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On the Invariant Uniform Roe Algebra as Crossed Product

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***Abstract*—**The uniform Roe C^* −algebra (also called uniform translation)C^*- algebra provides a link between coarse geometry and C^*- algebra theory. The uniform Roe algebra has a great importance in geometry, topology and analysis. We consider some of the elementary concepts associated with coarse spaces. A discrete group G has natural coarse structure which allows us to define the the uniform Roe algebra, $C_U^*(G)$. The reduced C^*- algebra $C_r^*(G)$ is naturally contained in $C_U^*(G)$. We show that the elements of $\ell^\infty(G) \rtimes_{alg} G$ which are invariant under $Ad\rho$ are of the form $(\ell^\infty(G))^{\rho(G)} \rtimes_{alg} G$. Finally we show that if X and Y are bounded geometry discrete metric spaces, then

$$C_u^*(X \times Y) \neq C_u^*(X) \otimes C_u^*(Y).$$

***Keywords*—**Invariant Approximation Property, Uniform Roe algebras.

I. Introduction

We assume that the reader is familiar with the basic notions in operator algebras and operator spaces, Roe [8], Kannan [4], [5], [6] and [7] Jolissaint [3], Brown and Ozawa [2], and Anantharaman-Delaroche [1] for the details on the invariant approximation property and the coarse geometry. The uniform Roe algebra $C_U^*(G)$ is the C^* −algebra completion of the algebra of bounded operators on $\ell^2(X)$ which have finite propagation. In other words: According to Roe [8] G has the invariant approximation property (IAP) if

$$C_\lambda^*(G) = C_U^*(G)^G .$$

In Section IV, we study the crossed product of C^* −algebra. In section IV we study stone-cech compactification. In section IV we study the following statements:

$$C_U^*(G) = C(\beta G) \rtimes_r G \cong \ell^\infty(G) \rtimes_{alg} G$$

and we show the induce action of G on $C(\beta G) \rtimes_r G$.

The main purpose of this paper is to prove that the Theorem 5.2, in section V. We also show that the elements of $\ell^\infty(G) \rtimes_{alg} G$ which are invariant under $Ad\rho$ are of the form $(\ell^\infty(G))^{\rho(G)} \rtimes_{alg} G$. Finally we show that if X and Y are bounded geometry discrete metric spaces, then

$$C_u^*(X \times Y) \neq C_u^*(X) \otimes C_u^*(Y).$$

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II. Preliminaries

Coarse geometry is the study of the large scale properties of spaces. The notion of large scale is quantified by means of a coarse structure. First we recall the following definitions:

Definition 1 [8]: Let X, Y be metric spaces and $f: X \longrightarrow Y$ a not necessarily continuous map.

1. The map f is called coarsely proper (or metrically proper), if the inverse image of a bounded set is bounded.
2. The map f is called coarsely uniform (or uniformly bornologous), if for every $r > 0$ there is $s(r) > 0$ such that for all $x, y \in X$,

$$d(x,y) \le r \Rightarrow d(f(x), f(y)) \le s(r)$$

3. The map f is called a coarse map, if it is coarsely proper and coarsely uniform.
4. Let S be a set. Two maps $f, g : S \longrightarrow X$, are called close if there is $C > 0$ such that for all

$$s \in S,\ \ d(f(s), g(s)) < C.$$

5. A subset E of $X \times X$ is called controlled (or entourage), if the coordinate projection a $\pi_1, \pi_1: E \longrightarrow X$ is close.

Definition 2 [8]: A coarse structure on a set X is a collection of subsets of X×X, called the controlled sets or entourages for the coarse structure, which contains the diagonal and is closed under the formation of subsets, inverses, products, and (finite) unions.

It is easy to see that the controlled sets associated to a metric space X have the following properties:

- Any subset of a controlled set is controlled;
- The transpose $E^t = \{(x,y): (y,x) \in E\}$ of a controlled set E is controlled;
- The composition $E_1 \circ E_2$ of controlled sets E_1 and E_2 is controlled; where

$$E_1 \circ E_2 = \begin{matrix}\{(x,z) \in X \times X : \exists\, y \in X, \\ (x,y) \in E_1 \text{ ,and } (y,z) \in E_2\}\end{matrix}$$

- A finite union of controlled sets is controlled;
- The diagonal $\Delta_x(X) := \{(x,x): x \in X\}$ is controlled.

