_x.\nu.1_x^{-1} - \varepsilon \leq \mu$, i.e., $int_x$ is continuous at $e$.

Where $\zeta_x : G \rightarrow G : z \mapsto x z$ ( resp. $R_x : G \rightarrow G : z \mapsto z x$ ) is the left ( resp. right ) translation.

**Theorem 2.2** [7] Let $(G, .)$ be a group and $\Im$ a family of fuzzy subset of $G$ such that the following hold:

(a) $\Im$ is a filterbasis, such that $\mu(e) = 1$ for all $\mu \in \Im$.

(b) For all $\mu \in \Im$ and for all $\varepsilon \in I_0$ there exists $\nu \in \Im$ such that $\nu - \varepsilon \leq \mu^{-1}$.

(c) For all $\mu \in \Im$ and for all $\varepsilon \in I_0$ there exists $\nu \in \Im$ such that $\nu.\nu - \varepsilon \leq \mu$.

(d) For all $\mu \in \Im$, for all $\varepsilon \in I_0$ and for all $x \in G$ there exists $\nu \in \Im$ such that $1_x.\nu.1_{x-1} - \varepsilon \leq \mu$.

Then there exists a unique T-neighborhood system $\beta$ such that $\Im$ is T-neighborhood basis for the T-neighbourhood system at $e$, $\beta(e)$ and $\beta$ is compatible with the group structure. This T-neighbourhood system is given by

$$\beta(x) = \{1_x.\mu \mid \mu \in \Im\}^{-1} = \{\mu.1_x \mid \mu \in \Im\}^{-1}, x \in G.$$

## III. T-Neighborhood Topologies Induced by T-Neighborhood Group Actions on Set

**Definition 3.1.** Let $(G, .)$ be a group acting on a set $X$, then for all $\Gamma \in I^G$, $\mu \in I^X$, $g \in G$ and $x \in X$ we define for all $y \in X$

$$\Gamma\mu(y) = \sup \{\Gamma(g) T \mu(x): (g, x) \in G \times X \text{ and } gx = y\} \tag{1}$$

**Proposition 3.1.** Let $(G, .)$ be a group acting on a set $X$ and $\Psi, \Gamma \in I^G$, $\mu \in I^X$. Then

(a) $\Psi(\Gamma\mu) \leq (\Psi.\Gamma)\mu$ In particular
$\Psi(\Gamma\mu)(y) \leq (\Psi.\Gamma)\mu(y)$

(b) $\Gamma 1_M = \bigvee_{x\in M} \Gamma 1_x$

(c) $\Gamma 1_M(y) = \sup\{\Gamma(g): g \in G \text{ and } g^{-1}y \in M\}$

(d) $\Gamma 1_x(y) = \sup\{\Gamma(g): g \in G \text{ and } gx = y\}$

(e) $1_g\mu(y) = \sup\{\mu(x): x \in X \text{ and } gx = y\}$
$= \mu(g^{-1}y)$

**Proof:** (b)-(e) follow immediately from Definition 3.1.

(a) For any $y \in X$:

$$\Psi(\Gamma\mu)(y) = \sup\{\Psi(g)\, T\, \Gamma\mu(x): (g, x) \in G\times X, gx = y\}$$
$$= \sup\{\Psi(g)\, T \sup\{\Gamma(h)\, T\, \mu(z): hz = x\}: gx = y\}$$
$$= \sup\{\Psi(g)\, T\, \Gamma(h)\, T\, \mu(z): ghz = y\}$$
$$(\Psi \odot_T \Gamma)\mu(y) = \sup\{(\Psi \odot_T \Gamma(k)\, T\, \mu(z): kz = y\}$$
$$= \sup\{\sup\{\Psi(g)\, T\, \Gamma(h): gh = k\}\, T\, \mu(z): kz = y\}$$
$$= \sup\{\Psi(g)\, T\, \Gamma(h)\, T\, \mu(z): (g, h, z) \in G\times G\times X \text{ and } ghz = y\}$$

Hence $\Psi(\Gamma\mu)(y) = (\Psi \odot_T \Gamma)\mu(y) \leq (\Psi.\Gamma)\mu(y)$.