Definition 3 [8]: A set equipped with a coarse structure is called a coarse space. Coarse geometry is the study of metric spaces (or perhaps more general objects) from a `large scale' point of view, so that two spaces which `look the same from a great distance' are considered equivalent.

Let X and Y be metric spaces. A (not necessarily continuous) map $f : X \longrightarrow Y$ is a coarse equivalence if there are constants C, A such that

$$d(x,y) \le C\, d(f(x), f(y)) + A$$

and

$$d(f(x), f(y)) \le C\, d(x,y) + A$$

for all x and y in X.

Definition 4 [8]: A coarse structure on X is *connected* if each point of $X \times X$ belongs to some controlled set.

Definition 5 [8]: Let (X, d) be a metric space, we say that the metric d induces a coarse structure on X, which is called a bounded coarse structure. More precisely, we can define the bounded coarse structure induced by the metric d as follows: Set

$$D_r := \{(x,y) \in X \times X : d(x,y) < r\}.$$

Then $E \subseteq X \times X$ is controlled, if $E \subseteq D_r$ for some $r > 0$. The following is an example of coarse structure.

Example 6 [8]: Let G be a finitely generated group. Then the bounded coarse structure associated to any word metric on G is generated by the diagonals

$$\Delta_g = \{(h, hg) : h \in G\},$$

as g runs over G.

Definition 7 [8]: Let X and Y be coarse spaces. A map $i : X \longrightarrow Y$ is a *coarse embedding* if it is a coarse equivalence between X and $i(X) \subseteq Y$.

We next recall some definitions about uniform Roe algebra and metric property of a discrete group. Let X be a discrete metric space.

Definition 8[8]: We say that discrete metric space X has bounded geometry if for all R there exists N in $\mathbb{N}$ such that for all $x \in X$, $|B_R(x)| < N$, where

$$B_R(x) = \{x \in X : d(y,x) \le R\}.$$

Definition 9[8]: A kernel $\varphi : X \times X \longrightarrow \mathbb{C}$

1. is bounded if there exists $M > 0$ such that $|\varphi(s,t)| < M$ for all $s, t \in X$
2. has finite propagation if, there exists $R > 0$ such that that $\varphi(s,t) = 0$, $d(s,t) > R$.

Next, we show the operator associated with a bounded kernel is bounded.

Lemma 10 [4]: Let X be bounded geometry metric space. An operator associated with a bounded finite propagation kernel is bounded.

We shall denote the finite propagation kernels on by $A^{\propto}(X)$.

Definition 11 [8]: The uniform Roe algebra of a metric space X is the closure of $A^{\propto}(X)$ in the algebra $B(\ell^2(X))$ of bounded operators on X.

If a discrete group G is equipped with its bounded coarse structure introduced in Example 2.4 then one can associated with it to uniform Roe algebra $C_U^*(G)$ by repeating the above.

Next we recall the left and right regular representation: An important class of C^*-algebra algebras arise in the study of groups. Let G be a discrete group, then the characteristic function $\delta_g(s)$ of $g, s \in G$ is defined as follows:

$$\delta_g(s) = \left\{\begin{matrix} 1 & g = s \\ 0 & otherwise. \end{matrix}\right\}$$

If we assume that the G is a discrete group then the functions $\delta_g(s)$ form a basis for the Hilbert space $\ell^2(G)$ of square summble functions on G.

The group ring $\mathbb{C}[G]$ consists of all finitely supported complex-valued functions on G, that is of all finite combinations

$$f = \sum_{s \in G} a_s s$$

with complex coefficients.

The convolution product and the adjoins are defined as follows:

$$\left(\sum_{s \in G} a_s s\right)\left(\sum_{s \in G} a_t t\right) = \left(\sum_{s \in G} a_s a_t st\right)$$

$$\left(\sum_{s \in G} a_s s\right)^* = \sum_{s \in G} \overline{a_s} s^{-1}$$

Denote by $B(\ell^2(G))$ the C^* − algebra of all bounded linear operator on the Hilbert space $\ell^2(G)$. We may distinguish between the left regular representation, which is induced by the left multiplication action, and the right regular representation, which is comes from the multiplication on the right.

Definition 12: The left regular representation

$$\lambda : \mathbb{C}[G] \longrightarrow B(\ell^2(G))$$

is defined by

$$\lambda(s)\delta_t(r) = \delta_t(s^{-1}r) = \delta_{st}(r)$$

for $s, r \in G$.

The right regular representation is given by

$$\rho(s)\delta_t(r) = \delta_t(rs) = \delta_{ts^{-1}}(r)$$

for $s, r \in G$.

The left regular representation is implemented using the familiar convolution formula

$$(\delta_g *_\lambda \delta_h)(s) = \sum_{t\in G} \delta_g(st^{-1})\delta_h(t) = \delta_{gh}(s)$$

It follows that for any function $f \in \ell^2(G)$ the left action by δ_g is given by

$$(\delta_g *_\lambda f)(s) = \sum_{t\in G} \delta_g(st^{-1})f(t) = f(g^{-1}s)$$

We can define the following right convolution:

$$(\delta_g *_\rho \delta_h)(s) = \sum_{t\in G} \delta_g(t^{-1}s)\delta_h(t) = \delta_{hg}(s)$$

which gives rise to the right regular representation:

$$(\delta_g *_\rho f)(s) = \sum_{t\in G} \delta_g(t^{-1}s)f(t) = f(sg^{-1})$$

We note that:

$$\begin{aligned}(\delta_g *_\lambda \delta_h)(s) &= \textstyle\sum_{t\in G} \delta_g(st^{-1})\delta_h(t)\\ &= \textstyle\sum_{t\in G} \delta_h\left(st'^{-1}\right)\delta_g(t')\end{aligned}$$

and hence: $(\delta_g *_\lambda \delta_h)(s) = (\delta_g *_\rho \delta_h)(s)$

Proposition 13: The left and right representations commute that is for all$s, t \in G$:

$$\rho(s)\lambda(t) = \lambda(t)\rho(s).$$

Proof: We have:

$$\begin{aligned}\rho(s)\lambda(t)\delta_r = \rho(s)\delta_{tr} = \delta_{trs^{-1}} = \lambda(t)\delta_{rs^{-1}}\\ = \lambda(t)\rho(s)\delta_r\end{aligned}$$

Thus

$$\rho(s)\lambda(t) = \lambda(t)\rho(s).$$

Remark 14: The left regular representation λ of the group ring $\mathbb{C}[G]$ assigns to each element $f \in C[G]$ a bounded operator $\lambda(f)$ which acts on any $\xi \in \ell^2(G)$ by convolution:

$$\lambda(f)(\xi) = f * \xi.$$

and

$$\lambda(f^*) = \lambda(f)^*$$

The image $\lambda(\mathbb{C}[G])$ of the group ring under the left regular representation is $a *-$ subalgebra of the algebra $B\left(\ell^2(G)\right)$ of bounded operators on $\ell^2(G)$.

Lemma 15: The left and right regular representations λ and ρ are $*-$ homomorphisms.

Proof: Let $f, g \in \mathbb{C}[G]$

$$\lambda(f)(\xi) = f * \xi.$$

and

$$\lambda(g)(\xi) = g * \xi.$$

Consider

$$\begin{aligned}\lambda(f * g)(\xi) = (f * g) * \zeta = f * (g * \zeta) = f * (\lambda(g)\zeta)\\ = (\lambda(f)\lambda(g)(\xi)\end{aligned}$$

Thus

$\lambda(f * g) = \lambda(f)\lambda(g)$ for all $f, g \in \mathbb{C}[G]$

Thus λ satisfies the product.Consider

$$\begin{aligned}\big(\lambda(f) + \lambda(g)\big)(\xi) = \lambda(f)(\xi) + \lambda(g)(\xi) = f * \xi + g * \xi\\ = (f + g) * \xi = \lambda(f + g)(\xi)\end{aligned}$$

Thus $\big(\lambda(f) + \lambda(g)\big) = \lambda(f + g)$ satisfies the sum. It is easy to prove scalar multiplication, and adjoin. Therefore λ satisfies the properties of an $*-$homomorphisms. The proof for $\rho$is similar.