If both $\Gamma$, $\mu$ are crisp, then $\Gamma\mu$ is also crisp and is given by $\Gamma\mu = \{gx: g \in \Gamma \text{ and } x \in \mu\}$.

Note that $\Gamma\mu$, $\Gamma 1_x$, $1_g\mu \in I^X$ and $\Gamma 1_x(y) = 0$ if $y \notin$ orbit of $x$.

**Theorem 3.1.** Let $G$ be a T-neighborhood group acting on a set $X$, and let $\mathfrak{R}$ be a fundamental system of $G$ at $e$. For each $x \in X$, let $\beta_x = \{\Gamma 1_x: \Gamma \in \mathfrak{R}\} \in I^X$. Then $\{\beta_x\}_{x\in X}$ is a T-neighborhood basis on $X$. The resulting T-neighborhood space is denoted by $\tau_X^{T-G}$. Its fuzzy closure operator $^{-}: I^X \to I^X$ is given by: For all $\eta \in I^X$, $x \in X$:

$$\bar{\eta}(x) = \inf_{\Gamma\in\mathfrak{R}} \sup_{g\in G} \Gamma(g)\, T\, \eta(gx) \quad (2)$$

**Proof.** First, we verify that $\{\beta_x\}_{x\in X}$ is a T-neighborhood basis in $X$. Let $x \in X$, $\Gamma$, $\Psi \in \mathfrak{R}$, $\mu = \Gamma 1_x \in \beta_x$, $\lambda = \Psi 1_x \in \beta_x$

(i) $\mu(x) = \Gamma 1_x(x) = \sup\{\Gamma(g): g \in G \text{ and } gx = x\}$
$\geq \Gamma(e) = 1$ (Because $ex = x$).

(ii) There exists $\Lambda \in \mathfrak{R}: \Gamma \wedge \Psi \geq \Lambda$. Hence
$\mu \wedge \lambda = \Gamma 1_x \wedge \Psi 1_x \geq \Lambda 1_x$,
which is in $\beta_x$.

(iii) T-kernel condition:

Recall that $\{\mathfrak{R} 1_g\}_{g\in G}$ is a T-neighborhood basis of the T-neighborhood group $G$ Theorem 2.2 . Let, as before, $\mu = \Gamma 1_x \in \beta_x$. By the T-kernel condition for $\Gamma \in \mathfrak{R}$, for all $\varepsilon \in I_0$ there exists a family $\{\Gamma_g 1_g \in \mathfrak{R}_g\}_{g\in G}$ such that for all $g, k \in G$

$$\Gamma_e(k)\, T\, (\Gamma_k 1_k)(g) \leq \Gamma(g) + \varepsilon. \quad (3)$$

We take $v_x = \Gamma_e 1_x$. For each $y \in X$, if $y \notin$ orbit of $x$, take for $v_y$ any element of $\beta_y = \mathfrak{R}_y$.

If $y \in$ orbit of $x$, choose some $h \in G$ such that $y = hx$, and

$$\delta + \Gamma_e(h) \geq \sup\{\Gamma_e(k): kx = y\} \quad (4)$$

where $\delta \in I_0$ is a real number that satisfies
$(b + \delta)\, T\, (c + \delta) \leq (b\, T\, c) + \varepsilon$
for all $b, c \in I$. Such $\delta$ exists by the uniform continuity of $T$. Take $v_y = \Gamma_h 1_y \in \beta_y$. Then, if $y \notin$ orbit of $x$, we find for all $z \in X$ that

$$2\varepsilon + \mu(z) \geq \upsilon_x(y)\, T\, \upsilon_y(z)$$

because then $v_x(y) = (\Gamma_e 1_x)(y) = 0$. And when $y \in$orbit of $x$, we find for all $z \in X$:

$$2\varepsilon + \mu(z) = 2\varepsilon + (\Gamma 1_x)(z)$$
$$= \varepsilon + \sup\{\varepsilon + \Gamma(g): gx = z\}$$
$$\geq \varepsilon + \sup\{\Gamma_e(h)\, T\, (\Gamma_h 1_h)(g): gx = z\} \text{ by (3)}$$
$$\geq (\Gamma_e(h) + \delta)\, T \sup\{(\Gamma_h 1_h)(g): gx = z\}$$
$$\geq \sup\{\Gamma_e(k): kx = y\}\, T \sup\{(\Gamma_h)(gh^{-1}): (gh^{-1})(hx) = z\}$$

by (4) Since $hx = y$, then

$$2\varepsilon + \mu(z) \geq (\Gamma_e 1_x)(y)\, T \sup\{(\Gamma_h)(t): ty = z\}$$
$$= (\Gamma_e 1_x)(y)\, T\, (\Gamma_h 1_y)(z)$$
$$= v_x(y)\, T\, v_y(z).$$

Thus, the kernel condition holds for $\mu \in \beta_x$ in both cases of $y$. Finally, for all $\eta \in I^X$

$$\bar{\eta}(x) = \inf_{\mu\in\beta} \sup_{y\in X} \mu(y)\, T\, \eta(y)$$
$$= \inf_{\Gamma\in\mathfrak{R}} \sup_{y\in X} \eta(y)\, T\, (\Gamma 1x)(y)$$
$$= \inf_{\Gamma\in\mathfrak{R}} \sup_{y\in orbitex} \eta(y)\, T \sup\{\Gamma(g): g \in G \text{ and } gx = y\}.$$

Because if $y \notin$ orbit $x$, then $(\Gamma 1_x)(y) = 0$. Thus,

$$\bar{\eta}(x) = \inf_{\Gamma\in\mathfrak{R}} \sup_{g\in G} \eta(gx)\, T\, \Gamma(g),$$

Rendering (2).

**Proposition 3.2.** Let $\Gamma \in I^G$, $\wp \subset I^G$, $g \in G$, $x \in X$ then

$(\Gamma.1_g)1_x = (\Gamma 1_{gx}) \in I^X$, and hence

$(\wp.1_g\})1_x = \wp 1_{gx} \subset I^X$.

**Proof:**

$$((\Gamma.1_g)1_x)(y) = \sup\{(\Gamma.1_g)(k): k \in G \text{ and } kx = y\}$$
$$= \sup\{\Gamma(kg^{-1}): k \in G \text{ and } kg^{-1}gx = y\}$$
$$= \sup\{\Gamma(t): t \in G \text{ and } tgx = y\}$$
$$= (\Gamma 1_{gx})(y).$$

This completes the proof.

**Proposition 3.3.** For each filterbasis $\mathrm{F}$ in $I^G$ and for $x \in X$.

$$\{\Gamma 1_x: \Gamma \in \mathrm{F}^{\sim}\} \subset \{\Psi 1_x : \Psi \in \mathrm{F}\}^{\sim} \subset I^X \quad (5)$$

**Proof:** Let $\Gamma \in \mathrm{F}^{\sim}$ Then for all $\varepsilon > 0$ there exists $\Gamma_\varepsilon \in \mathrm{F}$ such that $\Gamma + \varepsilon \geq \Gamma_\varepsilon$. Then for all $y \in X$ we have

$$\varepsilon + (\Gamma 1_x)(y) = \varepsilon + \sup\{\Gamma(g): gx = y\}$$
$$= \sup\{\varepsilon + \Gamma(g): gx = y\}$$

$\geq sup\ \{\Gamma_\varepsilon(g): gx = y\}$
$= (\Gamma_\varepsilon 1_x)(y)$.

Thus, $\varepsilon + \Gamma 1_x \geq \Gamma_\varepsilon 1_x \in \{\Psi 1_x: \Psi \in \mathrm{F}\}$. Hence $\Gamma 1_x \in \{\Psi 1_x: \Psi \in \mathrm{F}\}^{\sim}$. This proves (5).

**Proposition 3.4.** The fuzzy closure operator on $X$ defined in (2) does not depend on the particular choice of a fundamental system $\mathfrak{R}$ of $e$.