Lemma 16: The left and right regular representations λ and ρ are unitary bounded representations.

Proof: Let us define an operator

$$\lambda_g : \ell^2(G) \longrightarrow \ell^2(G)$$

which for any function $\xi \in \ell^2(G)$ is given by

$$\lambda_g\xi(t) = (\delta_g * \xi)(t) = \xi(g^{-1}t).$$

We have

$$\begin{aligned}\langle\lambda_g\xi, \eta\rangle = \sum_{t\in G} \lambda_g\,\xi(t)\overline{\xi(t)}\\ = \textstyle\sum_{t\in G} \xi(g^{-1}t)\,\overline{\xi(t)} = \langle\xi, \lambda_{g^{-1}}\eta\rangle.\end{aligned}$$

This means that

$$\lambda_g^* = \lambda_{g^{-1}}.$$

We have for every $g \in G, \xi \in \ell^2(G)$

$$\begin{aligned}\left\|\lambda_g\xi\right\|^2 &= \textstyle\sum_{t\in G}|\xi(g^{-1}t)|^2\\ &= \textstyle\sum_{t\in G}|\xi(t)|^2\\ &= \|\xi\|^2.\end{aligned}$$

Therefore, λ_g is a unitary bounded representation. The proof for $\rho$is similar.

Remark 17: The left regular representation λ is a faithful representation. The same argument can be used to show that $\rho$is a faithful representation as well.

The reduced $\mathrm{C}^* -$ algebra $C_\lambda^*(G)$ of a group G (which we shall assume to be discrete) arises from the study of the left regular representation λ of the group ring $\mathbb{C}[G]$ on the Hilbert space of square-summable functions on the group.

Definition 18: The reduced group C^* −algebra G, denoted by $C_\lambda^*(G)$ is the completion of $\mathbb{C}[G]$ in the norm given, for $c \in \mathbb{C}[G]$, by

$$\|c\|_\lambda = \overline{\lambda(\mathbb{C}[G])}$$

This mean that the closure of $\mathbb{C}[G]$ for the operator norm as a sub-algebra of $B(\ell^2(G))$ is called the reduced C^* −algebra $C_\lambda^*(G)$ of a group G. This is equivalently, it is the closure of $\mathbb{C}[G]$ is identified with its image under the left regular representation. i.e.

$$C_\lambda^*(G) \coloneqq \overline{\lambda(\mathbb{C}[G])}$$

Definition 19: The reduced group C^* −algebra G, denoted by $C_\rho^*(G)$ is the completion of $\mathbb{C}[G]$ in the norm given, for $c \in \mathbb{C}[G]$, by

$$\|c\|_\lambda = \overline{\rho(\mathbb{C}[G])}$$

This mean that the closure of $\mathbb{C}[G]$ for the operator norm as a subalgebra of $B(\ell^2(G))$ is called the reduced C^* −algebra $C_\rho^*(G)$ of a group G. This is equivalently, it is the closure of $\mathbb{C}[G]$ is identified with its image under the left regular representation. i.e.

$$C_\lambda^*(G) \coloneqq \overline{\rho(\mathbb{C}[G])}$$

III. Invariant Approximation Property

In this section we will give the definition of invariant approximation property. A discrete group G has a natural coarse structure which allows us to define the uniform Roe algebra $C_U^*(G)$.

A group G can be equipped with either the left or right-invariant of the metric. A choice of one of the determines whether $C_\lambda^*(G)$ or $C_\rho^*(G)$ is a sub-algebra of the uniform Roe algebra $C_U^*(G)$ of G as we now explain. If the metric of G is right-invariant then

$$C_\lambda^*(G) \subseteq C_U^*(G).$$

Let d_1 be the right-invariant metric on G.

$$d_1(x,y) = d_1(xg,yg) \forall g \in G.$$

The operator $\lambda(g)$ is given by the matrix. Let

$$A_g^\lambda(x,y) = \begin{matrix} 1 & ,if\ x = yg. \\ 0 & ,otherwise. \end{matrix}$$

Note that $A_g^\lambda(x,y)$ is right-invariant:

$$A_g^\lambda(xt,yt) = \begin{cases} 1 & xt = ygt \\ 0 & otherwise. \end{cases}$$

Therefore: $A_g^\lambda(xt,yt) = A_g^\lambda(x,y)$. If the metric on G is right-invariant, $A_g^\lambda(x,y)$ is of finite propagation and $A_g^\lambda(x,y) \in C_U^*(G)$ since $A_g^\lambda(x,y)$ is non-zero when $y^{-1}x = g$ and so $d_1(x,y) = d_1(xg,yg)$.

Hence any element of C[G] will finite propagation and this assignment extends to an inclusion $C_\lambda^*(G) \hookrightarrow C_U^*(G)$, similarly we can show that if the metric on G is left-invariant then

$$C_\rho^*(G) \subseteq C_U^*(G).$$

Let us now choose a right invariant metric for G so that $C_\lambda^*(G) \hookrightarrow C_U^*(G)$. The following important result as given in [8].

Lemma 20: If $T \in C_U^*(G)$ has kernel $A(x,y)$, then $Ad\rho(t)T$ has kernel $A(xt,yt)$.

Proof: We have that:

$$\begin{aligned}(Ad\rho(t)T\xi)(s) &= \rho(t)(T\rho(t)^*\xi)(s) \\ &= T\rho(t)^*\xi(st) \\ &= \textstyle\sum_{x\in G} A(st,x)(\rho(t)^{-1}\xi)(s) = \sum_{x\in G} A(st,x)\xi(xt^{-1})\end{aligned}$$

Now $A(st,x)$ is non-zero whenever $x,y,t \in G$ are such that $y = xt^{-1}$, so $x = yt$ and we have

$$(Ad\rho(t)T\xi)(s) == \textstyle\sum_{x\in G} A(st,yt)\xi(xt^{-1})$$

Thus $Ad\rho(t)T$ has kernel $A(st,yt)$.

In general, if $T \in C_U^*(X)$ then $\forall x,y \in G$,

$\langle Ad\rho(t)T\delta_x,\delta_y\rangle = \langle T\delta_{xt},\delta_{yt}\rangle$.

So the operator T is $Ad\rho-$ invariant if and only if

$\langle Ad\rho(t)T\delta_x,\delta_y\rangle = \langle T\delta_{xt},\delta_{yt}\rangle$, $\forall\, x,y \in X, \forall\, t \in G$.

We now define the invariant approximation property: (IAP).

Definition 21: We say that G has the invariant approximation property (IAP) if

$$C_U^*(G)^G = C_\lambda^*(G)$$

IV. Crossed Product of C^*- Algebras

Let G be a discrete group. Let $\alpha: G \curvearrowright H$ be an action of G on a C^*- algebra H: α is a homomorphism from group G into the group Aut(H) of automorphisms of H. This mean that for each $g \in G$ there is defined an automorphisms $\alpha(g)$ of H given by:

$$\alpha(x)\alpha(y) = \alpha(xy).$$

Any element of the algebraic crossed product of A by G is the formal sum $\sum a_t u_t$, where u_t is unitary, $a_t \in A$ and $t \in G$, and

$$u_{t_1}u_{t_2} = u_{t_{1}t_{2}}.$$

We denote by $H[G]$ the C^*- algebra of formal sums

$a = \sum a_t u_t$, where $t \longmapsto a_t$ is a map from G into H with finite support and where the operations are given by the following rules:

$$a_t b_s = a\alpha_t(b)ts$$
$$(a_t)^* = \alpha_{t^{-1}}(a)t^{-1}$$

for a, b in H and s, t in G.

Definition 22 [2]: A covariant representation of $: G \curvearrowright H$ is a pair (π, ρ), where π and ρ are unitary representation of G and representation of H in the same Hilbert space $\mathbb{H}$ respectively, satisfying the covariance rule

$$\forall\, a, t \in G,\ \pi(t)\rho(a)\pi(t)^* = \rho\alpha_t(a)),$$

where $\pi: G \longrightarrow U(\ell^2(G))$ and $\rho: H \longrightarrow B(\ell^2(G))$and $U(\ell^2(G))$ is unitary bounded operators.

Definition 23 [2]: The full crossed product of $H \times G$ associated with $\alpha: G \curvearrowright H$ is the *- algebra obtained as the completion of H[G] in the norm

$$\|a\| = \sup\|(\pi \times \sigma)(a)\|,$$

where $\pi \times \sigma$ runs over all covariant representation of $\alpha: G \curvearrowright H$

Next we describe the induced covariant representations.