**Proof:** All fundamental systems $\mathfrak{R}$ of $G$ at $e$ have the same saturation $\mathfrak{R}^{\sim}$. Also, for each $x \in X$

$$\beta_x = \{\Gamma 1_x: \Gamma \in \mathfrak{R}\} \subset \{\Gamma 1_x: \Gamma \in \mathfrak{R}^{\sim}\} \subset \{\Gamma 1_x: \Gamma \in \mathfrak{R}\}^{\sim} = \beta_x^{\sim}.$$

As $\{\beta_x\}$, $\{\beta_x^{\sim}\}$ induce the same fuzzy closure operator on $X$, then the fuzzy closure operator defined in (2) is also given by

$$\bar{\eta}(x) = \inf_{\Gamma\in\tilde{\mathfrak{R}}} \sup_{g\in G} \Gamma(g) T \eta(gx) \qquad (6)$$

Which is independent of the particular choice of a fundamental system $\mathfrak{R}$ of $e$.

The following definition is well phrased by virtue of Theorem 3.1, and Proposition 3.4;

**Definition 3.2.** Let $G$ be a T-neighborhood group acting on a set $X$. A T-G-action-topology on $X$ denoted by $\tau_X^{T-G}$ is introduced through its closure operator $\bar{}$, defined in (2).

**Proposition 3.5.** Let $\mathfrak{R}$ be a fundamental system at $e$ of $G$, $\mu \in \mathfrak{R}$. Then

$$1_g . \mathfrak{R} . 1_{g^{-1}} \subset \mathfrak{R}^{\sim} \qquad (7)$$

**Proof:** From condition (d) in Theorem 2.1, for all $\varepsilon > 0$ there exists $v_\varepsilon \in \mathfrak{R}$ such that

$v_\varepsilon - \varepsilon \leq 1_g . \mu . 1_{g^{-1}}$.

This proves that $1_g . \mu . 1_{g^{-1}} \in \mathfrak{R}^{\sim}$

**Notion:** In T-G-action topology

(1) We denote the T-neighborhood system at $x \in X$ by $\aleph(x)$.

(2) Let $\mathfrak{R}$ be the T-neighborhood system of $G$ at $e$, $x \in X$. We denote $\mathfrak{R} 1_x$ by $Ç(x)$. Recall that $Ç^{\sim} = \aleph$; i.e $Ç(x)$ is a T-neighborhood basis at $x$ for this space.

**Definition 3.3.** Let $(X, ., t(\beta))$ be a T-neighborhood space, $M$ *be* a non-empty set in $X$. Then $\mu \in I^X$ is said to be *a* T-neighborhood of $M$ if $\mu$ is a T-neighborhood of all points $x$ in $M$. It follows that the set of all T-neighborhoods of $M$ (called the T-neighborhood system of $M$) is the set $\bigwedge_{x\in M} \aleph(x)$.

**Proposition 3.6.** Let $\Gamma \in I^G$, $g \in G$, $z \in X$ then

$1_g^{-1} (\Gamma 1_z) = (1_g^{-1} . \Gamma) 1_z$

**Proof:**

$1_{g^{-1}}(\Gamma 1_z)(y) = (\Gamma 1_z)(gy)$
$= sup\ \{\Gamma(h): h \in G, hz = gy\}$
$= sup\ \{\Gamma(gk): k \in G, kz = y\}$

$(1_{g^{-1}} . \Gamma) 1_z(y) = sup\ \{(1_{g^{-1}}.\Gamma)(k): kz = y\}$
$= sup\ \{\Gamma(gk): k \in G, kz = y\}$