Definition 24 [2]: Let G be a discrete group. Let π be a representation of $\mathbb{H}$ on a Hilbert space $\mathbb{H}_0$ and

$$\mathbb{H} = (\ell^2(G), \mathbb{H}_0) = \ell^2(G) \otimes \mathbb{H}_0$$

We define a covariant representation $(\widehat{\pi}, \widehat{\lambda})$ of $G \curvearrowright H$ acting on $\mathbb{H}$ by

$$\widehat{\pi}(a)\xi(t) = \pi(\alpha_{t^{-1}}(a)))\xi(t)$$

and

$$\widehat{\lambda}(s)\xi(t) = \xi(s^{-1}t)$$

for all $a \in H$ and $s, t \in G$ and all $\xi \in (\ell^2(G), \mathbb{H}_0)$. The covariant representation $(\widehat{\pi}, \widehat{\lambda})$ is said to be induced by π .

Definition 25 [2]: The reduced crossed product of $H \rtimes_r G$ is the *- algebra obtained as the completion of H[G] in the norm

$$\|a\|_r = Sup\left\|\widehat{(\pi} \times \hat{\lambda})\ (a)\right\|$$

for $a \in H[G]$, where π is a representation of H.

We recall that the Stone - Cech compactification of a set X is a compact Hausdorff space, equipped with an inclusion of the discrete space X as an open dense subset and the following universal property: Every continuous function $f: X \longrightarrow Z$ extends uniquely to continuous function $\widehat{f}$: $\beta X \longrightarrow Z$, where is a compact Hausdorff space. In particular, every bounded complex-valued function on X extends uniquely to a continuous function on βX.

Next, we show the induced action of G on $C(\beta G) \rtimes_r G$.

V. Induced Action of G on $C(\beta G) \rtimes_R G$.

First we shall describe the following statement:

$$C_U^*(G) = C(\beta G) \rtimes_r G \cong \ell^\infty(X) \rtimes_{alg} G.$$

Any element of $f \in C(\beta G)[G] \subseteq C(\beta G) \rtimes_r G$ defined by

$$f = \sum f_t\, t$$

where $f_t \in C(\beta G)$ and $t \in G$. Here:

$$f = \sum f_t\, t \longmapsto \theta(f)$$
$$\beta G \times G \longrightarrow \mathbb{C}$$

$(x, t) \longmapsto f_t(x)$ and $\theta(f)(x, t) = f_t(x)$.

This mean that an isomorphism between the *- algebra $C(\beta G)[G]$ and the *- algebra $C_c(\beta G \times G)$ of continuous functions with compact support on $\beta G \times G$ given by

$$C(\beta G)[G] = C_c(\beta G \times G)$$

The operation of the *- algebra $C_c(\beta G \times G)$ on $\beta G \times G$ is given by the following: let $F, G \in C_c(\beta G \times G)$ then we have

$$(F * G)(x, s) = \sum F(x, t)G(t^{-1}x, t^{-1}s)$$

and

$$F(x, s) = \bar{F}(s^{-1}x, s^{-1})$$

In addition: $F \longmapsto F \circ J$

where $J: (s, t \longmapsto (s^{-1}, s^{-1}t)$ which gives

$C_c(\beta G \times G) =$ bounded kernel with finitepropagation on $G \times G\}$.

The following Theorem is from Roe [8].

Theorem 26: The map between $\pi: f \longrightarrow Op(\theta(f) \circ f)$ extends to an isomorphism between the $C^* -$ algebra

$$C_U^*(G) = C(\beta G) \rtimes_r G \cong \ell^\infty(X) \rtimes_{alg} G$$

The uniform Roe algebra, $C_U^*(G)$ acts on $\ell^2(G)$, G has unitary representation on $\ell^2(G)$. (e.g. a right regular representation):

$$C_U^*(G)^G = \{T \in C_U^*(G): Ad\rho(t) = T\ for\ all\ t\ \in T\}$$

where ρ is the left regular representation of G. Let $T \in C_U^*(G)$ and $g \in G$, then we have

$$\forall g\ \in\ G\ \ Ad(\rho(g))(T) = \rho_g T \rho_g^*.$$

We obtained $C(\beta G) \rtimes_r G$, which forms a covariant representation of $\ell^\infty(G) \rtimes_r G$. In Theorem 26, we will use

$H = \ell^{\infty}(G)$, and the algebraic crossed product of H by G is the *- algebras.