Then

$1_{g^{-1}}(\Gamma 1_z) = (1_{g^{-1}} . \Gamma) 1_z$

**Theorem 3.2.** Under this T-neighborhood topology the functions $\{\hat{g}\}$ are homeomorphisms on $X$.

**Proof:** Without loss of generality, we take $\mathfrak{R}$ the whole T-neighborhood system at e. Then from Proposition 3.5, $1_g . \mathfrak{R} . 1_{g-1} \subset \mathfrak{R}$. Given $x \in X$, $g \in G$, $\mathfrak{R} 1_{gx}$ is a T-neighborhood basis at $\hat{g}x$. Let $\mu \in \mathfrak{R} 1_{gx}$ we have $\hat{g}^{-1}(\mu)(y) = \mu(gy) = 1_{g-1} \mu(y)$, then $\hat{g}^{-1}(\mu) = 1_{g-1}\mu \in 1_g^{-1} \mathfrak{R} 1_{gx}$, and from Proposition 3.6

$1_g^{-1} (\mathfrak{R} 1_{gx}) = (1_g^{-1} . \mathfrak{R}) 1_{gx}$,
$= (1_g^{-1} . \mathfrak{R} . 1_g) 1_x$ by Proposition 3.2
$\subset \mathfrak{R}^{\sim} 1_x$ by Proposition 3.5
$\subset \aleph(x)$.

i.e., $\hat{g}^{-1}(\mu)$ is a T- neighborhood of $x$. So by Theorem 5.1 in [5] $\hat{g}$ is continuous at $x$ for all $x$, and hence it is continuous. Since $(g^{-1})^{\wedge} = (\hat{g})^{-1}$. Then $(\hat{g})^{-1}$ is also continuous. Thus $\hat{g}$ is a homeomorphism.

**Proposition 3.7.** For any symmetric T-neighborhood $\Delta$ of $e$, and any $M \subset X$; $x, z \in X$

$(\Delta 1_x)(z) = (\Delta 1_z)(x)$

$(\Delta 1_M)(x) = \sup_{y\in X} 1_M(y)\ T\ (\Delta 1_x)(y)$.

**Proposition 3.8.** For any subset $M$ of $X$ and any T-neighborhood $\Gamma$ of $e$, $\Gamma 1_M$ is a T-neighborhood of $M$, and

$$(1_M)^{-} \leq \Gamma 1_M \in I^X. \qquad (8)$$

**Proof:** Since $\Gamma 1_M = \bigvee_{x\in M} \Gamma 1_x$, then $\Gamma 1_M$ is a T-neighborhood of all points of $M$, hence $\Gamma 1_M$ is a T-neighborhood of $M$.

Next, let $\Gamma$ be a T-neighborhood of $e$. Then $\Gamma$ contains a symmetric T-neighborhood $\Delta$ of $e$. For any $x \in X$

$$(1_M)^-(x) = \inf_{\lambda \in \c{C}} \sup_{y \in X} 1_M(y) T \lambda(y)$$
$$\leq \sup_{y \in X} 1_M(y)\, T\, \Delta\, 1_x(y)$$
$$= \sup_{y \in M} \Delta\, 1_x(y)$$
$$= \sup_{y \in M} \Delta\, 1_y(x) \quad \text{by Proposition 3.7}$$
$$= (\Delta\, 1_M)(x)$$
$$\leq (\Gamma\, 1_M)(x).$$ This proves (8).

**Proposition 3.9.** Let $\Re$ be a fundamental system of T-neighborhoods of $e$. For any subset $M$ of $X$

$$(1_M)^- = \bigwedge_{\Gamma \in \Re} \Gamma 1_M$$

**Proof:** From Proposition 3.8, $(1_M)^- \leq \Gamma 1_M$ for every $\Gamma \in \Re$. Then

$$(1_M)^- \leq \bigwedge_{\Gamma \in O} \Gamma 1_M.$$

Next we prove that

$$\bigwedge_{\Gamma \in \Re} \Gamma 1_M = \bigwedge_{\Gamma \in \Re^\sim} \Gamma 1_M.$$

Since $\Re \subset \Re^\sim$, then

$$\bigwedge_{\Gamma \in \Re} \Gamma 1_M \geq \bigwedge_{\Gamma \in \Re^\sim} \Gamma 1_M$$

Also, let $\Gamma \in \Re^\sim$, for all $\varepsilon > 0$ there exists $\Gamma_\varepsilon \in \Re$ such that $\varepsilon + \Gamma \geq \Gamma_\varepsilon$,

$$\varepsilon + \Gamma 1_M \geq (\varepsilon + \Gamma) 1_M \geq \Gamma_\varepsilon 1_M \geq \bigwedge_{\Gamma \in \Re} \Gamma 1_M$$

Since this holds for all $\varepsilon > 0$, then

$$\Gamma 1_M \geq \bigwedge_{\Gamma \in \Re} \Gamma 1_M.$$

This inequality holds for all $\Gamma \in \Re^\sim$.
Consequently,

$$\bigwedge_{\Gamma \in \Re^\sim} \Gamma 1_M \geq \bigwedge_{\Gamma \in \Re} \Gamma 1_M$$

Hence, equality holds.
It is clear that if $O$ is the set of symmetric elements in $\Re^\sim$ then.

$$\bigwedge_{\Gamma \in \Re} \Gamma 1_M = \bigwedge_{\Gamma \in \Re^\sim} \Gamma 1_M \leq \bigwedge_{\Delta \in O} \Delta 1_M$$

Conversely, let $O$ is the set of symmetric elements in $\Re^\sim$. Then $O$ is a fundamental system at $e$:

$$\Big(\bigwedge_{\Delta \in O} \Delta 1_M\Big)(x) = \inf_{\Delta \in O} (\Delta 1_M)(x)$$
$$= \inf_{\Delta \in O} \sup_{y \in X} 1_M(y)\, T\, (\Delta 1x)(y) \text{ by Proposition 3.7}$$
$$= (1_M)^-(x)$$

because the set $\{\Delta 1_x : \Delta \in O\}$ is a T-neighborhood basis at $x$.

**Theorem 3.3.** A T-G-action topology on $X$ is a T-regular T-neighborhood topology.

**Proof:** Let $M \subset X$ and $x \in X$. We establish condition ($N^4$-T-regularity) of Theorem 3.2 in [6], which is equivalent to the T-regularity of $X$. For all

$M \subset X$, $x \in X$ such that

$$Inf\ hgt\ (\rho\, T\, v\colon \rho \in \c{C}(M),\ v \in O1_x)$$
$$\leq \inf_{\Delta \in O} hgt\ (\Delta 1_M T \Delta 1_x)$$
$$\leq \inf_{\Delta \in O} \sup\ \{((\Delta 1_M) \wedge \Delta 1_x)(y)\colon y \in X\}$$
$$= \inf_{\Delta \in O} \sup\ \{((\Delta 1_M)(y) \wedge (\Delta 1_x)(y))\colon y \in X\}$$
$$= \inf_{\Delta \in O} \sup\ \{\sup\ \{\Delta(h)\colon h \in G,\ hy \in M\}$$
$$\wedge \sup\{\Delta(k)\colon k \in G,\ y = kx\}\colon y \in X\}$$
$$= \inf_{\Delta \in O} \sup\ \{\sup\ \{\Delta(h)\colon h \in G,\ hy \in M\} \wedge$$
$$\sup\{\Delta(k)\colon k \in G,\ y = kx\}\colon y \in orbit\ x\}$$

So. (call $y \in kx$)

$$Inf\ hgt\ (\rho\, T\, v\colon \rho \in \c{C}(M),\ v \in O1_x)$$
$$\leq \inf_{\Delta \in O} \sup\ \{\sup\ \{\Delta(h)\colon hkx \in M\} \wedge \Delta(k)\colon k \in G\}$$
$$= \inf_{\Delta \in O} \sup\ \{\Delta(h) \wedge \Delta(k)\colon h, k \in G \text{ and } hkx \in M\}$$
$$= \inf_{\Delta \in O} \sup\ \{(\Delta\Delta)(g)\colon g \in G \text{ and } gx \in M\}$$
$$= \inf_{\Delta \in O} \sup\ ((\Delta\Delta)(1_M))(x)$$