Theorem 27: Assume that the algebraic crossed product $\ell^{\infty}(G) \rtimes_{alg} G$ is given by:

- the pointwise action of $\ell^{\infty}(G)$ on $\ell^{2}(G)$

$$(a\zeta)(s) = a(s)\zeta(s), a \in \ell^{\infty}(G), \xi \in \ell^{2}(G),$$

- the left regular representation λ of G on $\ell^{2}(G)$

$$(\lambda(g)\xi)(s) = \zeta(g^{-1}s),$$

Then the elements of $\ell^{\infty}(G) \rtimes_{alg} G$ which are invariant under $Ad\rho$ are of the form $(\ell^{\infty}(G))^{\rho(G)} \rtimes_{alg} G$. And also any element $(\ell^{\infty}(G))^{\rho(G)} \rtimes_{alg} G$. is of the form $a\lambda(g)$, where $a \in \ell^{\infty}(G)$ and $g \in G$.

Proof: Let $\xi \in \ell^{2}(G)$ and $g, h \in G$. Then

$$\begin{aligned}(\rho(h)a\lambda(g)\rho(h)^{*}\xi)(s) &= (\rho(h)a\rho(h)^{*}\lambda(g)\xi)(s)\\ &= (\rho(h)a\rho(h)^{*}\xi)(g^{-1}s)\\ &= (a\rho(h)^{*}\xi)(g^{-1}sh)\\ &= (a(g^{-1}sh))\rho(h)^{*}\xi(g^{-1}sh)\\ &= (\rho(h)a)(g^{-1}s)\zeta(g^{-1}s)\\ &= \big((\rho(h)a)\zeta\big)(g^{-1}s)\\ &= \rho(h)a\lambda(g)\zeta)(s).\end{aligned}$$

Since a is operator on $\ell^{2}(G)$

$$((\rho(h)a)\xi\ )(s)=a(sh)\xi(s) = a(s)\xi(s)$$

and

$$\rho(h)a = a$$

We also have

$$\rho(h)a\lambda(g)\rho(h)^{*} = a\lambda(g).$$

We define the set of fixed points $(\ell^{\infty}(G))^{\rho(G)} \rtimes_{alg} G$ of this action in the whole of $(\ell^{\infty}(G))^{\rho(G)} \rtimes_{alg} G = \Gamma(\text{say})$ as

$$\ell^{\infty}(G))^{\rho(G)} \rtimes_{alg} G \equiv \{a\lambda(g) \in \Gamma : \rho(h)a\lambda(g)\rho(h)^{*} = a\lambda(g)\}$$

The induce action of Γ on both side yields:

$$C_{U}^{*}(G)^{G} = (\ell^{\infty}(G) \rtimes_{alg} G)^{G} = (\ell^{\infty}(G))^{\rho(G)} \rtimes_{alg} G$$

Proposition 28: Let X and Y be bounded geometry discrete metric space. Then in general

$$C_{u}^{*}(X \times Y) \neq C_{u}^{*}(X) \otimes C_{u}^{*}(Y).$$

Proof: To show this, note that

$$C_{u}^{*}(X) \cong C(\beta X) \rtimes_{r} X \cong \ell^{\infty}(X) \rtimes_{r} X$$

So that

$$\begin{aligned}C_{u}^{*}(X \times Y) &\cong C\big(\beta(X \times Y)\big) \rtimes_{r} (X \times Y)\\ &\cong \ell^{\infty}(X \times Y) \rtimes_{r} (X \times Y)\end{aligned}$$

Since

$$\ell^{\infty}(X) \subseteq C_{u}^{*}(X)$$

and

$$\ell^{\infty}(X) \otimes \ell^{\infty}(X) \cong C\big(\beta(X \times Y)\big)$$

We have

$$\psi : \beta(X \times Y) \overset{\neq}{\rightarrow} \beta X \times \beta Y$$

This mean ψ is not isomorphism.

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