But by Theorem 2.2 in [7], for every $\Delta \in O$, $\varepsilon \geq 0$ there exists $\Delta_1 \in O$ such that $\Delta_1 \Delta_1 \leq \Delta + \varepsilon$. Hence,

$$Inf\ hgt\ (\rho\, T\, v\colon \rho \in \c{C}(M),\ v \in O1_x)$$
$$\leq \inf_{\Delta \in O, \varepsilon > 0} ((\Delta + \varepsilon)\, 1_M)(x)$$
$$= \inf_{\Delta \in O} (\Gamma 1_M)(x)$$
$$= \bigwedge_{\Gamma \in O} (\Gamma 1_M)(x) = (1_M)^-(x) \quad \text{by Proposition 3.9.}$$

The opposite inequality is always valid.

**Theorem 3.4.** A T-G-action-topology $\tau_X^{T-G}$ coincides with the final T-neighborhood topology $\tau_f$ on $X$ defined by the set of functions

$$\{\hat{x} : G \to X\colon x \in X\},\ \hat{x}(g) = gx$$

**Proof:** For any $x \in X$, the function

$\hat{x} : G \to (X, \tau_X^{T-G})$ is continuous, because for all $g \in G$ and for each neighborhood $\Gamma(1_{gx})$ in the fundamental system $\Re 1_{gx}$ of $\hat{x}(g) = gx$, where $\Gamma \in \Re$, we have

$$\hat{x}\ (\Gamma .\ 1_g)(y) = \sup\ \{(\Gamma.\ 1_g)(h)\colon h \in G,\ \hat{x}(h) = y\}$$
$$= \sup\ \{(\Gamma.1_g)(h)\colon h \in G,\ hx = y\}$$
$$= (\Gamma .\ 1_g)1_x(y)$$

then $\hat{x}\ (\Gamma.1_g) = (\Gamma.1_g\ )1_x = \Gamma 1_{gx}$ and $\Gamma.1_g$ is a T-neighborhood of $g$ by Theorem 2.3 in [7]. Therefore

$\tau_X^{T-G} \subset \tau_f$ since $\tau_f$ is the finest T-neighborhood topology making all $\hat{x}$ continuous.

Next, let x $\in$ X, $\mu$ a T-neighborhood of $x$ in $\tau_f$. Then $\hat{x}^{-1}(\mu)$ a T-neighborhood of $e$ in $G$; i.e. $(\hat{x}^{-1}(\mu))1_x$ is a T-neighborhood of $x$ in $\tau_X^{T-G}$.

But $(\hat{x}^{-1}(\mu))1_x = \hat{x}(\hat{x}^{-1}(\mu)) = \mu \wedge 1_{range\hat{x}} \leq \mu$. This proves that $\mu$ is a T-neighborhood of $x$ in $\tau_X^{T-G}$. Then $\tau_f \subset \tau_X^{T-G}$. Hence, equality holds.

## References

[1] T.M.G. Ahsanullah, On fuzzy neighbourhood groups, J. Math. Anal. Appl.130 (1988)237-251.
[2] N. Bourbaki, “General Topology Part I,” Addision- Wesley, Reading, MA, 1966.
[3] D.H. Foster, Fuzzy topological groups, J. Math. Anal. Appl. 67 (1979) 549-564.
[4] K. A. Hashem and N. N. Morsi, Fuzzy T- neighborhood spaces, Part I: T- proximities, Fuzzy Sets and Systems 127 (2002) 247-264.
[5] K. A. Hashem and N. N. Morsi, Fuzzy T- neighborhood spaces, Part II: T- neighborhood systems, Fuzzy Sets and Systems 127 (2002) 265-280.
[6] K. A. Hashem and N. N. Morsi, Fuzzy T- neighborhood spaces, Part III: T- separation axioms, Fuzzy Sets and Systems 133 (2002) 333-361.
[7] H. A. Khorshed and M. A. El Gendy, On Fuzzy T- neighbourhood groups, 3rd Internationa Conference of Mathematics and Engineering Physics (2006) 24-32.
[8] A. Patronis, Colloq. Math. Soc. Janos Bolyai 23(1978) 939-944.